Tropopause in a Steady Zonal Wind Field

By H. Arakawa

(Meteorological Research Institute)

Summary

The relation of temperature and velocity to the slope of the tropopause in a steady zonal wind field is derived in spherical polar coordinates. Then, the tropopause “dip” just north of the jet streams in the westerlies is discussed.

The relation of temperature and velocity to the slope of the tropopause in a geostrophic wind field was first derived by V. Bjerknes. Further the relation of temperature and velocity to the slope of the tropopause in a gradient wind field was given by H. Ertel. The case treated by H. Ertel is admittedly a simple one, so it may not be adequate for the problem of the tropopause funnel. However, such a consideration will be of value as a preliminary to the discussion of a steady zonal wind field, though it must be admitted that the analysis will not be essentially new from that given by V. Bjerknes.
We shall define that the tropopause is a discontinuity of the first order with respect to the temperature. Let the index 1 denote the stratosphere, and the index 2 the troposphere, then the tropopause is defined by

$$T_1 - T_2 = 0. \quad (1)$$

If $\phi$ be the geocentric latitude, $\lambda$ the longitude taken as positive when measured to east, and $r$, $\phi$ and $\lambda$ the spherical coordinates, then, upon differentiating the equation of tropopause in a steady zonal wind field, it follows that

$$\left( \frac{\partial T}{\partial r} \right)_1 - \left( \frac{\partial T}{\partial r} \right)_2 \, dr + \left( \frac{\partial T}{\partial \phi} \right)_1 - \left( \frac{\partial T}{\partial \phi} \right)_2 \, r \, d\phi = 0.$$

The slope of the tropopause to the horizontal $\beta$ is given by

$$\tan \beta = \frac{dr}{r \, d\phi} = \frac{\left( \frac{\partial T}{\partial r} \right)_1 - \left( \frac{\partial T}{\partial r} \right)_2}{\left( \frac{\partial T}{\partial \phi} \right)_1 - \left( \frac{\partial T}{\partial \phi} \right)_2}. \quad (2)$$

The equations for steady zonal wind are

$$- \frac{v_\lambda^2}{r} - 2 \omega \cos \phi \cdot v_\lambda = -g \cdot \frac{1}{\rho} \frac{\partial p}{\partial r}, \quad \frac{v_\lambda^2}{r} \cdot t \phi + 2 \omega \sin \phi \cdot v_\lambda = - \frac{1}{\rho} \frac{\partial p}{\partial \phi}, \quad (3a), (3b)$$

where $v_\lambda$ is the velocity along the horizontal taken as positive when measured to east, $g$ the acceleration of gravity, $\rho$ the density, $p$ the pressure and $\omega$ the angular velocity of the earth's rotation.

Upon introducing the gas equation $p=\rho RT$ into these two equations, we get

$$\left( \frac{\partial (\log p)}{\partial r} \right) = \frac{2 \omega \cos \phi \cdot v_\lambda}{RT} + \frac{v_\lambda^2}{r \cdot RT} - \frac{g}{RT}, \quad \left( \frac{\partial (\log p)}{\partial \phi} \right) = - \frac{2 \omega \sin \phi \cdot v_\lambda}{RT} - \frac{v_\lambda^2}{r \cdot RT} \cdot t \phi. \quad (3a'), (3b')$$

Upon differentiating $(3a')$ partially with respect to $\phi$, and $(3b')$ with respect to $r$, it is found that

$$\frac{\partial T}{\partial \phi} \left( g - 2 \omega \cos \phi \cdot v_\lambda - \frac{v_\lambda^2}{r} \right) = \frac{\partial T}{\partial r} \left( 2 \omega \sin \phi \cdot v_\lambda + \frac{v_\lambda^2}{r} \cdot t \phi \right)$$

$$- T \frac{\partial v_\lambda}{\partial \phi} \left( 2 \omega \cos \phi + \frac{v_\lambda^2}{r} \right) - T \frac{\partial v_\lambda}{\partial r} \left( 2 \omega \sin \phi + \frac{2 v_\lambda}{r} \cdot t \phi \right) - \frac{T}{r} \cdot 2 \omega \sin \phi \cdot v_\lambda.$$

Because the temperature and the wind speed are the same on both sides of the tropopause,

$$\left( \frac{\partial T}{\partial \phi} \right)_1 - \left( \frac{\partial T}{\partial \phi} \right)_2 = \frac{2 \omega \sin \phi \cdot v_\lambda + \frac{v_\lambda^2}{r} \cdot t \phi}{g - 2 \omega \cos \phi \cdot v_\lambda - \frac{v_\lambda^2}{r}} \left( \frac{\partial T}{\partial r} \right)_1 - \left( \frac{\partial T}{\partial r} \right)_2$$

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Upon substituting this equation in Eq. (2), it follows that

$$\tan C = \frac{\psi}{\cos \phi \cdot v_A - \frac{v_A^2}{r}}$$

The slope of the isobaric surface is given by

$$\tan \alpha = \frac{\frac{\partial p}{\partial \phi}}{\frac{\partial p}{\partial r}} = \frac{v_A (2\omega \sin \phi + \frac{v_A}{r} \cdot \tan \phi)}{g - 2\omega \cos \phi \cdot v_A - \frac{v_A^2}{r}}, \tag{5}$$

according to Eqs. (3a) and (3b), where $\alpha$ is the slope of the isobaric surface to the horizontal. Thus the slope of the tropopause to the horizontal is

$$\tan \beta = \frac{2\omega \sin \phi + \frac{v_A}{r} \cdot \tan \phi}{g - 2\omega \cos \phi \cdot v_A - \frac{v_A^2}{r}} \left[ \frac{\frac{\partial p}{\partial \phi}}{\frac{\partial p}{\partial r}} \right] \tag{6}$$

If we assume static equilibrium in the vertical direction $\frac{\partial p}{\partial r} = -\rho g$ instead of Eq. (3a), then the slope of the tropopause to the horizontal will be given by

$$\tan \beta = \frac{2\omega \sin \phi + \frac{v_A}{r} \cdot \tan \phi}{g} \left[ \frac{\frac{\partial p}{\partial \phi}}{\frac{\partial p}{\partial r}} \right], \tag{6a}$$

where

$$\tan \alpha = \frac{v_A (2\omega \sin \phi + \frac{v_A}{r} \cdot \tan \phi)}{g}, \tag{5a}$$

The observed distribution of temperature and wind in the vertical shows that the difference of temperature lapse-rate $\left(\frac{\partial T}{\partial r}\right)_1 - \left(\frac{\partial T}{\partial r}\right)_2$ is positive and the difference $\left(\frac{\partial v_A}{\partial r}\right)_1 - \left(\frac{\partial v_A}{\partial r}\right)_2$ is negative. Solving Eq. (8b) in usual way, we find

$$v_A = -\omega \cos \phi \cdot r \pm \sqrt{\omega^2 \cos^2 \phi \cdot r^2 - \frac{\rho \cdot \cot \phi}{\rho} \cdot \frac{\partial p}{\partial \phi}},$$
The upper sign gives \( v_\lambda = \frac{-1}{2\omega \sin \phi \cdot \rho} \frac{\partial p}{\partial \phi} \) when \( r = \infty \), so yields a solution which is continuous near straight isobars, while the lower sign gives an arrangement which is not available in the earth’s atmosphere. Thus the only solution of Eq. (3b) which can be extended out to a region of straight isobars is given by

\[
v_\lambda = -\omega \cos \phi \cdot r + \sqrt{\omega^2 \cos^2 \phi \cdot r^2 - \frac{\cot \phi}{\rho} \frac{\partial p}{\partial \phi}},
\]

or

\[
v_\lambda + \omega \cos \phi \cdot r \geq 0, \quad \text{i.e.,} \quad \omega \sin \phi \cdot \frac{v_\lambda}{r} + \eta \phi \geq 0.
\]

Thus the slope of the tropopause is generally steeper than the slopes of near-by isobaric surfaces in the temperate latitudes of the northern hemisphere.

Several recent papers from the University of Chicago have attempted to infer the vertical circulation pattern associated with well developed jet streams in the westerlies. If there is to be a finite wind maximum, it is necessary that near the jet centre the slope of the isobaric surfaces steepens and above the jet centre the slope of the isobaric surfaces decreases upward. Since the atmosphere preserves hydrostatic equilibrium to a high degree of approximation, the meridional temperature gradient above the jet stream core must be opposite in direction from that of the lower levels. Thus, well developed jet streams are likely to happen in the region of the tropopause. Isobaric surfaces and the tropopause with steep slopes will be associated with the jet core in the westerlies. Thus, the tropopause dip will be found just north of the jet stream.

References


On page 107 of this book, Eq. (250), \( \eta g(a, z) = \frac{2\omega \sin \phi}{g} \cdot v + \frac{\nu^2}{r} \ldots \ldots \) should be read

\[ \eta g(a, z) = \left( 2\omega \sin \phi \cdot v + \frac{\nu^2}{r} \right) / g \ldots \ldots \]