Research on the Tertiary Scattering of the Sun's Ray in the Earth's Atmosphere

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Part 1. General Aspect.

By the author's research the explanation of twilight which had not been hitherto researched became possible up to about 100° of the sun's zenith distance by considering up to the secondary scattering (1). But, as for greater zenith distances, the consideration of the tertiary scattering is indispensable for the explanation of twilight. Even in the day-time radiation this scattering is also necessary for rigorous calculation. This is the object of the research.

Let O be the earth's centre and O' a point on its surface. Take a coordinate system $x_1y_1z_1$ with its origin at O , x_1 -axis being directed towards the sun, z_1 -axis normal to it on the plane containing O , O' and the sun, y_1 -axis normal to x_1 and z_1 . Let a_0 be the earth's radius. Let the cylinder

$$y_1^2 + z_1^2 = a_0^2$$

be called G -cylinder, and let its part of $x_1 > 0$, *i. e.* the sun's side be called the bright cylinder, or briefly B -cylinder. Let the upper limit of the earth's atmosphere be called G -sphere, and let the domain bounded by B -cylinder, the earth's surface and G -sphere, *i. e.* the inner part of B -cylinder be denoted by $B(i)$.

Let the domain bounded by B -cylinder and G -sphere, *i. e.* the outer part of the former be denoted by $B(o)$.

Let the half part of G -cylinder satisfying $x_1 < 0$, *i. e.* the shadow side be called dark cylinder or, briefly D -cylinder. Let the inner part of D -cylinder be denoted by $D(i)$, and let its outer part be $D(o)$.

Take a point E in the whole sky dome seen from O' . Let a cone with its centre in the atmosphere and touching the earth be A -cone. Any point in the domain bounded by G -sphere, A -cone with its vertex at E (which shall be called T -cone) and the earth's surface can be seen from E . So, let this domain be noted by T -domain. Any point F in the same domain with its vertex at T can be seen from T . Let it be F -domain.

Let A -cone with vertex at T be F -cone. Take a sunlit point F in F -domain.

The domain of such point F is not necessarily the whole F -domain and is different

according as T is in $B(o)$, $B(i)$, $D(o)$ or $D(i)$, moreover, these four domains of T are dependent on the position of E .

Take a unit volume of atmosphere in F , T and E . Receiving direct solar rays, F sends the primary scattering to T . Receiving it, T sends the secondary scattering to E . Receiving it, E sends out the tertiary scattering to O' . Let the tertiary scattering intensity received by O' be dS_3 .

Take OO' as Z -axis, X -axis normal to it on the plane determined by O , O' and the sun, Y -axis normal to XZ -plane.

By using the polar coordinates, the section of each of the above defined spheres, cones and cylinders by a plane normal to x_1 -axis is denoted as follows:

$$y_1 = r_1 \cos \alpha, \quad z_1 = r_1 \sin \alpha$$

on the plane,

$$G\text{-sphere}, \quad r_1 = \varphi_1(x_1\alpha) = \sqrt{(a_0 + l)^2 - x_1^2} = q(x_1),$$

$$G\text{-cylinder}, \quad r_1 = a_0 = \varphi_0(x_1\alpha),$$

$$\text{the earth}, \quad r_1 = \varphi_2(x_1\alpha),$$

$$A\text{-cone}, \quad r_1 = \varnothing(x_1\alpha),$$

$$G\text{-cone}, \quad r_1 = f(x_1\alpha) \text{ (to be explained later);}$$

and the value of α of the intersection points

of $\varnothing$ and φ_2 ,	be α_2, α_2' ,
of $\varnothing$ and φ_1 ,	be α_2, α_1' ,
of $\varnothing$ and φ_0 ,	be $\alpha_0, \alpha_3, \alpha_3', \alpha_0'$,
of f and φ_1 ,	be α_4, α_4' .

The small circle in which A -cone touches the earth is denoted by C -circle, and the greatest value of x_1 of the points on C -circle by x_c , and the same value corresponding to the intersection curve of A -cone and G -cylinder by x_b . Let d be the sun's dip, and $d_1 = \cos^{-1} \frac{a_0}{a_0 + l}$, where l is the depth of the earth's atmosphere.

Part 2. The Case When $0 \leq d \leq d_1$.

Chapter 1. The separation of T -domain.

Section 1. When E is in the bright zone and $x_1 > 0$.

§ 1. In $B(o)$.

The domain of T in $B(o)$ is denoted by $T(pbo)$, and

$$T(pbo) = \int_0^{q(a_0)} dx_1 \left[\left\{ \int_{\alpha_1}^{\alpha_0} + \int_{\alpha_0'}^{\alpha_1'} \right\} d\alpha \int_{\varnothing}^{\varphi_1} r_1^2 dr_1 + \int_{\alpha_0}^{\alpha_0'} d\alpha \int_{\alpha_0}^{\varphi_1} r_1^2 dr_1 \right].$$

See Figs. 1 and 5, the latter is, strictly speaking, applied to the case where E is in the bright zone and $x_1 < 0$, but may be used for $x_1 > 0$ by analogy.

Fig. 1. It illustrates the integration

$$\left(\int_{\alpha_1}^{\alpha_0} + \int_{\alpha_0'}^{\alpha_1'} \right) \int_{\Phi}^{\varphi_1} + \int_{\alpha_0}^{\alpha_0'} \int_{\sigma_0}^{\varphi_1}$$
 represented in $T(pbo)$, $T(nbo)$, $T(pdo)$, $T(ndo)$, 1st term of $T(ddo)$, 4th term of $T(g\phi)$, 1st term of $T(uido)$, 2nd term of $T(gi)$, 1st term of $F(bo)$ and 1st term of $F(di)$, without regard to the position of E .

Fig. 3. It illustrates the integration of

$$\int_{\alpha_1}^{\alpha_0'} \int_f^{\varphi_1}$$
 represented in 1st term of $T(g\phi)$ and 1st term of $T(ji)$.

§ 2. In $B(i)$.

Let it be $T(pbi)$. Let $OE=e$

$$X=e \sin \gamma \cos A, \quad Y=e \sin \gamma \sin A, \quad Z=e \cos \gamma,$$

$$0 \leq \gamma \leq \frac{\pi}{2}, \quad 0 \leq A \leq 2\pi.$$

Now, let the axes x_2 , y_2 and z_2 be defined by the following relations to X , Y and Z , z_2 being chosen as OE ,

	x_2	y_2	z_2
X	$\cos \gamma \cos A$	$-\sin A$	$\sin \gamma \cos A$
Y	$\cos \gamma \sin A$	$\cos A$	$\sin \gamma \sin A$
Z	$-\sin \gamma$	0	$\cos \gamma$

Then the x_2 axis is normal to OE on the plane OEZ , and the y_2 axis normal to $x_2 z_2$. Let $x_2' y_2' z_2'$ be a new system with its origin at E and parallel to the $x_2 y_2 z_2$ system. Let the coordinates of T be

Fig. 2. This illustrates the integration of

$$\int_{\alpha_1}^{\alpha_1'} \int_{\Phi}^{\varphi_1}$$
 represented in $T(dbo)$, 2nd term of $T(ddo)$. 3rd term of $T(g\phi)$, 5th term of $T(g\phi)$, 2nd term of $T(ji)$, 3rd term of $T(gi)$, 3rd term of $F(bo)$ and 2nd term of $T(uido)$.

Fig. 4. It illustrates the integration of

$$\left(\int_{\alpha_1}^{\alpha_2} + \int_{\alpha_2'}^{\alpha_1'} \right) \int_{\Phi}^{\varphi_1} + \int_{\alpha_2}^{\alpha_2'} \int_{\varphi_2}^{\varphi_1}$$
 represented in 2nd term of $T(g\phi)$ and 2nd term of $F(bo)$.

 Fig. 5. This makes it easy for us to understand the integration of $T(nbo)$, $T(pbo)$, $T(nbi)$, $T(ndo)$, $T(pdo)$ and $T(nbi)$, when $0 \leq d \leq d_1$.

$$x_2' = r \sin \theta_1 \cos A_1, \quad y_2' = r \sin \theta_1 \sin A_1, \quad z_2' = -r \cos \theta_1,$$

which define θ_1 and A_1 .

The circle $y_1^2 + z_1^2 = a_0^2$, $x_1 = 0$ shall be called G -circle. Let A_1' and A_2' be the values of A_1 determined by the intersection points of G - and C -circles, where A_2' is applied to the point with smaller value of y_1 and A_1' applied to the point with greater value, and the method of obtaining them was discussed in (2).

Let θ_1'' be the value of θ_1 of a point on the generating lines passing these intersection points, and define

$$\sin \theta_0 = \frac{a_0}{e}.$$

Let the circle $y_1^2 + z_1^2 = a_0^2$, $x_1 = q(a_0)$ be called B circle.

Let A_1'' and A_2'' be the values of A_1 determined by the intersection points of B -circle and the above generating lines. Let r_0 be the distance of a point on the earth from E , and r_2 be the corresponding value of G -sphere. And let r_1' , r_1'' ($r_1' < r_1''$) be the distances from the two intersection points of G -cylinder penetrated by a line passing E .

Let θ_2' be the value of θ_1 determined by a line connecting E and a point on B -circle, and θ_1' be that corresponding to G -circle.

Then the domain of T contained in $B(i)$ is

$$\begin{aligned} T(pbi) = & \left(\int_{A_2'}^{A_2''} + \int_{A_1''}^{A_1'} \right) \left[\int_0^{\theta_0} \int_{r_1'}^{r_0} + \int_{\theta_0}^{\theta_1''} \int_{r_1''}^{r_1'} \right] \\ & + \int_{A_1'}^{A_2'} \int_0^{\theta_1'} \int_{r_1'}^{r_0} + \int_{A_2''}^{A_1''} \left[\int_0^{\theta_0} \int_{r_1'}^{r_0} + \int_{\theta_0}^{\theta_2'} \int_{r_1'}^{r_2} \right]. \end{aligned}$$

See Fig. 5.

§ 3. In $D(o)$.

In this case the domain is

$$T(pdo) = \int_{-q(a_0)}^0 \left[\left\{ \int_{\alpha_1}^{\alpha_0} + \int_{\alpha_0'}^{\alpha_1'} \right\} \int_{\theta}^{\varphi_1} + \int_{\alpha_0}^{\alpha_0'} \int_{a_0}^{\varphi_1} \right] d\alpha.$$

See Figs. 1 and 5.

§ 4. In $D(i)$.

$A_1' A_2''$, θ_2' regarding B -circle should be expressed by $\bar{A}_1'$, $\bar{A}_2''$, $\bar{\theta}_2'$ when D -circle is concerned. Then

$$T(pdi) = \left(\int_{A_1'}^{\bar{A}_1''} + \int_{\bar{A}_2''}^{A_2'} \right) \left(\int_{\theta_1'}^{\theta_0} \int_{r_1'}^{r_0} + \int_{\theta_0}^{\theta_1''} \int_{r_1''}^{r_1'} \right) + \int_{\bar{A}_1''}^{\bar{A}_2''} \left(\int_{\theta_1'}^{\theta_0} \int_{r_1'}^{r_0} + \int_{\theta_0}^{\bar{\theta}_2'} \int_{r_1'}^{r_2} \right).$$

See Fig. 5.

Section 2. When E is in the bright zone and $x_1 < 0$.

§ 5. $B(o)$.

The desired domain $T(nbo)$ is equal to $T(pbo)$.

the outer part of G -cone, T in the inner part of J -cone cannot have F . When E is in the inner part of G -cone, all of T in the T -domain can have F .

a) When E is in the outer part of G -cone.

In this case, the domain of T which has F and is in $D(i)$ is denoted by $T(go)$. Note $x_J = \frac{1}{2} x_e$, which is the value of x_1 for all points on J -circle. Then

$$T(go) = \int_{x_e}^{x_f} \int_{\alpha_4}^{\alpha_1'} \int_{\varphi}^{\varphi_1} + \int_{x_f}^{x_c} \left[\left\{ \int_{\alpha_1}^{\alpha_2} + \int_{\alpha_2'}^{\alpha_1'} \right\} \int_{\varnothing}^{\varphi_1} + \int_{\alpha_2}^{\alpha_2'} \int_{\varphi_2}^{\varphi_1} \right] \\ + \int_{x_c}^{x_d} \int_{\alpha_1}^{\alpha_1'} \int_{\varnothing}^{\varphi_1} - \left[\int_{-q(a_0)}^{x_b} \left\{ \left(\int_{\alpha_1}^{\alpha_0} + \int_{\alpha_0'}^{\alpha_1'} \right) \int_{\varnothing}^{\varphi_1} + \int_{\alpha_0}^{\alpha_0'} \int_{a_0}^{\varphi_1} \right\} - \int_{x_b}^{x_d} \int_{\alpha_1}^{\alpha_1'} \int_{\varnothing}^{\varphi_1} \right]$$

See Figs. 3, 4, 1, 2 and 6.

b) When E is in the inner part of G -cone.

$T(gi)$ denotes the domain of T which is in $D(i)$ and has F . Then

$$T(gi) = \int_0^{2\pi} dA_1 \left\{ \int_0^{\theta_0} \int_0^{r_0} + \int_{\theta_0}^{\pi/2} \int_0^{r_2} \right\} \\ - \left[\int_{-q(a_0)}^{x_b} \left\{ \left(\int_{\alpha_1}^{\alpha_0} + \int_{\alpha_0'}^{\alpha_1'} \right) \int_{\varnothing}^{\varphi_1} + \int_{\alpha_0}^{\alpha_0'} \int_{a_0}^{\varphi_1} \right\} + \int_{x_b}^{x_d} \int_{\alpha_1}^{\alpha_1'} \int_{\varnothing}^{\varphi_1} \right]$$

See Figs. 1, 2 and 6.

In this chapter A -cone should be used as T -cone.

Chapter 2. The integration of F .

In this chapter A -cone should be used as F -cone.

Section 4. Problem to obtain the domain of sunlit F in the F -domain.

The separation of the discussing domain in $B(o)$, $B(i)$, $D(o)$ and $D(i)$ is unnecessary because the tertiary scattering is concerned.

§ 13. When T is in $B(o)$.

The domain denoted by $F(bo)$ is the sum of $T(pbo)$, $T(pbi)$ and $T(pdo)$, besides, it is expressed by

$$F(bo) = \int_{-q(a_0)}^0 \left[\left\{ \int_{\alpha_1}^{\alpha_0} + \int_{\alpha_0'}^{\alpha_1'} \right\} \int_{\varnothing}^{\varphi_1} + \int_{\alpha_0}^{\alpha_0'} \int_{\varphi_0}^{\varphi_1} \right] \\ + \int_0^{x_c} \left[\left\{ \int_{\alpha_1}^{\alpha_2} + \int_{\alpha_2'}^{\alpha_1'} \right\} \int_{\varnothing}^{\varphi_1} + \int_{\alpha_2}^{\alpha_2'} \int_{\varphi_2}^{\varphi_1} \right] + \int_{x_c}^{x_d} \int_{\alpha_1}^{\alpha_1'} \int_{\varnothing}^{\varphi_1}$$

See Figs. 1, 4, 2 and 5.

§ 14. When T is in $B(i)$.

F -domain is the answer.

$$F(bi) = \int_0^{2\pi} dA_1 \left\{ \int_0^{\theta_0} \int_0^{r_0} + \int_{\theta_0}^{\pi/2} \int_0^{r_2} \right\}$$

Here, the origin of $\theta_1 A_1$ is taken as T .

§ 15. When T is in $D(o)$.

The answer is the sum of $T(nbo)$, $T(nbi)$ and $T(ndo)$, besides, it is expressed by $F(bo)$

$$F(do) = F'(bo).$$

§ 16. When T is in $D(i)$.

$$F(di) = \int_{-q(a_0)}^{x_0} \left[\left\{ \int_{\alpha_1}^{\alpha_0} + \int_{\alpha_0'}^{\alpha_1'} \right\} \int_{\emptyset}^{\varphi_1} + \int_{\alpha_0}^{\alpha_0'} \int_{a_0}^{\varphi_1} \right] + \int_{x_0}^{x_d} \int_{\alpha_1}^{\alpha_1'} \int_{\emptyset}^{\varphi_1}$$

See Figs. 1, 2 and 6.

Chapter 3. The integration of E .

§ 17. General aspect.

In this part the intersection point of the Z -axis and G -sphere (called G' -point) is in the bright zone.

The value of θ determined by a line connecting O' and a point on D -circle is denoted by Θ , and its length by R_0 , the distance between O' and a point on G -sphere by R_1 .

A' denotes the value of A of a point on D -circle and on the horizontal plane.

§ 18. When E is in the bright zone and $x_1 > 0$, i. e. $B(o)$.

The plane $x_1=0$ is transformed into

$$X \cos d - Z \sin d = 0, \quad X' \cos d - (Z' + \alpha_0) \sin d = 0,$$

where the $X'Y'Z'$ system is produced from the XYZ system by translating the origin O to O' .

$$\text{Now, let} \quad X' = R \cos \theta \cos A, \quad Y' = R \cos \theta \sin A, \quad Z' = R \sin \theta,$$

then

$$R(\cos \theta \cos A \cos d - \sin \theta \sin d) = \alpha_0 \sin d,$$

can give us the lower limit of integration with respect to R . Let it be R_0' . This is the distance of a point on the $x_1=0$ plane from O' . A_0' denotes the value of A of the intersection point of $x_1=0$ and the horizontal line

$$X'^2 + Y'^2 = q^2(a_0).$$

Then,

$$\text{tg } A_0' = \frac{\alpha_0 \sec d}{q(a_0)}$$

It is the limit of A .

How about the range of θ , then? This is determined by a line connecting O' and a point on the intersection circle (denoted by A -circle) of G -sphere and $x_1=0$. G -sphere $y_1^2 + z_1^2 = (\alpha_0 + l)^2$ is reduced to

$$Y'^2 + (X' \sin d + (Z' + \alpha_0) \cos d)^2 = (\alpha_0 + l)^2,$$

and to $(R \cos \theta \sin A)^2 + (R \cos \theta \cos A \sin d + (R \sin \theta + \alpha_0) \cos d)^2 = (\alpha_0 + l)^2$.

$$\text{From the last and} \quad R(\cos \theta \cos d \cos A - \sin d \sin \theta) = \alpha_0 \sin d,$$

we can obtain θ for one A . Let it be Θ_0 . Then the domain of E is

$$E(p) = 2 \int_0^{A_0'} dA \int_0^{\Theta_0} \cos \theta d\theta \int_{R_0'}^{R_1} R^2 dR.$$

See Fig. 5.

§ 19. When E is in the bright zone and $x_1 < 0$.

The domain is by omitting dA , $\cos \theta d\theta$, $R^2 dR$ under the integral signs,

$$E(n) = 2 \int_0^{A_0'} \left\{ \int_0^{\theta_0} \int_{R_0}^{R_0'} + \int_{\theta_0}^{\pi/2} \int_{R_0}^{R_1} \right\} + 2 \int_{A_0'}^{A'} \int_0^{\pi/2} \int_{R_0}^{R_1} + 2 \int_{A'}^{\pi} \int_{\theta}^{\pi/2} \int_{R_0}^{R_1}$$

See Fig. 5.

§ 20. When E is in the dark zone and inner part of G -cone.

R_2 denotes the distance of a point on G -cone from O' . Then

$$E(gi) = 2 \left[\int_0^{A'} \int_0^{\pi/2} \int_0^{R_0} + \int_{A'}^{\pi} \left(\int_0^{\theta} \int_0^{R_2} + \int_{\theta}^{\pi/2} \int_0^{R_0} \right) \right]$$

See Fig. 6.

§ 21. When E is in dark zone and outer part of G cone.

$$E(go) = 2 \int_{A'}^{\pi} \int_0^{\theta} \int_{R_0}^{R_1}$$

See Fig. 6.

Chapter 4. Horizontal intensity due to the tertiary scattering.

D_3 denotes the intensity, and let the last expressions in the preceding paragraphs be denoted by the number of the same paragraphs. Then

$$D_3 = [\text{§ 18}(1.13+2.14+3.15+4.16) + \text{§ 19}(5.13+6.14+7.15+8.16) \\ + \text{§ 20}(9.13+10.14+11.15+12.b.16) + \text{§ 21}(9.13+10.14+11.15+12.a.16)] dS_3.$$

More fully

$$D_3 = [E(p) \{ T(pbo)F(bo) + T(pbi)F(bi) + T(pdo)F(do) + T(pdi)F(di) \} + E(n) \{ T(nbo)F(bo) \\ + T(nbi)F(bi) + T(ndo)F(do) + T(ndi)F(di) \} + E(gi) \{ T(dbo)F(bo) + T(ddo)F(do) \\ + T(gi)F(di) \} + E(go) \{ T(dbo)F(bo) + T(ddo)F(do) + T(go)F(di) \}] dS_3.$$

For the integration of T and F , the x_1 - y_1 - z_1 and x_2 '- y_2 '- z_2 ' systems are used. In the results of these two integrations the systems should be converted to the X' - Y' - Z' system, and we may integrate with respect to E by using this system.

Part 3. The Case When $d_1 \leq d \leq 2d$.

Chapter 5. General aspect.

G' -point is in the dark zone. In this case the mere division of the dark zone into the inner and outer parts of G -cone is not sufficient.

Let U -cone be a cone guided by A -circle and touching the earth on the side of $x_1 < 0$, then it is tangent to the horizontal plane just when $d=d_1$, but when d reaches the value larger than d_1 it will float on the plane, and the T -cone for E within U -cone can not invade $B(o)$. The cone for other E can do it. Otherwise by the same reason as in the preceding Part the domain of E should be divided by G -cone. Hence the domain of E , i.e. the sky dome must be divided into four parts.

Chapter 7. The integration of T .

Section 5. When E is in the birght zone and $x_1 < 0$.

- § 28. $B(o)$. It is equal to $T(nbo)$.
 § 29. $B(i)$. It is equal to $T(nbi)$.
 § 30. $D(o)$. It is equal to $T(ndo)$.
 § 31. $D(i)$. It is equal to $T(ndi)$.

Section 6. When E is in the dark zone and $E(gouo)$.

- § 32. $B(o)$. It is equal to $T(dbo)$.
 § 33. $B(i)$. None.
 § 34. $D(o)$. It is equal to $T(ddo)$.
 § 35. $D(i)$. It is equal to $T(goi)$.

section 7. When E is in the dark zone and is $E(goui)$.

- § 36. $B(o)$.

The integration vanishes as explained in Chapter 5.

- § 37. $B(i)$.

As E is in the dark zone, there exists no $B(i)$.

- § 38. $D(o)$.

Let it be $T(uido)$, then as $x_a < 0$, so

$$E(uido) = \int_{-q(a_0)}^{x_0} \left[\left\{ \int_{\alpha_1}^{\alpha_0} + \int_{\alpha_0'}^{\alpha_1'} \right\} \int_{\emptyset}^{\varphi_1} + \int_{\alpha_0}^{\alpha_0'} \int_{\alpha_0}^{\varphi_1} \right] + \int_{x_0}^{x_{it}} \int_{\alpha_1}^{\alpha_1'} \int_{\emptyset}^{\varphi_1}$$

See Figs. 1, 2 and 7.

- § 39. $D(i)$.

Whether E is within U -cone or not, is only related with whether T exists in $B(o)$ or not, not related with $D(i)$. Hence the integration is equal to § 35 and $T(goi)$.

Section 8. When E is in the dark zone and is $E(giuo)$.

- § 40. $B(o)$.

Whether E is within G -cone or not is not related with $B(o)$, so it is the same as in § 32 and is $T(dbo)$.

- § 41. $B(i)$. None.

- § 42. $D(o)$.

It is the same as in § 34 and is $T(ddo)$.

- § 43. $D(i)$. It is equal to $T(gi)$.

Section 9. When E is in the dark zone and is $E(giui)$.

- § 44. $B(o)$. It vanishes.

- § 45. $B(i)$. None.

- § 46. $D(o)$. It is equal to $T(uido)$.

- § 47. $D(i)$. It is equal to $T(gi)$.

Chapter 8. The integration of F .

This is the same as in the paragraphs from 13 to 16 in Chapter 2.

Chapter 9. The horizontal intensity due to the tertiary scattering when $d_1 \leq d \leq 2d_1$.

$$\begin{aligned}
 D_3 = & [\S 23 \begin{pmatrix} 28.13 & 29.14 & 30.15 & 31.16 \\ (5) & (6) & (7) & (8) \end{pmatrix} + \S 24 \begin{pmatrix} 32.13 & 33.14 & 34.15 & 35.16 \\ (9) & (10) & (11) & (12a) \end{pmatrix} \\
 & + \S 25 \begin{pmatrix} 36.13 & 37.14 & 38.15 & 39.16 \\ (12a) & & & \end{pmatrix} + \S 26 \begin{pmatrix} 40.13 & 41.14 & 42.15 & 43.16 \\ & (9) & (11) & (12b) \end{pmatrix} \\
 & + \S 27 \begin{pmatrix} 44.13 & 45.14 & 46.15 & 47.16 \\ (38) & (12b) & & \end{pmatrix}] dS_3.
 \end{aligned}$$

$$\begin{aligned}
 = & [E(n)(T(nbo)F(bo) + T(nbi)F(bi) + T(ndo)F(do) + T(ndi)F(di)) + E(gouo)(T(dbo)F(bo) \\
 & + T(ddo)F(do) + T(goi)F(di)) + E(goui)(T(uido)F(do) + T(goi)F(di)) \\
 & + E(giuo)(T(dbo)F(bo) + T(ddo)F(do) + T(gi)F(di)) + E(giui) + (T(uido)F(do) \\
 & + T(gi)F(di))] dS_3.
 \end{aligned}$$

This formula shows us:

The difference of ui and uo consists in that T invades $B(o)$ or not, and in the conditions under which T invades $D(o)$, and yet when uo it is equal to $T(ddo)$ which represents the conditions of T in $D(o)$ when E is in the dark zone in the case when $0 \leq d \leq d_1$, notwithstanding in go or gi .

The difference of go and gi consists in the conditions of $D(i)$.

Part 4. The Case When $2d_1 \leq d \leq 3d_1$.

Chapter 10. General aspect.

When $d=2d_1$, the bright zone just vanishes and the whole sky is contained in the dark zone. Moreover it is all without G -cone.

See Figs. 9 and 10. Hereafter, the inner and outer parts of G -cone shall be denoted by $G(i)$ and $G(o)$, etc.

$G(i)$ in Parts 2 and 3 means a part of G -cone bounded by D -cylinder and G -

Fig. 9. It is convenient to understand the elevation of the domains $E(gouo)$, $E(goui)$ and $E(jiui)$, and $T(ji)$, when $2d_1 \leq d \leq 3d_1$.

Fig. 10. It is convenient to understand the plan of each of the domains $E(gouo)$, $E(goui)$ and $E(jiui)$, when $2d_1 \leq d \leq 3d_1$.

cone. But, when d becomes larger than $2d_1$, $G(i)$ in this meaning vanishes, but the whole sky dome is divided into two portions; the one is $G(o)$ and the other $J(i)$ in § 12. And in both portions, all of T in T -domain have not F . That $J(i)$ has such character is a different point to $G(i)$ in the preceding two parts.

$J(i)$ is contained in $U(i)$, but $G(o)$ is divided into $U(i)$ and $U(o)$. Hence the whole sky dome is divided into three portions as in Fig. 8. And T -cone for E in $J(i)$ cannot invade $D(o)$, but, $J(o)$ can.

Chapter 11. The integration of E .

See Figs. 9 and 10.

§ 48. When E is in $E(gouo)$. The domain of E

$$E(gouo) = 2 \left[\int_0^{A''} \int_0^{\theta'} \int_{R_3}^{R_1} \right]$$

§ 49. When E is in $E(goui)$.

$$E(goui) = 2 \left[\int_0^{A''} \left\{ \int_0^{\theta'} \int_{R_2}^{R_3} + \int_{\theta'}^{\pi/2} \int_{R_2}^{R_1} \right\} + \int_{A''}^{A_0'} \int_0^{\pi/2} \int_{R_2}^{R_1} + \int_{A_0'}^{\pi} \int_{\theta_0'}^{\pi/2} \int_{R_2}^{R_1} \right]$$

§ 50. When E is in $E(jiui)$.

$E(jiui)$ means that E is within J - and U -cones. Then

$$E(jiui) = 2 \left[\int_0^{A_0'} \int_0^{\pi/2} \int_0^{R_2} + \int_{A_0'}^{\pi} \left\{ \int_0^{\theta_0'} \int_0^{R_1} + \int_{\theta_0'}^{\pi/2} \int_0^{R_2} \right\} \right]$$

Chapter 12. The integration of T .

Section 10. When E is in $E(gouo)$.

§ 51. The integration of $B(o)$. It is equal to $T(dbo)$ of § 9.

§ 52. The integration of $D(o)$. It is equal to $T(ddo)$ in § 11.

§ 53. The integration of $D(i)$. It is equal to $T(go)$ in § 12 a.

Section 11. When E is in $E(goui)$.

§ 54. $B(o)$.

As E is in $U(i)$, so T -cone cannot invade $B(o)$, and the integration is zero. See Fig. 9.

§ 55. $D(o)$. It is equal to $T(uido)$ in § 38.

§ 56. $D(i)$. It is equal to $T(go)$ in § 12 a.

Section 12. When E is in $E(jiui)$.

§ 57. $B(o)$. None.

§ 58. $D(o)$. None.

§ 59. $D(i)$.

The greatest value of x_1 of the points on the intersection curve of T - and of J -cones is denoted by x_j , then the integration of T is

$$T(ji) = \int_{x_0}^{x_j} \int_{\alpha_1}^{\alpha_1'} \int_f^{\varphi_1} + \int_{x_j}^{x_1} \int_{\alpha_1}^{\alpha_1'} \int_{\emptyset}^{\varphi_1}$$

See Fig. 9.

Chapter 13. The integration of F .

The integration of F when T is in $B(o)$, $B(i)$, $D(o)$, and when $D(i)$ is the same as in Part 2.

Chapter 14. The horizontal intensity due to the tertiary scattering when $2d_1 \leq d \leq 3d_1$.

In this case we get

$$\begin{aligned}
 D_3 = & [\S 48 (51.13 + 52.15 + 53.16) + \S 49 (54.13 + 55.15 + 56.16) + \S 50 (57.13 + 58.15 + 59.16)] dS_3. \\
 & (9) \quad (11) \quad (12.a) \quad (38) \quad (12.a) \\
 = & E(gouo) \{ T(dbo) T(bo) + T(ldo) F(do) + T(go) F(di) \} + E(goui) \{ T(uido) F(do) \\
 & + T(go) F(di) \} + E(jiui) T(ji) F(di)] dS_3.
 \end{aligned}$$

Part 5. The Case When $3d_1 \leq d \leq 4d_1$.

Chapter 15. General aspect.

See Fig. 11. When $d = 3d_1$, there exists no portion in the whole sky dome which is in $U(o)$, hence T -cone cannot invade $B(o)$. The whole sky is contained in $U(i)$ and divided in $G(o)$ and $J(i)$. When it becomes $d > 3d_1$ the above conditions is invariable, but G' -point is in $J(i)$.

Fig. 11. It is convenient to understand the domains $E(goui)$ and $E(jiui)$ when $3d_1 \leq d \leq 4d_1$.

Chapter 10. The integration of E .

§ 60. $E(goui)$.

$$E(goui) = 2 \int_0^{A_0'} \int_0^{\Theta_0'} \int_{R_2}^{R_1}$$

§ 61. $E(jiui)$.

$$E(jiui) = 2 \left[\int_0^{A_0'} \left\{ \int_0^{\Theta_0'} \int_0^{R_2} + \int_{\Theta_0'}^{\pi/2} \int_0^{R_1} \right\} + \int_{A_0'}^{\pi} \int_0^{\pi/2} \int_0^{R_1} \right]$$

Chapter 17. The integration of T .

Section 13. When E is in $E(goui)$.

§ 62. $D(o)$. It is equal to $T(uido)$ in § 38.

§ 63. $D(i)$. It is equal to $T(go)$ in § 12 a.

Section 14. When E is in $E(jiui)$.

§ 64. $D(o)$. None.

§ 65. $D(i)$. It is equal to $T(ji)$ in § 59.

Chapter 18. The integration of F .

It is the same as in Part 2.

Chapter 19. The horizontal intensity due to the tertiary scattering, when
 $3d_1 \leq d \leq 4d_1$.

$$D_3 = [\S 60 (62.15 + 63.16) + \S 61 (64.15 + 65.16)] dS_3.$$

$$(38) \quad (12a) \quad (59)$$

$$= [E(goui) \{T(uido) F(do) + T(go) F(di)\} + E(jiui) T(ji) F(di)] dS_3.$$

Part 6. The Case When $4d_1 \leq d \leq 5d_1$.

Chapter 20 General aspect.

See Fig. 12. When $d = 4d_1$, $G(o)$ just vanishes in the sky dome, and it is all in $J(i)$ and naturally in $U(i)$. When it reaches $d > 4d_1$, the above is the same, but there is produced a portion in which E cannot send the tertiary scattering to O' . Let K -cone be the cone (not J -cone itself) guided by a small circle, in which J -cone and G -sphere intersect, and touching the earth. The domain of E within K -cone is denoted by $E(ki)$, then E in it cannot indifferent to the tertiary scattering, i. e. T -domain for this E has not F . E in $E(ko)$ can produce the tertiary scattering.

Fig. 12. It illustrates the conditions when $d \geq 4d_1$.

Chapter 21. The integration of E .

§ 66. G' -point is in $E(ko)$.

Let A_0'' be the value of A determined by the intersection point of the horizontal line and the intersection curve of K -cone and the horizontal plane, and R_4 the distance of a point on K -cone from O' , and Θ_0'' the value of θ given by a point on the intersection curve of K -cone and G -sphere. Then

$$E(ko) = 2 \left[\int_0^{A_0''} \left\{ \int_0^{\pi/2} \int_{R_4}^{R_1} + \int_{A_0''}^{\pi} \int_{\Theta_0''}^{\pi/2} \int_{R_4}^{R_1} \right\} \right]$$

Chapter 22. The integration of T .

§ 67. As $E(ko)$ is in $J(i)$, the domain of T is equal to $T(ji)$ in § 59.

Chapter 23. The integration of F .

It is the same as in Part 2.

Chapter 24. The horizontal intensity due to the tertiary scattering when
 $4d_1 \leq d \leq 5d_1$.

$$D_3 = [\S 66, 67, 16] dS_3.$$

$$(59)$$

$$= [E(ko) T(ji) F(di)] dS_3.$$

Part 7. The Case When $5d_1 \leq d \leq 6d_1$.

Chapter 25. The integration of E .

§ 68. $E(ko)$.

When $d > 5d_1$, G' -point is in $E(ki)$. $E(ko) = 2 \int_0^{A_0''} \int_0^{\theta_0''} \int_{R_4}^{R_1}$

Chapter 26. The integration of T .

§ 69. When E is in $E(ko)$. It is equal to $T(ji)$ in § 67 and § 59.

Chapter 27. The integration of F .

It is equal to $F(di)$ in Part 2.

Chapter 28. The horizontal intensity due to the tertiary scattering when $5d_1 \leq d \leq 6d_1$.

$$\begin{aligned} D_3 &= [\S 68, 69, 16] dS_3. \\ (58) \\ &= [E(ko)T(ji)F(di)] dS_3. \end{aligned}$$

Part 8. The Case When $d \geq 6d_1$.

Chapter 29. General aspect.

When $d = 6d_1$, the whole sky dome is contained in K -cone, so it cannot produce the tertiary scattering. $D_3 = 0$.

Part 9. In the Case $0 \leq h \leq d_1$.

Chapter 30. General aspect.

h denotes the altitude of the sun in daytime. The whole sky is contained in the bright zone. See Fig. 5. Let x_b' be the smallest value of x_1 of the points on the intersection curve of A -cone and B -cylinder, x_c' the converted value from greatest to smallest in the definition of x_c .

Chapter 31. The integration of T .

Section 15. When E is in $x_1 < 0$.

§ 70. The domain of T in $B(o)$.

Let $T'(nbo)$ be the answer, then it is equal to $T(pdo)$ in § 3 with the sign of x_1 converted, and so equal to $T(pbo)$ in § 1.

$$T'(nbo) = T(nbo) = T(pbo).$$

§ 71. The domain of T in $B(i)$.

This is equal to $T(pdi)$ in § 4, A_1 being substituted for $A_1 + \pi$. Let $T'(pdi, A_1 + \pi)$ be the result of substitution, then

$$T'(nbi) = T'(pdi, A_1 + \pi).$$

§ 72. The domain of T in $D(o)$.

It is equal to $T(pbo)$ in § 1 with the sign of x_1 converted, so equal to $T(pdo)$ in § 3.

$$T'(ndo) = T(pdo) = T(ndo).$$

§ 73. The domain of T in $D(i)$.

It is equal to $T(pbi)$ in § 2, A_1 being substituted for $A_1 + \pi$.

$$T'(ndi) = T(pbi, A_1 + \pi).$$

Section 16. When E is in $x_1 > 0$ and $B(o)$.

§ 74. T in $B(o)$.

It is equal to $T(ndo)$ in § 7 with the sign of x_1 converted, and so equal to $T(pbo) = T(nbo)$.

$$T'(pbo) = T(pbo).$$

§ 75. T in $B(i)$.

It is equal to $T(ndi)$ in § 8, A_1 being substituted for $A_1 + \pi$.

$$T'(pbi) = T(ndi, A_1 + \pi).$$

§ 76. T in $D(o)$.

It is equal to $T(nbo)$ in § 5 with the sign of x_1 converted, and so equal to $T(pdo)$ in § 3.

$$T'(pdo) = T(pdo).$$

§ 77. T in $D(i)$.

It is equal to $T(nbi)$ in § 6, A_1 being substituted for $A_1 + \pi$.

$$T'(pdi) = T(nbi, A_1 + \pi).$$

Section 17. When E is in $x_1 > 0$ and $B(i)$.

§ 78. T in $B(o)$.

$$T'(bbo) = T(nbo).$$

§ 79. T in $B(i)$.

$$T'(bbi) = \int_0^{2\pi} dA_1 \left\{ \int_0^{\theta_0} \int_0^{r_0} + \int_{\theta_0}^{\pi/2} \int_0^{r_2} \right\} \\ - \left[\int_{x_a'}^{x_b'} \int_{\alpha_1}^{\alpha_1'} \int_{\emptyset}^{\varphi_1} + \int_{x_b'}^{x_a'} \int_{\alpha_1}^{g(\alpha_0)} \left\{ \left(\int_{\alpha_1}^{\alpha_0} + \int_{\alpha_0'}^{\alpha_1'} \right) \int_{\emptyset}^{\varphi_1} + \int_{\alpha_0}^{\alpha_0'} \int_{\alpha_0}^{\varphi_1} \right\} \right].$$

§ 80. T in $D(o)$.

$$T'(bdo) = \int_{x_a'}^0 \int_{\alpha_1}^{\alpha_1'} \int_{\emptyset}^{\varphi_1}$$

§ 81. T in $D(i)$. None.

Chapter 32. The integration of F .

§ 82. When T is in $B(o)$. It is equal to $F(bo)$ in § 13.

$$F'(bo) = F(bo).$$

§ 83. When T is in $B(i)$. It is equal to $F(bi)$ in § 14.

$$F'(bi) = F(bi).$$

§ 84. When T is in $D(o)$. It is equal to $F(do)$ in § 15.

$$F'(do) = F(do).$$

§ 85. When T is in $D(i)$. It is equal to $F(di)$ in § 16.

$$F'(di) = F(di).$$

Chapter 33. The integration of E .

§ 86. When E is in $x_1 < 0$.

It is equal to $E(p)$ in § 18, A being substituted for $A + \pi$. Let it be $E(p, A + \pi)$, then the answer $E'(n)$ is

$$E'(n) = 2 \int_{\pi}^{A_0 + \pi} dA \int_0^{\Theta_0} \cos \theta d\theta \int_{R_0}^{R_1} R^2 dR = E(p, A + \pi).$$

§ 87. When E is in $x_1 > 0$ and $B(o)$.

It is equal to $E(n)$ in § 19, substituting A for $A + \pi$.

$$E'(p) = E(n, A + \pi).$$

§ 88. When E is in $x_1 > 0$ and $B(i)$.

$$E'(b) = 2 \left[\int_{\pi}^{A' + \pi} \int_0^{\pi/2} \int_0^{R_0} + \int_{A' + \pi}^{2\pi} \left\{ \int_0^{\Theta} \int_0^{R_1} + \int_{\Theta}^{\pi/2} \int_0^{R_0} \right\} \right].$$

Chapter 34. The horizontal intensity due to the tertiary scattering when $0 \leq h \leq d_1$.

$$B_3 = [\text{§ 86 } (70.13 + 71.14 + 72.15 + 73.16) + \text{§ 87 } (74.13 + 75.14 + 76.15 + 77.16) \\ + \text{§ 88 } (78.13 + 79.14 + 80.15 + 81.16)] dS_3.$$

$$= [E'(n) \{T'(nbo)F(bo) + T'(nbi)F(bi) + T'(ndo)F(do) + T'(ndi)F(di)\} + E'(p) \{T'(pbo)F(bo) \\ + T'(pbi)F(bi) + T'(pdo)F(do) + T'(pdi)F(di)\} + E'(b) \{T'(bbo)F(bo) + T'(bbi)F(bi) \\ + T'(bdo)F(do) + T'(bdi)F(di)\}] dS_3.$$

Part 10. The Case When $d_1 \leq h \leq 2d_1$.

Chapter 35. General aspect.

See Fig. 7. In this case the whole sky dome loses the portion of $x_1 < 0$ and is divided into two parts, $B(o)$ and $B(i)$.

Chapter 36. The integration of T .

Section 18. When E is in $B(o)$.

§ 89. T in $B(o)$. It is the same as in § 74.

§ 90. T in $B(i)$. It is the same as in § 75.

§ 91. T in $D(o)$. It is equal to § 76.

§ 92. T in $D(i)$. It is the same as in § 77.

Section 19. When E is in $B(i)$.

§ 93. T in $B(o)$. It is the same as in § 78.

§ 94. T in $B(i)$. It is the same as in § 79.

§ 95. T in $D(o)$. It is the same as in § 80.

§ 96. T in $D(i)$. It is the same as in § 81.

Chapter 37. The integration of E .

§ 97. When E is in $B(o)$.

It is equal to the substitution of A for $A+\pi$ in § 23.

$$E'(p) = 2 \int_{\pi}^{A'+\pi} dA \int_0^{\theta} \cos \theta d\theta \int_{R_0}^{R_1} R^2 dR.$$

§ 98. When E is in $B(i)$.

$$E'(b) = 2 \left[\int_{\pi}^{A'+\pi} \left\{ \int_0^{\theta} \int_0^{R_0} + \int_{\theta}^{\pi/2} \int_0^{R_1} \right\} + \int_{A'+\pi}^{2\pi} \int_0^{\pi/2} \int_0^{R_1} \right]$$

Chapter 38. The horizontal intensity due to the tertiary scattering when

$$d_1 \leq h \leq 2d_1.$$

$$B_3 = [\text{§ 97 (89.13+60.14+91.15+92.16)} + \text{§ 98 (93.13+94.14+95.15+96.16)}] dS_3.$$

(74) (75) (76) (77) (78) (79) (80) (81)

$$= [E'(p)(T'(pbo)F(bo) + T'(pbi)F(bi) + T'(pdo)F(do) + T'(pdi)F(di)) \\ + E'(b)(T'(bbo)F(bo) + T'(bbi)F(bi) + T'(bdo)F(do) + T'(bdi)F(di))] dS_3.$$

Part 11. The Case When $2d_1 \leq h \leq 3d_1$.

Chapter 39. General aspect.

Let J' -cone be the cone guided by B -circle and touching the earth in the sun's side. The whole sky is contained in $G(i)$ and is divided into two parts: within and without J' -cone, i. e. $J'(i)$ and $J'(o)$. See Fig. 9.

Chapter 40. The integration of T .

Section 20. When E is in $J'(o)$.

T -domain can invade $B(o)$ and $B(i)$ only.

§ 99. T in $B(o)$.

$$T'(j'obo) = \int_{x_a'}^{x_b'} \int_{\alpha_1'}^{\alpha_1'} \int_{\phi}^{\phi_1} + \int_{x_b'}^{x_a'} \left\{ \left(\int_{\alpha_1}^{\alpha_0} + \int_{\alpha_0'}^{\alpha_1'} \right) \int_{\phi}^{\phi_1} + \int_{\alpha_0}^{\alpha_0'} \int_{\alpha_0}^{\phi_1} \right\}$$

§ 100. T in $B(i)$.

$$T'(j'obi) = \int_0^{2\pi} dA_1 \left\{ \int_0^{\theta_0} \int_0^{r_0} + \int_{\theta_0}^{\pi/2} \int_0^{r_2} \right\} \\ - \left[\int_{x_a'}^{x_b'} \int_{\alpha_1'}^{\alpha_1'} \int_{\phi}^{\phi_1} + \int_{x_b'}^{x_a'} \left\{ \left(\int_{\alpha_1}^{\alpha_0} + \int_{\alpha_0'}^{\alpha_1'} \right) \int_{\phi}^{\phi_1} + \int_{\alpha_0}^{\alpha_0'} \int_{\alpha_0}^{\phi_1} \right\} \right]$$

Section 21. When E is in $J'(i)$.

In this case all the T -domain is contained in $B(i)$.

§ 101. T in $B(i)$. Let it be $T'(j'i)$.

$$T'(j'i) = \int_0^{2\pi} dA_1 \left\{ \int_0^{\theta_0} \int_0^{r_0} + \int_{\theta_0}^{\pi/2} \int_{R_2}^{r_2} \right\}$$

Chapter 41. The integration of E .

§ 102. When E is in $J'(o)$. $E'(j'o)$ denotes the domain.

$$E'(j'o) = 2 \left[\int_{\pi}^{A_0' + \pi} \int_0^{\pi/2} \int_{R_2}^{R_1} + \int_{A_0' + \pi}^{2\pi} \int_{\theta_0'}^{\pi/2} \int_{R_2}^{R_1} \right]$$

§ 103. When E is in $J'(i)$.

Let it be $E'(j'i)$, then it is equal to $E(jiui)$ in § 50, A being substituted for $A + \pi$.

$$E'(j'i) = E(jiui, A + \pi).$$

Chapter 42. The horizontal intensity due to the tertiary scattering when

$$2d_1 \leq h \leq 3d_1.$$

$$B_3 = [\text{§ 102 (99.13 + 100.14)} + \text{§ 103 (101.14)}] dS_3.$$

$$= [E'(j'o)(T'(j'obo)F(bo) + T'(j'obi)F(bi)) + E'(j'i)T'(j'i)F(bi)] dS_3.$$

Part 12. The Case When $3d_1 \leq h \leq 4d_1$.

Chapter 43. General aspect.

Also in this case, the whole sky dome is divided into $J'(o)$ and $J'(i)$, but G -point is in $J'(i)$.

Chapter 44. The integration of T .

Section 22. When E is in $J'(o)$.

§ 104. T in $B(o)$. It is the same as in § 99.

§ 105. T in $B(i)$. It is the same as in § 100.

Section 23. When E is in $J'(i)$.

§ 106. T in $B(i)$. It is the same as in § 101.

Chapter 45. The integration of E .

§ 107. When E is in $J'(o)$.

It is equal to $E(goui)$ in § 60, A being substituted for $A + \pi$.

$$E'(j'o) = 2 \int_{A_0' + \pi}^{2\pi} \int_0^{\theta_0'} \int_{R_2}^{R_1}$$

§ 108. When E is in $J'(i)$.

It is equal to the substitution of A for $A + \pi$ in § 61.

Chapter 46. The horizontal intensity due to the tertiary scattering when
 $3d_1 \leq h \leq 4d_1$.

$$B_3 = \left[\begin{array}{ccc} \S 107 & (104.13 + 105.14) & + \S 108, 106.14 \end{array} \right] dS_3. \\ \qquad \qquad \qquad (99) \qquad \qquad (100) \qquad \qquad (101)$$

$$= [E'(j'o)(T')j'obo)F(b_o) + T'(j'obi)F(b_i) + E'(j'i)T'(j'i)F(b_i)]dS_3.$$

Part 13. The Case When $4d_1 \leq h \leq 90$.

Chapter 47. General aspect.

Just when $h=4d_1$ the whole sky dome is contained in $J'(i)$.

Chapter 48. The integration of E .

§ 109. The integration of $E'(j'i)$

$$E'(j'i) = \int_0^{2\pi} b_A \int_0^{2\pi} \cos \theta i \theta \int_0^{R_1} k^2 dR.$$

Chapter 49. The integration of T .

§ 110. T in $B(i)$.

E is only in $J'(i)$ and T only in $B(i)$, so the answer is the same as in § 101.

Chapter 50. The horizontal intensity due to the tertiary scattering when
 $4d_1 \leq h \leq 90$.

$$B_3 = [\S 109, 110.14]dS_3 = E'(j'i)T'(j'i)F(b_i)dS_3.$$

(To be continued.)

Literature

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- (2) *Loc. cit.* Vol. 2, No. 1, Series 5.