Notes on the Energy Dispersion in the Atmosphere*

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— Abstract —

In this paper, the dispersion of atmospheric disturbances has been studied analytically. Particularly, the formation of a new trough over North America following an intense cyclogenesis in the Gulf of Alaska and the blocking action of atmospheric disturbances have been investigated, the former as a boundary value problem and the latter as an initial value problem. In the former problem, the method of Sommerfeld and Brillouin which was used in their study of the propagation of signals in a dispersive medium has been utilised.

In the latter problem, using Airy's integral, we have obtained approximate solutions, some of which agree well with those obtained numerically by T. C. Yeh.

§ 1. Introduction

Concerning the energy dispersion in the free atmosphere, many studies have been made. C. G. Rossby (1945, 1949) has noticed an important role of the dispersion which occurs in the atmosphere, and clearly elucidated the physical mechanism of the dispersion process in the atmosphere. According to him, the energy is propagated with the group velocity which can be expressed in terms of phase velocity or wave length. Recently, T. C. Yeh (1949) has investigated the mechanism of the dispersion process in detail with four kinds of models of the atmosphere and has shown the relation between the group velocity, phase velocity and wave length. As the applications of these theoretical investigations, he has given the following two statements:

(1) The formation of a new trough over North America following an intense cyclogenesis in the Gulf of Alaska can be interpreted as a result of continuous dispersion from a point source of cyclonic relative vorticity.

(2) The blocking action can be explained by the dispersion of an initial solitary wave.

Since the foregoing conclusions of Yeh have been derived from numerical calculations, we cannot see the general processes involved in the phenomenon of dispersion, although his investigation is an elaborate one.

In this paper we shall study the formation of a new trough over North America as a boundary value problem and the blocking action as an initial value problem. In addition, we shall try to explain the physical processes of dispersion in connection with the relation

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between the group velocity, phase velocity and wave length.

§ 2. Group velocity, phase velocity and wave length in the atmosphere

we assume that the atmosphere is uniform and incompressible, and has a free surface and that there is a basic current with a uniform and constant velocity U .

In this model, the vorticity equation may be written in the following form (T. Yeh 1949):

$$\frac{\partial^3 D}{\partial x^2 \partial t} + U \frac{\partial^3 D}{\partial x^3} + \beta \frac{\partial D}{\partial x} - \lambda^2 \frac{\partial D}{\partial t} = 0, \quad (2.1)$$

where D is the deformation of the free surface, β the rate of northward change of the Coriolis parameter, $\lambda^2 = f^2/gD_0$ and D_0 the undisturbed depth. In order to obtain a non-dimensional equation, we put

$$\xi = kx, \quad \tau = t\sqrt{\beta U}, \quad A^2 = \frac{f^2 U}{gD_0 \beta}, \quad k = \sqrt{\frac{\beta}{U}}.$$

Then (2.1) becomes as follows;

$$\frac{\partial^3 D}{\partial \xi^2 \partial \tau} + \frac{\partial^3 D}{\partial \xi^3} + \frac{\partial D}{\partial \xi} - A^2 \frac{\partial D}{\partial \tau} = 0. \quad (2.2)$$

Assuming a solution of the form $e^{i(k\xi - n\tau)}$, the frequency equation from (2.2) is

$$C = \frac{n}{k} = \frac{k^2 - 1}{k^2 + A^2}, \quad (2.3)$$

where k is the wave number, n the frequency and C the phase velocity. In this model, the group velocity may be written as

$$C_g = \frac{\partial n}{\partial k} = \frac{k^4 + k^2(1 + 3A^2) - A^2}{(A^2 + k^2)^2}.$$

The values of C and C_g are graphically shown in Fig. 1. Although the relation between C , C_g and wave length has been investigated by Rossby and Yeh,* we recapitulate their results for the convenience of the following discussions. As may

be seen in the figure, both C and C_g have the same lower limit $-1/A^2$, corresponding to $k=0$. With increasing wave numbers, the phase velocity C increases from $C = -1/A^2$ to $C = 1$ asymptotically, and at $k=1$, C becomes zero. The group velocity C_g first increases with wave number and reaches a maximum value at $k = \sqrt{3}A$. At this point $(C_g)_{\max} = \frac{9A^2 + 1}{8A^2}$.

Fig. 1

* Rossby and Yeh expressed C_g in another form, i. e.,

$$C_g = \frac{U + \beta L^2 + 2\lambda^2 L^2 C}{1 + \lambda^2 L^2}.$$

although this expression and ours are identical.

After this wave number is reached, C_g decreases asymptotically to the value 1. From the above considerations, it is evident that longer waves ($k \approx 0$) can be propagated westward with a negative group velocity and shorter ones ($k \gg 1$) can be propagated eastward with a positive group velocity. If we represent the above relation on the $(\xi-\tau)$ plane as in Fig. 2, the shadowed part corresponds to the domain of disturbance resulting from an instantaneous point source. The domain of disturbance is between $\xi = C_{g \min} \tau$ and $\xi = C_{g \max} \tau$. Here $C_{g \max}$ and $C_{g \min}$ are given by

$$C_{g \max} = \frac{9A^2 + 1}{8A^2},$$

$$C_{g \min} = -\frac{1}{A^2}$$

respectively and the straight line $\xi = C_g \tau$ in Fig. 2 corresponds to an arbitrary group velocity, *i. e.*, $C_g = C_g(k)$. For an arbitrary wave number k , we get the figure shown in Fig. 3. If we put $k = k_0$ at which the group velocity vanishes, the wave number k is uniform in the domain of negative ξ , provided that

$$0 < k < k_0.$$

For $k_0 < k < \infty$, the wave numbers which satisfy the following relations:

$$k_0 < k < \sqrt{3} A,$$

$$\sqrt{3} A < k < \infty,$$

are distributed, overlapping each other in the domain of positive ξ .

These properties of the group velocity for various k 's are important in the course of the following study.

§ 3. Initial value problem in a dispersive system

Concerning this problem, C. Eckart (1947) has discussed in a general form. In this section we shall give a concrete explanation of its application.

As mentioned above, there are an infinite number of straight lines originating from the origin on the $\xi-\tau$ diagram as shown in Fig. 4, each line corresponding to one specific group velocity. Here $C_g(k)$ satisfies the equation

Fig. 2

Fig. 3

$$C_g(k) = \frac{\xi}{\tau} = \frac{k^4 + k^2(1 + 3A^2) - A^2}{(A^2 + k^2)^2}. \quad (3.1)$$

In addition, the straight lines for $C_{g \max}$ and $C_{g \min}$ represent the fronts associated with the maximum and minimum group velocities.

It is an important fact that the group velocity at the fronts satisfies the equation

$$\frac{\partial C_g}{\partial k} = 0. \quad (3.2)$$

Taking these properties concerning the group velocity into account, we consider an initial value problem. Putting a particular solution of (2.2) as follows:

$$D = F(k) \exp \{i(k\xi - H(k)\tau)\}, \quad (3.3)$$

where F and H are arbitrary functions of k , we get from (2.3)

$$H(k) = \frac{k^3 - k}{k^2 + A^2}. \quad (3.4)$$

Hence the general solution is

$$D(\xi, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{i(k\xi - H(k)\tau)} dk. \quad (3.5)$$

If τ is put to be zero, (3.5) becomes

$$D(\xi, 0) \equiv D_0(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ik\xi} dk. \quad (3.6)$$

The application of the Fourier integral theorem to this equation results in

$$F(k) = \int_{-\infty}^{\infty} D_0(\xi) e^{-ik\xi} d\xi. \quad (3.7)$$

Thus, $F(k)$ is determined by the initial value of D , and *vice versa*.

The value of the integral of the type (3.5) is determined almost entirely by its value near the stationary points of the phase Θ , which is given by

$$\Theta = k\xi - H(k) \cdot \tau, \quad (3.8)$$

provided that $F(k)$ changes by only a small fraction of it, when $F(k)$ changes by 2π .

Generally there will be several values of k , i. e.,

$$k = \kappa_1(\xi, \tau), \quad k = \kappa_2(\xi, \tau), \quad \dots, \quad (3.9)$$

which satisfy the equation

$$\frac{\partial}{\partial k} \Theta(k, \xi, \tau) = 0. \quad (3.10)$$

Fig. 4

In our problem, from (3.8) and (3.10), $k = \kappa_i$ are the roots of the equation

$$\xi - \frac{\partial \Theta}{\partial k} \cdot \tau = 0,$$

or
$$C_g(k) = \frac{\xi}{\tau} = \frac{\partial \Theta}{\partial k} = \frac{k^4 + k^2(1 + 3A^2) - A^2}{(A^2 + k^2)}. \quad (3.11)$$

For the sake of simplicity, we shall assume that there is only one value $k = \kappa(\xi, \tau)$.

Near this point, Θ may be written as follows:

$$\Theta(k, \xi, \tau) = (\Theta)_{k=\kappa} + \left(\frac{\partial^2 \Theta}{\partial k^2} \right)_{k=\kappa} \cdot \frac{(k-\kappa)^2}{2} + \dots \quad (3.12)$$

Hence (3.5) becomes approximately

$$\begin{aligned} D(\xi, \tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{i(\Theta(k, \xi, \tau))}_{k=\kappa} e^{-i \left(\frac{\partial^2 \Theta}{\partial k^2} \right)_{k=\kappa} \cdot \frac{(k-\kappa)^2}{2}} dk, \\ &= F(\kappa) e^{i\Theta(\kappa, \xi, \tau)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i \left(\frac{\partial^2 \Theta}{\partial k^2} \right)_{k=\kappa} \cdot \frac{(k-\kappa)^2}{2}} dk, \\ &= \frac{1}{\sqrt{2\pi i R}} F(\kappa) e^{i\Theta(\kappa, \xi, \tau)}, \end{aligned} \quad (3.13)$$

where

$$R = - \left(\frac{\partial^2 \Theta}{\partial k^2} \right)_{k=\kappa} \quad (3.14)$$

This solution represents the progressive or retrograding wave having a constant amplitude and the velocity $C_g(\kappa)$. This approximation, however, cannot be admitted near the fronts, because R is there written as follows,

$$R = - \frac{\partial^2 \Theta}{\partial k^2} = - \frac{\partial C_g}{\partial k}$$

and from (3.2),

$$R = - \frac{\partial C_g}{\partial k} \simeq 0 \quad \text{near the fronts.}$$

Thus we see that near the fronts, we must retain terms of higher order of k in the expansion of $\Theta(k)$. Since we can put $\frac{\partial^2 \Theta}{\partial k^2} \simeq 0$ near the fronts, we have

$$\Theta = \Theta(k_0, \xi, \tau) + \left(\frac{\partial \Theta}{\partial k} \right)_{k=k_0} \cdot (k - k_0) + \frac{1}{3!} \left(\frac{\partial^3 \Theta}{\partial k^3} \right)_{k=k_0} \cdot (k - k_0)^3 + \dots, \quad (3.15)$$

where $k = k_0$ is the root of

$$\frac{\partial^2 \Theta}{\partial k^2} = \frac{\partial^2 H(k)}{\partial k^2} = \frac{\partial C_g}{\partial k} = 0. \quad (3.16)$$

(3.16) means that $k = k_0$ represents the wave numbers at $C_g = C_{g \max}$ and $C_g = C_{g \min}$. In this case, (3.5) becomes approximately

$$D(\xi, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{i\Theta(k)} dk = F(k_0) e^{i\Theta(k_0)} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \left[(k - k_0) A + \frac{1}{3} (k - k_0)^3 \delta^3 \right]} dk, \quad (3.17)$$

where

$$\left. \begin{aligned} \Theta(k_0) &= k_0 \xi - H(k_0) \tau, \\ \Delta &= \xi - (C_0)_{k=k_0} \tau, \\ \delta &= - \left\{ \frac{1}{2} \left(\frac{\partial^2 C_0}{\partial k^2} \right)_{k=k_0} \cdot \tau \right\}^{\frac{1}{2}} \end{aligned} \right\} \quad (3.18)$$

If we introduce Airy's integral (E. Eckart)

$$A_i(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\pi u + \frac{1}{3} u^3)} du,$$

(3.17) becomes

$$D(\xi, \tau) = \frac{F(k_0)}{|\delta|^{\frac{1}{2}}} A_i\left(\frac{\Delta}{\delta}\right) e^{i\Theta(k_0, \xi, \tau)}. \quad (3.19)$$

In this way, near the fronts an approximate solution is given by (3.19) and at the points remote from the fronts, (3.13) gives an approximate solution.

§ 4. An approximate solution near the fronts

Although the approximate solution near the fronts is given formally by (3.19), we shall investigate the behavior of this solution in connection with the blocking action in the westerlies.

Fig. 5

At first we consider Airy's integral, which is graphically shown in Fig. 5. The integral tends to zero rapidly for positive x , and oscillates slowly and tends to zero gradually for negative x . Since the argument x is given by

$$x = \frac{\Delta}{\delta},$$

we have

$$x = - \frac{\xi - (C_0)_{k=k_0} \tau}{\left\{ \frac{1}{2} \left(\frac{\partial^2 C_0}{\partial k^2} \right)_{k=k_0} \cdot \tau \right\}^{\frac{1}{2}}}. \quad (4.1)$$

In our model, we have the following relations (cf. section 2):

$$(I) \quad k_0 = 0, \quad C_{g \min} = \frac{-1}{A^2}, \quad \left(\frac{\partial^2 C_g}{\partial k^2} \right)_{k=0} = 6 \left(\frac{1+A^2}{A^4} \right) \quad \text{for} \quad C_g = C_{g \min}.$$

$$(II) \quad k_0 = \sqrt{3} A, \quad C_{g \max} = \frac{9A^2+1}{8A^2}, \quad \left(\frac{\partial^2 C_g}{\partial k^2} \right)_{k=\sqrt{3}A} = -\frac{3(1+A^2)}{16A^4} \quad \text{for} \quad C_g = C_{g \max}.$$

Therefore, in the neighborhood of $C_{g \max}$, $(\partial^2 C_g / \partial k^2)_{k=k_0}$ becomes negative and the advancing front takes the form shown in Fig. 6 whose schematic representation is easily understood from (3.19) and (4.1). On the other hand, $(\partial^2 C_g / \partial k^2)_{k=k_0}$ becomes positive in the neighborhood of $(C_g)_{\min}$ and hence the receding front takes the form shown in Fig. 7. Thus,

Fig. 6

Fig. 7

at the rear of the front we see always a trough and a ridge. Next we consider the slope of $D(\xi, \tau)$ to the ξ -axis. If the slope is sharp, we see a conspicuous wave front.

Let us define A and B on the ξ -axis, as may be seen in Fig. 8. Then

Fig. 8

Fig. 9

$$\overline{AB} = \xi_A - \xi_B,$$

$$x_A = (x)_{\xi=\xi_A} = \frac{\xi_A - C_g \tau}{\left\{ -\frac{1}{2} \left(\frac{\partial^2 C_g}{\partial k^2} \right) \tau \right\}^{\frac{1}{3}}}.$$

$$x_B = (x)_{\xi=\xi_B} = \frac{\xi_B - C_g \tau}{\left\{ -\frac{1}{2} \left(\frac{\partial^2 C_g}{\partial k^2} \right) \tau \right\}^{\frac{1}{3}}}$$

and

$$|x_A - x_B| \approx 4.0$$

(cf. the graph of Airy's integral and Fig. 9).

Hence

$$\overline{AB} \simeq 4 \left\{ -\frac{1}{2} \left(\frac{\partial^2 C_g}{\partial k^2} \right)_{k=k_0} \cdot \tau \right\}^{\frac{1}{2}}. \quad (4.2)$$

In this way, the slope of $D(\xi, \tau)$ to the ξ -axis is proportional to

$$\tau^{\frac{1}{2}} (\tau = t\sqrt{\beta U}), \quad \left(\frac{\partial^2 C_g}{\partial k^2} \right)_{k=k_0} \quad \text{and} \quad \frac{F(k_0)}{|\delta|}.$$

As a numerical example of $(\partial^2 C_g / \partial k^2)_{k=k_0}$, we put $U=17$ m/s, $D=8$ km. Then we have the following results:

(i) Near the receding front*:

$$\begin{aligned} \text{at } \varphi=40^\circ\text{N}, \quad A^2=0.109, \quad C_g=-9.2, \quad \frac{\partial^2 C_g}{\partial k^2}=92.4; \\ \text{at } \varphi=70^\circ\text{N}, \quad A^2=0.523, \quad C_g=-1.9, \quad \frac{\partial^2 C_g}{\partial k^2}=5.56. \end{aligned}$$

These results represent that the receding front — this is regarded as the blocking action in the westerlies — moves faster at 40°N than at 70°N and the slope of $D(\xi, \tau)$ to the ξ -axis is more gradual at 70°N than at 40°N . These are the main reason that the speed of the blocking wave decreases with latitudes and the intensity of the blocking wave increases with latitudes. (Yeh. 1949.)

(ii) Near the advancing front:

$$\begin{aligned} \text{at } \varphi=40^\circ\text{N}, \quad C_g=1.36, \quad \frac{\partial^2 C_g}{\partial k^2}=-17.3; \\ \text{at } \varphi=70^\circ\text{N}, \quad C_g=2.27, \quad \frac{\partial^2 C_g}{\partial k^2}=-1.04. \end{aligned}$$

These results mean that the advancing front moves faster at 70°N than at 40°N , while the slope of $D(\xi, \tau)$ to the ξ -axis is more gradual at 40°N than at 70°N . The latter result means that the advancing front apparently disappears more frequently at 40°N than at 70°N . In order to discuss the explicit representation of $D(\xi, \tau)$, we must consider the amplitude $F(k_0)/|\delta|$ (cf. (3.17)). For example, if we put

$$D(\xi, 0) \propto e^{-\xi^2/4a^2}, \quad (4.3)$$

we have

$$F(k) \propto e^{-a^2 k^2}. \quad (4.4)$$

Since $k_0 = \sqrt{3}A$ for $C_g = C_{g \max}$, we have

$$\begin{aligned} \text{at } \varphi=40^\circ\text{N}, \quad k_0^2=0.33, \\ \text{at } \varphi=70^\circ\text{N}, \quad k_0^2=1.57. \end{aligned}$$

If we assume the value $a^2=1.5$, from the last equation of (3.18) and (4.4) we have

$$* \quad A^2 = \frac{f^2}{g D_2} \frac{U}{\beta} \propto \frac{\sin^2 \varphi}{\cos \varphi}, \quad C_g = -\frac{1}{A^2} \propto \frac{\cos \varphi}{\sin^2 \varphi}, \quad \frac{\partial^2 C_g}{\partial k^2} = \frac{1+A^2}{A^4} \simeq \frac{1}{A^4} \propto \frac{\cos^2 \varphi}{\sin^4 \varphi}.$$

$$\left[\frac{F(k_0)}{|\delta|} \right]_{\varphi=40^\circ\text{N}} \propto \frac{0.61}{2.05 \tau^{1/3}} \approx 0.31 \tau^{-1/3},$$

$$\left[\frac{F(k_0)}{|\delta|} \right]_{\varphi=70^\circ\text{N}} \propto \frac{0.10}{0.80 \tau^{1/3}} \approx 0.13 \tau^{-1/3}.$$

From the consideration concerning the amplitude mentioned above and the discussion on $(\partial^2 C_0 / \partial k^2)$, we cannot expect a clear distinction in the apparent disappearance of the advancing front at 40°N and at 70°N such as in the case of the receding front. (The numerical examples will be given in the next section.) On the other hand, since $k_0=0$ near the receding front, we have

$$[F(k_0)]_{\varphi=40^\circ\text{N}} = [F(k_0)]_{\varphi=70^\circ\text{N}}.$$

Hence the amplitude $F(k_0)/|\delta|$ is proportional to $|\delta|^{-1}$, i.e., $\{(\partial^2 C_0 / \partial k^2) \cdot \tau\}^{-1/3}$. This result means nothing but that the intensity of blocking increases with latitudes. Concerning the oscillatory term of $D(\xi, \tau)$, i.e., $e^{i\Theta(k_0)}$, we have the following relations:

(i) Near the receding front

$$\Theta(k_0) = [k\xi - H(k)\tau]_{k=0} = 0.$$

This means that the "wave-length" of disturbance is determined by the oscillations of Airy's integral and determined empirically by measuring the interval between its consecutive maxima. Thus the "wave-length" near the fronts will increase as the cubic root of τ .

(ii) Near the advancing front

$$\Theta(k_0) = k_0(\xi - C_0\tau),$$

where

$$k_0 = \sqrt{3} A, \quad C_0 = \frac{3A^2 - 1}{4A^2} = (C)_{k=\sqrt{3}A}.$$

The wave is described as a carrier wave with the phase velocity C_0 , whose amplitude is modulated.

In this way, the disturbance at $\tau=0$ is separated into the progressive and the retrograding wave of two entirely different types. The intensity of the retrograding wave increases with latitudes, while the progressive wave has no such character. At the points remote from the front, there are waves with constant amplitudes and with the phase velocity $C_0(k)$.

§ 5. Comparison with the numerical method adopted by Yeh

Near the receding front, an approximate solution is given by

$$D(\xi, \tau) = \frac{F(k_0)}{|\delta|} A_i \left(\frac{\Delta_r}{\delta_r} \right), \quad (3.19)'$$

where

$$k_0 = 0, \quad \delta_r = \left\{ -\frac{6}{2} \left(\frac{1+A^2}{A^4} \right) \tau \right\}^{\frac{1}{3}}, \quad \Delta_r = \xi + \frac{1}{A^2} \tau.$$

On the other hand, in the neighborhood of the advancing front, an approximate solution is given by

$$D(\xi, \tau) = \frac{F(k_0)}{|\delta|} A^i \left(\frac{\Delta_a}{\delta_a} \right) \cos k_0(\xi - C_0 \tau), \quad (3 \cdot 19)''$$

where $k_0 = \sqrt{3} A$, $\delta_a = \left\{ \frac{1}{2} \left(\frac{3}{16} \right) \left(\frac{1+A^2}{A^4} \right) \tau \right\}^{\frac{1}{2}}$, $\Delta_a = \xi - \left(\frac{9A^2+1}{8A^2} \right) \tau$,
 $C_{0 \max} = \frac{9A^2+1}{8A^2}$, $C_0 = \frac{3A^2-1}{4A^2}$.

Here the subscripts r and a refer to the receding and the advancing fronts respectively.

In this section, we compare these approximate solutions with the result obtained by the numerical method.

As an example, we take the functional form of $D(\xi, \tau)$ as follows:

$$D(\xi, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{i(k\xi - H(k) \cdot \tau)} dk.$$

In addition, an initial solitary wave is assumed to be represented by

$$D(\xi, 0) = \frac{B}{2} \sqrt{\pi} a^{-1} e^{-\xi^2/4a^2},$$

where a^2 is an arbitrary non-dimensional constant and B another constant of the dimension of length. This form has been given by Yeh, who carried out the numerical computations for $a^2=1.5$. In our example, a^2 is also taken as 1.5.

For the convenience of graphical representation, Yeh assumed 7.0 as the value of $B\sqrt{\pi}/2a$.

Then we have
$$B = 7.0 \times \frac{2a}{\sqrt{\pi}},$$

and furthermore
$$F(k) = 14a\sqrt{\pi} e^{-a^2 k^2}$$

If we put $U=17$ m/s, $D_0=8$ km, we have

$$\begin{aligned} \text{at } \varphi &= 40^\circ \text{N}, & A^2 &= 0.109; \\ \text{at } \varphi &= 70^\circ \text{N}, & A^2 &= 0.523. \end{aligned}$$

In this way, the parameters of (3·19)' and (3·19)'' become as follows:

$$\begin{aligned} \text{at } \varphi &= 40^\circ \text{N}, \\ & \frac{1+A^2}{A^4} = 92.4, & \frac{1}{A^2} &= 91.7, & \frac{9A^2+1}{8A^2} &= 2.27, \\ & \delta_r = -6.7\tau^{1/3}, & \Delta_r &= \xi + 9.2\tau, & k_0 &= 0, & \delta_a &= 2.05\tau^{1/3}, \\ & \Delta_a = \xi - 2.3\tau, & k_0 &= 0.55, & C_0 &= -1.52, & C_{0 \max} &= 2.3; \\ \text{at } \varphi &= 70^\circ \text{N}, \\ & \frac{1+A^2}{A^4} = 5.56, & \frac{1}{A^2} &= 1.91, & \frac{9A^2+1}{8A^2} &= 1.36, \\ & \delta_r = -2.6\tau^{1/3}, & \Delta_r &= \xi + 1.9\tau, & k_0 &= 0, & \delta_a &= 0.80\tau^{1/3}, \\ & \Delta_a = \xi - 1.4\tau, & k_0 &= 1.23, & C_0 &= 0.27, & C_{0 \max} &= 1.4. \end{aligned}$$

Thus, near the advancing front, we have

$$\text{at } 70^\circ\text{N: } D_I(\xi, \tau) = 17\sqrt{\pi} \left(\frac{0.10}{0.80\tau^{1/3}} \right) A_i \left(\frac{\xi - 1.4\tau}{0.80\tau^{1/3}} \right) \cos 1.23(\xi - 0.27\tau),$$

$$\text{at } 40^\circ\text{N: } D_{II}(\xi, \tau) = 17\sqrt{\pi} \left(\frac{0.61}{2.05\tau^{1/3}} \right) A_i \left(\frac{\xi - 2.3\tau}{2.50\tau^{1/3}} \right) \cos 0.55(\xi + 1.52\tau).$$

On the other hand, near the receding front, we have

$$\text{at } 70^\circ\text{N, } D_{III}(\xi, \tau) = 17\sqrt{\pi} \left(\frac{1}{2.6\tau^{1/3}} \right) A_i \left(\frac{\xi + 1.9\tau}{2.6\tau^{1/3}} \right),$$

$$\text{at } 40^\circ\text{N, } D_{IV}(\xi, \tau) = 17\sqrt{\pi} \left(\frac{1}{6.7\tau^{1/3}} \right) A_i \left(\frac{\xi + 9.2\tau}{6.7\tau^{1/3}} \right)$$

Fig. 10

Fig. 11

The values of $D(\xi, \tau)$ for various ξ 's are shown in Figs. 10~13. From the figures we see that in general the approximation is considerably good, compared with that of the numerical computation. But the approximation is not so good for the progressive wave at $\varphi = 70^\circ\text{N}$. This may be attributed to the following reason. The curve on the $(k-C_g)$ diagram is so flat (cf. Fig. 14) that the approximation up to 3rd order of $(k-k_0)$ is not sufficient because $\frac{\partial^2 C_g}{\partial k^3} = \frac{\partial^3 \Theta}{\partial k^3} \approx 0$

Fig. 12

$$\psi = 70^\circ N$$

$$\tau = 2$$

Fig. 13

Fig. 14

at $k = \sqrt{3}A$. If we want to get a better approximation in this case, we must retain the term of the 4th order of $(k - k_0)$ in the expansion of Θ .

§ 6. Boundary value problem in a dispersive system

In this section we make basic considerations concerning the boundary value problem in a dispersive system. For the sake of simplicity, we assume that there is no divergence nor convergence in the atmosphere. Then for a small one-dimensional motion, the vorticity equation may be written as follows;

$$\frac{\partial^2 v}{\partial x \partial t} + U \frac{\partial^2 v}{\partial x^2} + \beta v = 0, \quad (6.1)$$

where v is the perturbation velocity transverse to the x -direction, U denotes the basic current parallel to the x -direction and β the rate of northward change of the Coriolis parameter. Assuming a solution of the form $e^{i(kx - nt)}$, we get

$$k = \frac{n \pm \sqrt{n^2 + 4\beta U}}{2U}. \quad (6.2)$$

Hence the general solution of v is given by

$$v(x, t) = \int F(n) e^{-int + iG(n)x} dn, \quad (6.3)$$

where

$$G(n) = \frac{n \pm \sqrt{n^2 + 4\beta U}}{2U}.$$

The integral of (6.3) is formally obtained, provided that $v(x, 0)$ and $\partial v(x, 0)/\partial x$ are given. This integration corresponds to that of (3.5), viz.,

$$F(k) \longrightarrow F(n), \quad \Theta(k, \xi, \tau) \longrightarrow -nt + G(n)x, \quad n \longrightarrow k.$$

Therefore a first approximation of (6.3) may be obtained by the method of steepest descent.

In this section, however, we use another method which was employed by Sommerfeld (1914) in the problem of the propagation of a signal in a dispersive medium.

As the conditions we put as follows:

$$\begin{aligned} v &= 0 & \text{at } t &= 0, \\ v &= 0, & \frac{\partial v}{\partial x} &= F(t) & \text{at } x &= 0. \end{aligned}$$

For the convenience of discussion, we put

$$F(t) = C \cos \frac{2\pi}{\tau} t \quad (C = \text{const.}).$$

If we put $\tau \rightarrow \infty$, $F(t)$ becomes constant. In this case, $F(t)$ may be written on the complex n -plane as follows:

$$F(t) = \frac{C}{2\pi} \Im \int_u e^{-int} \frac{dn}{n - \frac{2\pi}{\tau}}, \quad (6.4)$$

where $\Im$ denotes the imaginary part of the integral which follows it. The integration path u in the complex n -plane is shown in Fig. 15. If $t < 0$, the integration path u is indicated by a in Fig. 15, and then $F(t)$ vanishes. If $t > 0$, the integration path u is replaced by b and the small circle b_0 enclosing the singular point $n = 2\pi/\tau$. Then, considering the residue at the pole $n = 2\pi/\tau$, we get readily

$$F(t) = \frac{C}{2\pi} \Im \{ 2\pi i e^{-\frac{2\pi}{\tau} t} \} = C \cos \frac{2\pi}{\tau} t.$$

In this way, the solution which satisfies the boundary conditions is given by

$$v = \frac{C}{2\pi} \Re \int_u \frac{U e^{-int}}{\sqrt{n^2 + 4\beta U}} (e^{i \frac{n - \sqrt{n^2 + 4\beta U}}{2U} x} - e^{i \frac{n + \sqrt{n^2 + 4\beta U}}{2U} x}) dn, \quad (6.5)$$

Fig. 15

Fig. 16

where $\Re$ denotes the real part of the integral which follows it. The integration path of (6.5) is shown in Fig. 16, and $n = \pm 2i\sqrt{\beta U}$ are the branch points.

The cut is now the line AA' , and the pole $n = 2\pi/\tau$ lies on the real axis of n -plane.

Furthermore, it can easily be shown that if we replace u by a in Fig. 17 the integration (6.5) for $t < 0$ becomes as follows:

$$\begin{aligned} v(x, 0) &= 0 & \text{for } x > 0, \\ \frac{\partial}{\partial x} v(x, 0) &= F(t) = C \cos \frac{2\pi}{\tau} t & \text{for } t > 0, \\ &= 0 & \text{for } t < 0, \end{aligned}$$

For $t > 0$, we divide the integral along u as follows:

$$\int_u = \int_{b_0} + \int_{b_1} + \int_{b_2} + \int_{b_3} + \int_{b_4} + \int_{b_4'} + \int_{b_5} + \int_{b_5'}$$

$$\text{Since } \int_{b_4} + \int_{b_4'} = 0, \quad \int_{b_5} + \int_{b_5'} = 0,$$

we get (cf. Fig. 18)

$$\int_u = \int_{b_0} + \int_{b_1} + \int_{b_2} + \int_{b_3}$$

Furthermore, we can easily obtain $\int_{b_0}$ as follows:

$$\begin{aligned} v_{b_0} = \int_{b_0} &= \frac{C}{2\pi} \Re \left\{ \frac{2\pi i U}{4\beta U + \left(\frac{2\pi}{\tau}\right)^2} e^{i \frac{2\pi}{\tau} t + i \frac{\pi}{U \tau} x} \right. \\ &\quad \left. \times \left(-2i \sin \frac{\sqrt{4\beta U + \left(\frac{2\pi}{\tau}\right)^2}}{U} x \right) \right\} \end{aligned}$$

$$\text{Hence } v_{b_0} = \frac{2UC}{\sqrt{4\beta U + \left(\frac{2\pi}{\tau}\right)^2}} \cos \frac{2\pi}{\tau} \left(t + \frac{x}{2U} \right) \sin \frac{\sqrt{4\beta U + \left(\frac{2\pi}{\tau}\right)^2}}{2U} x. \quad (6.6)$$

Next we consider the integration $\int_{b_3}$

Since $e^{-int} \cdot e^{i \frac{n - \sqrt{n^2 + 4\beta U}}{2U} x}$ becomes e^{-int} , when n tends to ∞ , we have

$$\int_{b_3} \frac{U e^{-int}}{\sqrt{n^2 + 4\beta U}} e^{i \frac{n - \sqrt{n^2 + 4\beta U}}{2U} x} dn = 0 \quad \text{for } t > 0.$$

Similarly we have

$$\int_{b_3} \frac{U e^{-int}}{\sqrt{n^2 + 4\beta U}} e^{i \frac{n + \sqrt{n^2 + 4\beta U}}{2U} x} dn = 0 \quad \text{for } t - \frac{x}{U} > 0.$$

Now we put

$$v = v_1 - v_2,$$

where

$$v_1 = \frac{C}{2\pi} \Re \int \frac{U e^{-int}}{\sqrt{n^2 + 4\beta U}} e^{i \frac{n - \sqrt{n^2 + 4\beta U}}{2U} x} dn,$$

Fig. 17

Fig. 18

$$v_2 = \frac{C}{2\pi} \Re \int \frac{U e^{-int}}{\sqrt{n^2 + 4\beta U}} e^{t \frac{n + \sqrt{n^2 + 4\beta U}}{2U} x} dn.$$

From the above considerations, we have the following values:

$$\begin{aligned} v_1 &= \int_{b_1} + \int_{b_2} + \int_{b_0} & \text{for } t > 0, \\ v_1 &= 0 & \text{for } t < 0, \\ v_2 &= \int_{b_1} + \int_{b_2} + \int_{b_0} & \text{for } t - \frac{x}{U} > 0, \\ v_2 &= 0 & \text{for } t - \frac{x}{U} < 0. \end{aligned}$$

On the other hand, we consider the integrals $\int_{b_1} + \int_{b_2}$. Putting $n = 2i\sqrt{\beta U} \sin \varphi$, we get*

$$(v_1)_{b_1 b_2} = \int_{b_1} + \int_{b_2} = \frac{UC}{\pi} \Re \int_{-\pi}^{\pi} e^{-\sqrt{\frac{\beta}{U}} (x-2Ut) \sin \varphi + i \sqrt{\frac{\beta}{U}} x \cos \varphi} \frac{d\varphi}{\left(2i\sqrt{\beta U} \sin \varphi - \frac{2\pi}{\tau}\right)}. \quad (6.7)$$

Here $(v_1)_{b_1 b_2}$ means the value of integration of v_1 along the integration paths b_1 and b_2 . In (6.7) we put

$$\begin{aligned} (x-2Ut) \sqrt{\frac{\beta}{U}} &= i \sqrt{\frac{\beta}{U} x^2 - (x-2Ut)^2} \frac{\beta}{U} \cdot \sin \beta, \\ \sqrt{\frac{\beta}{U}} x &= \sqrt{\frac{\beta}{U} x^2 - (x-2Ut)^2} \frac{\beta}{U} \cdot \cos \beta. \end{aligned}$$

Then we have $(v_1)_{b_1 b_2} = \frac{UC}{\pi} \Re \int_{-\pi}^{\pi} e^{i \sqrt{4\beta t(x-2Ut)} \cdot \cos(\varphi + \beta)} \frac{d\varphi}{\left(2i\sqrt{\beta U} \sin \varphi - \frac{2\pi}{\tau}\right)} \quad (6.8)$

Finally we have** $(v_1)_{b_1 b_2} = UC J_0(\sqrt{4\beta t(x-2Ut)}) \Re \left(\frac{1}{2i\sqrt{\beta U} \sin \varphi - \frac{2\pi}{\tau}} \right) \quad (t > 0),$
 $= 0 \quad (t < 0),$

where the symbol "bar" means the mean value of $\left(\frac{1}{2i\sqrt{\beta U} \sin \varphi - \frac{2\pi}{\tau}} \right)$ in the interval $[-\pi, \pi]$. Quite similarly we get

* Here we assume $\Im(n) > 0$ for $\Re(n) > 0$, i. e., on b_1 we put

$$\sqrt{n^2 + 4\beta U} = 2\sqrt{\beta U} \cos \varphi \quad n = 2i\sqrt{\beta U} \sin \varphi,$$

while on b_2 we put $\sqrt{n^2 + 4\beta U} = 2\sqrt{\beta U} \cos \varphi, \quad n = -2i\sqrt{\beta U} \sin \varphi.$

** We use the following relation of Bessel function,

$$\int_{-\pi}^{\pi} e^{iz \cos(\varphi + \beta)} d\varphi = J_0(z).$$

$$(v_2)_{b_1 b_2} = UC J_0(\sqrt{4\beta t(-Ut)}) \Re \left(\frac{1}{2i\sqrt{\beta U} \sin \varphi - \frac{2\pi}{\tau}} \right), \quad \left(t - \frac{x}{U} < 0 \right)$$

$$= 0, \quad \left(t - \frac{x}{U} > 0 \right)$$

From the considerations above mentioned, we get readily

$$v = v_{b_0}, \quad \text{for} \quad t - \frac{x}{U} > 0; \quad (6.9)$$

$$v = UC J_0(\sqrt{4\beta t(x-Ut)}) \Re \left(\frac{1}{2i\sqrt{\beta U} \sin \varphi - \frac{2\pi}{\tau}} \right) + (v_1)_{b_0}, \quad \text{for} \quad t - \frac{x}{U} < 0. \quad (6.10)$$

Here $(v_1)_{b_0}$ means the value of integration of v_1 along the integration path v_{b_0} , *i.e.*,

$$(v_1)_{b_0} = \frac{C}{2\pi} \Re \left\{ \frac{2\pi i U}{\sqrt{4\beta U + \left(\frac{2\pi}{\tau}\right)^2}} e^{-i\left(\frac{2\pi}{\tau}\right)t} e^{i\frac{\left(\frac{2\pi}{\tau}\right) - \sqrt{4\beta U + \left(\frac{2\pi}{\tau}\right)^2}}{2U} x} \right\}$$

$$= \frac{UC}{\sqrt{4\beta U + \left(\frac{2\pi}{\tau}\right)^2}} \sin \left\{ \frac{\sqrt{4\beta U + \left(\frac{2\pi}{\tau}\right)^2} - \left(\frac{2\pi}{\tau}\right)}{2U} \cdot x - \frac{2\pi}{\tau} t \right\} \quad (6.11)$$

v_{b_0} is given by (6.6), and if we rewrite it in another form, we get

$$v_{b_0} = \frac{UC}{\sqrt{4\beta U + \left(\frac{2\pi}{\tau}\right)^2}} \left\{ \sin \left[\frac{\sqrt{4\beta U + \left(\frac{2\pi}{\tau}\right)^2} + \left(\frac{2\pi}{\tau}\right)}{2U} \cdot x + \frac{2\pi}{\tau} t \right] \right.$$

$$\left. + \sin \left[\frac{\sqrt{4\beta U + \left(\frac{2\pi}{\tau}\right)^2} - \left(\frac{2\pi}{\tau}\right)}{2U} x - \frac{2\pi}{\tau} t \right] \right\}. \quad (6.12)$$

From (6.9) and (6.12), we see that in the domain $x < Ut$, there are progressive and retrograding waves, which have the same period as the initial disturbance. These waves are caused as the result of the contour integration at the pole. On the other hand, in the domain $x > Ut$, there are a progressive wave resulting from contour integration at the pole and the main disturbance resulting from the contour integrations at the branch points.

The main disturbance is expressed by a Bessel function. If τ tends to infinity, *i.e.*, the initial disturbance becomes constant with respect to the time, we have the following standing waves from (6.11) and (6.12),

$$(v)_{b_0} = C \sqrt{\frac{U}{\beta}} \sin \sqrt{\frac{\beta}{U}} x, \quad \text{for} \quad x < Ut,$$

$$(v_1)_{b_0} = \frac{C}{2} \sqrt{\frac{U}{\beta}} \sin \sqrt{\frac{\beta}{U}} x, \quad \text{for} \quad x > Ut.$$

This result was noticed by Yeh, so that we do not need to go further in detail here. The function $J_0(\sqrt{4\beta t(x-Ut)})$ becomes as follows,

Fig. 19

$$J_0(\sqrt{4\beta t(x-Ut)}) = J_0\left(4\pi\sqrt{\frac{Ut}{L_s}\left(\frac{x}{L_s} - \frac{Ut}{L_s}\right)}\right),$$

where

$$L_s = 2\pi\sqrt{\frac{U}{\beta}}$$

In order to estimate the change of $J_0(\sqrt{4\beta t(x-Ut)})$ with respect to t and x , we put as an example as follows,

$$\beta = 1.6 \times 10^{-13}, \quad L_s = 5.31 \times 10^8 \quad (U = 11.6 \text{ m/s} = 100 \text{ km/day}).$$

$$\text{Then we have } J_0(\sqrt{4\beta t(x-Ut)}) = J_0(12.56 \times \sqrt{0.188t(\xi - 0.188t)}),$$

where the unit of t is a day and $\xi = \frac{x}{L_s}$. The graphical representation of $J_0(\sqrt{4\beta t(x-Ut)})$ is given in Fig. 19. It is remarkable, as Yeh has pointed out in his paper, that the establishment of a ridge immediately east of the source of cyclogenesis is followed or accompanied by the development of a second ridge still farther to the east, with a new trough developing between the two ridges.

On the other hand, in the domain $x < Ut$, when the period τ of the initial disturbance becomes null there appears no disturbance, because $(v)_0 \simeq \frac{U\tau C}{2\pi} \cos \frac{2\pi}{\tau} \left(t + \frac{x}{2U}\right) \sin \frac{\pi}{U\tau} x \simeq 0$ for small values of τ . In the domain $x > Ut$, we have

$$v \simeq -\frac{C\tau}{2\pi} U \left\{ J_0 \sqrt{4\beta t(x-Ut)} + \sin \frac{2\pi}{\tau} t \right\}, \quad (6.13)$$

for small values of τ .

It is also interesting to compare the order of $(2\pi/\tau)^2$ with the order of $4\beta U$, since $(v)_0$ contains the variables $(2\pi/\tau)^2$ and $4\beta U$ in the form $(2\pi/\tau)^2 + 4\beta U$, which is additive in these variables.

If we put
$$\left(\frac{2\pi}{\tau}\right)^2 \approx 4\beta U$$

then we have for $\beta = 1.6 \times 10^{-13}$ and $U = 10$ m/s,

$$\tau \approx 0.90 \text{ day.}$$

Thus, if the initial disturbance has the period shorter than one day, the effect of the initial disturbance are small in the domain $x < Ut$, and in the domain $x > Ut$, the effects of the initial disturbance and the main disturbance are given by (6.13). If the initial disturbance has the period longer than one day, the disturbance in the domain $x < Ut$ approaches to a standing wave as the period becomes longer.

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