論文 Memoirs

On Numerical Prediction (III)*

S. Syōno and K. Gambo

(Geophysical Institute, Tokyo University)

— Abstract —

In this paper the prediction of the spatial correlation coefficient and the time correlation coefficient of the atmospheric disturbances has been discussed in connection with the distribution function of the fluctuations in pressure patterns. In the case of wind-tunnel turbulence, the spatial correlation coefficients are easily transformed to the time correlation coefficients, since the time coordinate t is transformed to the space coordinate x , multiplying the general flow U by t . In the case of atmospheric disturbances, however, the transformation is not uniquely determined and hence there occurs an apparent difference between the spatial and time correlation coefficients.

§ 1. Introduction

In our previous paper (1950), we discussed the basic principle for numerical prediction of geophysical phenomena and put forward a practical method, with which we predicted the change in pressure patterns. As shown in the previous paper, the observed pressure patterns are separated into components of Fourier series. Each component wave thus obtained is propagated independently of each other. By synthesis of each component of Fourier series after the displacement following the law of each component, we obtain the pressure patterns to be predicted.

In the present paper we intend to predict the correlation coefficient of the change in pressure patterns. Generally, there are two kinds of correlation coefficients *i.e.*, the spatial and time correlation coefficients. Here we treat the atmosphere as a dispersive system. *i.e.*, in such an atmosphere the phase-velocity of propagation depends upon the wave length. The functional forms of the two kinds of correlation coefficients do not coincide with each other, since the spatial coordinate can not be transformed uniquely to the time coordinate, multiplied by a constant factor as may be done in the case of wind-tunnel turbulence. By the above reason, there occurs an apparent difference on the nature of the state function, when we observe it at a fixed point and at a fixed time.

§ 2. Double Fourier series

We consider a function Z of time t and spatial coordinates (x, y) . The function Z

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satisfies the differential equation

$$L[Z] = 0, \quad (2.1)$$

where L is a linear operator with constant coefficients:

$$L = \sum_{l,m,n} A_{l,m,n} \frac{\partial^{l+m+n}}{\partial t^l \partial x^m \partial y^n}. \quad (2.2)$$

In this equation we put

$$Z \propto e^{i(p_\alpha x + \beta^2 t - \alpha x - \beta y)}. \quad (2.3)$$

Then, substituting it into (2.1), we get

$$\sum_{l,m,n} (-1)^{m+n} (i)^{l+m+n} A_{l,m,n} (p_\alpha, \beta)^l \alpha^m \beta^n = 0. \quad (2.4)$$

This is a frequency equation which determines p_α, β 's as functions of α and β . For the sake of simplicity we confine ourselves to the case where $l=1$ and $n=0$. Then the function Z may be written in the form as follows:

$$Z = \sum_{\alpha=1}^{\infty} A_\alpha \cos(p_\alpha t - \alpha x) + \sum_{\alpha=1}^{\infty} B_\alpha \sin(p_\alpha t - \alpha x), \quad (2.5)$$

where α denotes positive integers, and p_α the quantity to be determined by (2.4) for α . On the other hand, we may rewrite (2.5) as follows:

$$Z = \sum C_\alpha(x) \cos p_\alpha t + \sum D_\alpha(x) \sin p_\alpha t, \quad (2.5)'$$

where $C_\alpha(x) = A_\alpha \cos \alpha x + B_\alpha \sin \alpha x$, $D_\alpha(x) = A_\alpha \sin \alpha x - B_\alpha \cos \alpha x$,

and A_α 's and B_α 's are determined by the initial conditions.

Next we consider a time interval $[0-T]$, in which $\cos p_\alpha t$, $\sin p_\alpha t$ shall be expanded into Fourier series. Then we get

$$\cos p_\alpha t = \sum_{\gamma=1}^{\infty} C_{\alpha,\gamma}' \cos \frac{2\pi}{T} \gamma t + \sum_{\gamma=1}^{\infty} D_{\alpha,\gamma}' \sin \frac{2\pi}{T} \gamma t, \quad (2.6)$$

$$\sin p_\alpha t = \sum_{\gamma=1}^{\infty} C_{\alpha,\gamma}'' \cos \frac{2\pi}{T} \gamma t + \sum_{\gamma=1}^{\infty} D_{\alpha,\gamma}'' \sin \frac{2\pi}{T} \gamma t, \quad (2.7)$$

where γ denotes positive integers.

Here the period T should be chosen, according to the time interval in which we are necessitated to treat our problem. For instance, if we treat long waves T must be long, while if we treat short waves T must be short.

In order to obtain simple relations between p_α and the coefficients of (2.6) and (2.7) we consider the following equations:

$$\begin{aligned} \cos \lambda x &= \frac{2\lambda \sin \lambda \pi}{\pi} \left\{ \frac{1}{2\lambda^2} - \frac{\cos x}{\lambda^2 - 1} + \frac{\cos 2x}{\lambda^2 - 2^2} - \frac{\cos 3x}{\lambda^2 - 3^2} + \dots \right\}, \\ \sin \lambda x &= \frac{2 \sin \lambda \pi}{\pi} \left\{ \frac{\sin x}{\lambda^2 - 1} - \frac{2 \sin 2x}{\lambda^2 - 2^2} + \frac{3 \sin 3x}{\lambda^2 - 3^2} - \dots \right\}, \end{aligned}$$

where λ is not an integer but an arbitrary rational value.

Substituting $\lambda = \frac{T}{2\pi} p_\alpha$ and $x = \frac{2\pi}{T} t$ into these equations, we obtain the following equations:

$$\begin{aligned}\cos p_\alpha t &= \frac{2\left(\frac{T}{2\pi} p_\alpha\right)}{\pi} \sin\left(\frac{Tp_\alpha}{2}\right) \left\{ \frac{2\pi^2}{T^2 p_\alpha^2} - \frac{4\pi^2}{(T^2 p_\alpha^2 - 4\pi^2)} \cos \frac{2\pi}{T} t + \dots \right\}, \\ \sin p_\alpha t &= \frac{2}{\pi} \sin\left(\frac{Tp_\alpha}{2}\right) \left\{ \frac{4\pi^2}{(T^2 p_\alpha^2 - 4\pi^2)} \sin \frac{2\pi}{T} t - \frac{8\pi^2}{(T^2 p_\alpha^2 - 8\pi^2)} \sin \frac{2\pi}{T} 2t + \dots \right\}.\end{aligned}$$

Thus we have

$$\left. \begin{aligned}G_{\alpha\gamma}' &= \left(\frac{4\pi^2}{T^2 p_\alpha^2 - 4\pi^2 \gamma^2} \right) \left(\frac{p_\alpha T}{\pi^2} \sin \frac{p_\alpha T}{2} \right) (-1)^\gamma, & D_{\alpha\gamma}' &= 0, \\ D_{\alpha\gamma}'' &= \left(\frac{4\pi^2}{T^2 p_\alpha^2 - 4\pi^2 \gamma^2} \right) \left(\frac{2}{\pi} \sin \frac{p_\alpha T}{2} \right) (-1)^{\gamma+1}, & C_{\alpha\gamma}'' &= 0.\end{aligned} \right\} \quad (2.8)$$

Substituting (2.6) and (2.7) into (2.5), we get

$$\left. \begin{aligned}Z &= \sum_{\alpha} \sum_{\gamma} G_{\alpha\gamma} \cos \frac{2\pi}{T} \gamma t \cos \alpha x + \sum_{\alpha} \sum_{\gamma} G_{\alpha\gamma}' \cos \frac{2\pi}{T} \gamma t \sin \alpha x \\ &+ \sum_{\alpha} \sum_{\gamma} G_{\alpha\gamma}'' \sin \frac{2\pi}{T} \gamma t \sin \alpha x + \sum_{\alpha} \sum_{\gamma} G_{\alpha\gamma}''' \sin \frac{2\pi}{T} \gamma t \cos \alpha x,\end{aligned} \right\} \quad (2.9)$$

where $G_{\alpha\gamma} = A_\alpha C_{\alpha\gamma}'$, $G_{\alpha\gamma}'' = A_\alpha D_{\alpha\gamma}''$, $G_{\alpha\gamma}' = B_\alpha C_{\alpha\gamma}''$ and $G_{\alpha\gamma}''' = -B_\alpha D_{\alpha\gamma}''$.

On the other hand, we consider a two-dimensional space, whose coordinates are t and x . In such a space we consider a domain $[0-T]$, $[0-2\pi]$. Then Z may be expanded into a double Fourier series as follows:

$$\begin{aligned}Z &= \sum_{\alpha} \sum_{\gamma} H_{\alpha\gamma} \cos \frac{2\pi}{T} \gamma t \cos \alpha x + \sum_{\alpha} \sum_{\gamma} H_{\alpha\gamma} \cos \frac{2\pi}{T} \gamma t \sin \alpha x \\ &+ \sum_{\alpha} \sum_{\gamma} H_{\alpha\gamma}'' \sin \frac{2\pi}{T} \gamma t \sin \alpha x + \sum_{\alpha} \sum_{\gamma} H_{\alpha\gamma}''' \sin \frac{2\pi}{T} \gamma t \cos \alpha x,\end{aligned} \quad (2.10)$$

where α, γ are positive integers.

Comparing (2.9) with (2.10), we get

$$\left. \begin{aligned}H_{\alpha\gamma} &= A_\alpha C_{\alpha\gamma}', & H_{\alpha\gamma}' &= B_\alpha C_{\alpha\gamma}'', \\ H_{\alpha\gamma}'' &= A_\alpha D_{\alpha\gamma}'', & H_{\alpha\gamma}''' &= -B_\alpha D_{\alpha\gamma}''.\end{aligned} \right\} \quad (2.11)$$

§ 3. Correlation coefficients

As a spatial correlation coefficient, we put

$$R_\xi = \frac{\overline{Z(x)Z(x+\xi)}}{\overline{Z(x)^2}}. \quad (3.1)$$

Similarly, as a time correlation coefficient we put as follows:

$$R_\tau = \frac{\overline{Z(t)Z(t+\tau)}}{\overline{Z(t)^2}}. \quad (3.2)$$

In the present section we attempt to obtain the relation between R_ξ and R_τ , so that

we rewrite (3.1) and (3.2) in other forms. For the turbulence in a wind-tunnel, there is generally one to one correspondence concerning the space and the time, because the variable x is transformed to the variable t by $x=Ut$, where U is the mean velocity of the flow. In our problem, however, we have not the same relation mentioned above, since the atmosphere is a dispersive medium for the propagation of pressure patterns.

Now, from (2.5) we obtain easily the following relation:

$$\overline{Z(x)^2} = \frac{1}{2\pi} \int_0^{2\pi} Z(x)^2 dx = \sum_{\alpha=0}^{\infty} (A_{\alpha}^2 + B_{\alpha}^2). \quad (3.3)$$

Next we define the distribution function F such as $\overline{Z(x)^2}$ is the sum of the contributions of $\overline{Z(x)^2}F(\alpha)$ to α . Then we get

$$\overline{Z(x)^2} \cdot F(\alpha) = A_{\alpha}^2 + B_{\alpha}^2 \quad (3.4)$$

$$\text{and} \quad \sum_{\alpha=0}^{\infty} F(\alpha) = 1. \quad (3.5)$$

On the other hand, we get easily the following relation:

$$\begin{aligned} \overline{Z(x)Z(x+\xi)} &= \frac{1}{2\pi} \int_0^{2\pi} Z(x)Z(x+\xi) dx \\ &= \sum_{\alpha=0}^{\infty} (A_{\alpha}^2 + B_{\alpha}^2) \cos \alpha \xi. \end{aligned} \quad (3.6)$$

Thus, from (3.4) and (3.6) we get

$$\overline{Z(x)Z(x+\xi)} = \sum_{\alpha=0}^{\infty} \overline{Z(x)^2} F(\alpha) \cos \alpha \xi.$$

Hence R_{ξ} becomes as follows:

$$R_{\xi} = \sum_{\alpha=0}^{\infty} F(\alpha) \cos \alpha \xi. \quad (3.7)$$

Similarly we define the distribution function E such as $\overline{Z(t)^2}$ is the sum of the contributions of $\overline{Z(t)^2}E(\gamma)$ to γ .

On the other hand, from (2.10) we get

$$\begin{aligned} \overline{Z(t)^2} &= \frac{1}{T} \int_0^T Z(t)^2 dt \\ &= \left\{ \sum_{\alpha} \sum_{\gamma} (H_{\alpha\gamma}^2 \cos^2 \alpha x + H_{\alpha\gamma}'^2 \sin^2 \alpha x + H_{\alpha\gamma}''^2 \sin^2 \alpha x + H_{\alpha\gamma}'''^2 \cos^2 \alpha x) \right\}. \end{aligned} \quad (3.8)$$

Hence we get

$$\overline{Z(t)^2} \cdot E(\gamma) = \left\{ \sum_{\alpha=0}^{\infty} (H_{\alpha\gamma}^2 \cos^2 \alpha x + H_{\alpha\gamma}'^2 \sin^2 \alpha x + H_{\alpha\gamma}''^2 \sin^2 \alpha x + H_{\alpha\gamma}'''^2 \cos^2 \alpha x) \right\}, \quad (3.9)$$

and

$$\sum_{\gamma=0}^{\infty} E(\gamma) = 1. \quad (3.10)$$

Thus R_{τ} may be written the same as R_{ξ} in the following form:

$$R_\tau = \sum_{\gamma=0}^{\infty} B(\gamma, x) \cos \frac{2\pi}{T} \gamma \tau. \quad (3.11)$$

Since $E(\gamma, x)$ is a function of x as may be seen in (3.9), R_τ is generally a function of x . So that we consider the mean value of $E(\gamma)$ for x and we put

$$\overline{E(\gamma)} = \frac{1}{2\pi} \int_0^{2\pi} E(\gamma, x) dx. \quad (3.12)$$

Hence, from (2.11) and (3.9) we have the following equations:

$$\begin{aligned} \overline{E(\gamma)Z(t)^2} &= \sum_{\alpha=0}^{\infty} (H_{\alpha\gamma}^2 + H_{\alpha\gamma}'^2 + H_{\alpha\gamma}''^2 + H_{\alpha\gamma}'''^2), \\ &= \sum_{\alpha=0}^{\infty} \{(C_{\alpha\gamma}'^2 + D_{\alpha\gamma}''^2)(A_\alpha^2 + B_\alpha^2)\}. \end{aligned} \quad (3.13)$$

The mean value of R_τ for x becomes as follows:

$$\overline{R}_\tau = \sum_{\gamma=0}^{\infty} \overline{E(\gamma)} \cos \frac{2\pi}{T} \gamma \tau. \quad (3.14)$$

Next we consider the relation between R_ξ and $\overline{R}_\tau$. If we give the distribution function $F(\alpha)$ initially, then R_ξ is obtained from (3.7). $\overline{R}_\tau$ is also obtained from (3.13) and (3.14), provided that $C_{\alpha\gamma}'$ and $D_{\alpha\gamma}''$ are given for various α 's.

Since the relation between $C_{\alpha\gamma}'$, $D_{\alpha\gamma}''$ and p_α are given by (2.8), we may readily obtain the values of $C_{\alpha\gamma}'$ and $D_{\alpha\gamma}''$ for various α 's from the given values of p_α and T .

Next we consider the method to obtain p_α from the observed data. When we observe $Z(0, x)$ and $[\partial Z(t, x)/\partial t]_{t=0}$, we expand Z and $\partial Z/\partial t$ into a Fourier series as follows,

$$Z(0, x) = \sum G_\alpha(0) \cos \alpha x + \sum G_{-\alpha}(0) \sin \alpha x, \quad (3.15)$$

$$[\partial Z(t, x)/\partial t]_{t=0} = \sum G_\alpha^I(0) \cos \alpha x + \sum G_{-\alpha}^I(0) \sin \alpha x. \quad (3.16)$$

Since $Z(t, x) \propto e^{i(p_\alpha t - \alpha x)}$, we easily obtain

$$p_\alpha = - \frac{G_{-\alpha}^I(0)}{G_\alpha(0)} \quad \text{or} \quad p_\alpha = - \frac{G_\alpha^I(0)}{G_{-\alpha}(0)}. \quad (3.17)$$

When we observe $Z(-t, x)$ and $Z(0, x)$, we expand $Z(-t, x)$ and $Z(0, x)$ into the following Fourier series:

$$Z(-t, x) = \sum G_\alpha(-t) \cos \alpha x + \sum G_{-\alpha}(-t) \sin \alpha x, \quad (3.18)$$

$$Z(0, x) = \sum G_\alpha(0) \cos \alpha x + \sum G_{-\alpha}(0) \sin \alpha x, \quad (3.19)$$

and we have

$$\begin{aligned} G_\alpha(-t) \cos p_\alpha t - G_{-\alpha}(-t) \sin p_\alpha t &= G_\alpha(0), \\ G_\alpha(-t) \sin p_\alpha t + G_{-\alpha}(-t) \cos p_\alpha t &= G_{-\alpha}(0). \end{aligned}$$

Thus

$$\cos p_\alpha t = \frac{G_\alpha(0)G_\alpha(-t) + G_{-\alpha}(0)G_{-\alpha}(-t)}{[G_\alpha(-t)^2 + G_{-\alpha}(-t)^2]}. \quad (3.20)$$

§ 4. Numerical examples

As an example, we make use of the data of the height profiles of the 500-mb isobaric surface at 45°N, given by J. G. Charney (1949). Performing Fourier analysis of the height profile Z , all components of Fourier series for various α 's are obtained. Since the distribution $F(\alpha)$ is proportional to $(A_\alpha^2 + B_\alpha^2)$, we take $(A_\alpha^2 + B_\alpha^2)$ as $F(\alpha)$, and in Fig 1. $(A_\alpha^2 + B_\alpha^2)$ are plotted for various α 's during four days. In this calculation we took 48 terms of Fourier series. As the mean values of $F(\alpha)$ for various α 's we take the following values which are tabulated on Table 1 and also shown by the continuous curve in Fig. 1.* Using this distribution function $F(\alpha)$ and the equation (3.7), we calculated R for various ξ 's which is shown by a continuous curve in Fig. 2. The plotted

Table 1

α	1	2	3	4	5	6
$F(\alpha)$	$\times 10^7 \text{ feet}^2$ 200	150	80	50	30	17

α	7	8	9	10	11	
$F(\alpha)$	$\times 10^7 \text{ feet}^2$ 10	7	5	4	3

Fig. 1

Fig. 2

Fig. 3

* In our previous paper (1952) we have shown that the mean values of $F(\alpha)$ for various α 's are represented by $F(\alpha) \propto \alpha^{-7/3}$ for $10 \geq \alpha \geq 2$. In this paper, however, we use the values shown on Table 1 as that of $F(\alpha)$.

points in Fig. 2. denote R_{ξ}' for various ξ 's which are obtained from the data of the 8th, 9th, 11th and 12th of Jan.

The correlation coefficient R_{ξ} calculated from the observed data shows a considerable agreement with that obtained theoretically. So we consider that $F(\alpha)$ is a basic condition of Z and changes little from day to day. On the other hand, $\bar{R}_{\tau}$ changes from day to day, according to the physical conditions of the atmosphere, as the intensity of the westerlies U and the meridional width of the atmospheric disturbance, although R_{ξ} holds the same form.

To obtain $\bar{R}_{\tau}$ simply from R_{ξ} , we calculate $C_{\alpha\gamma}$'s and $D_{\alpha\gamma}$'s in this paper, using the following relation, in place of calculating them from the observed data,

$$p_{\alpha} = \frac{\alpha\beta + U(\alpha^3 + \alpha m^2)}{\alpha^2 + (m^2 + \lambda^2)}. \quad (4.1)$$

This relation is obtained from the following planetary wave equation:

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \left(\frac{\partial^2 Z}{\partial x^2} - m^2 Z\right) + \beta \frac{\partial Z}{\partial x} - \lambda^2 \frac{\partial Z}{\partial t} = 0. \quad (4.2)$$

The symbols of this equation are the same as in the Charney's paper (1949). Next we calculate the period at a fixed point, assuming the special form of the dispersion for the phase velocity such as (4.1), and the periods T are shown in Fig. 3 for various α 's, U 's and α^2 's. ($\alpha^2 = m^2 + \lambda^2$) As may be seen in the figure, there is an apparent prevailing period, provided that we observe the planetary wave at a fixed point. Concerning the observation of the period of each wave, the above consideration is correct, but we must consider the intensity of each wave, i. e., the distribution function $F(\alpha)$, when we treat the apparent period which occurs from the superposition of waves. The above considerations are represented in an analytical form in (3.14).

For practical calculation of $\bar{R}_{\xi}$, we put $T=50$ days. Next we obtain from (2.8) $C_{\alpha\gamma}$'s

Fig. 4

Fig. 5

Fig. 6

Fig. 7

and $D_{\alpha\gamma}''$'s for various α 's, γ 's and p_α i. e., C_{11}' , $C_{12}' \dots$, D_{11}'' , $D_{12}'' \dots$ and C_{21}' , $C_{22}' \dots$, D_{21}'' , $D_{22}'' \dots$. Using these values of $C_{\alpha\gamma}'$ and $D_{\alpha\gamma}''$, we may obtain $\bar{E}(\gamma)$ from (3.14). For the sake of simplicity, we took the unit of the time as one day, and obtained $\bar{E}(\gamma)$ approximately. For various U 's and α^2 's, the relation between $\bar{R}_\tau$ and τ , obtained by the method mentioned above, are shown in Figs. 4, 5, 6 and 7. As may be seen in the figures, there occur apparent differences of $\bar{R}_\tau$ when the physical condition of the atmosphere changes. For instance, holding $\alpha^2=10$, the apparent period which is shown by $(\bar{R}_\tau - \tau)$ diagram varies from 7 days to (2, 4, 8) days and to (4, 6, 10) days, following the change of U from 0 radian/day to $1/3$ radian/day and to 1 radian/day, respectively. For $U=1/3$ radian/day and $\alpha^2=10$, which are reasonable values for middle latitudes, we see that period of 4 days is most predominant. The above consideration concerning the predominant period of 4 days may explain qualitatively the apparent period of 4 days at the fixed station, which has been studied by Japanese meteorologists.

References

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