重調和解析法について (I)

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On the Multi-harmonic Analysis

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— Abstract —

Harmonic analyses for many terms are troublesome to perform by the ordinal scheme, and in this paper we showed that they are easily obtained by combining sets of simpler analyses.

When integers m and n are incommensurable, we represent $m \times n$ terms $y_0, y_1, y_2, \dots, y_p, \dots, y_{mn-1}$ as follows.

$$\left(\begin{array}{l} y_0, y_n, y_{2n}, \dots, y_{(m-1)n} \\ y_m, y_{n+m}, y_{2n+m}, \dots, y_{(m-1)n+m} \\ y_{2m}, y_{n+2m}, \dots, \dots \\ \dots \\ y_{(n-1)m}, \dots, y_{(m-1)n+(n-1)m} \end{array} \right) \dots \dots \dots (I)$$

where m is even.

(i) Calculate the Fourier coefficients of the j -th line of (I) using the analyses for m orders whose results are

$j \backslash i$	(0)	(1)		$\left(\frac{m}{2}+1\right)$		($m-1$)
0	α_{00}	α_{01}		β_{01}		
1	α_{10}	α_{11}		β_{11}		 (II)
.....						
.....						
$n-1$						

(ii) Calculate the Fourier coefficients of the i -th column of (II) using the analyses for n orders whose result are

$i \backslash j$	(0)	(1)		$\left(\frac{m}{2}+1\right)$		($m-1$)
(0)	$A_{00}, A_{01}, \dots, B_{01}$					
(1)	$A_{10}, A_{11}, \dots, B_{11}$					
.....						 (III)
.....						
$\left(\frac{m}{2}+1\right)$	$a_{10}, a_{11}, \dots, b_{11}$					
.....						
($m-1$)						

(iii) The Fourier coefficients of the entire series $P(\nu)$, and $Q(\nu)$ are shown by,

$$\begin{aligned}\frac{mn}{2}P(\nu) &= \sum_{\rho=0}^{mn-1} y_{\rho} \cos \nu \rho \frac{2\pi}{mn} \\ &= \sum_{j=0}^{n-1} \alpha_j(\nu) \cos \nu_j \frac{2\pi}{n} - \sum_{j=0}^{n-1} \beta_j(\nu) \sin \nu_j \frac{2\pi}{n}\end{aligned}$$

and

$$\begin{aligned}\frac{mn}{2}Q(\nu) &= \sum_{\rho=0}^{mn-1} y_{\rho} \sin \nu \rho \frac{2\pi}{mn} \\ &= \sum_{j=0}^{n-1} \beta_j(\nu) \cos \nu_j \frac{2\pi}{n} + \sum_{j=0}^{n-1} \alpha_j(\nu) \sin \nu_j \frac{2\pi}{n}\end{aligned}$$

where

$$\nu=0, 1, 2, \dots, \frac{mn}{2}$$

As examples we showed schematically the analyses for $36(4 \times 9)$, $60(4 \times 15)$, $72(8 \times 9)$, $120(8 \times 15)$ and $180(5 \times 36)$ orders, the results of which were given by the tables (IV. 36), (IV. 60), (IV. 72), (IV. 120), and (IV. 180) in the text. In each case mentioned above m is even, but in the case when both m and n are odd the similar scheme can be used.

§ 1. ま え が き

ある離散的な時系列 y_i が与えられた場合、その r 項を取り出して調和解析を行うことは、その物理的意味についてはとも角、 r の任意の数について

$$\begin{aligned}\left. \begin{aligned} a_{\nu} \\ b_{\nu} \end{aligned} \right\} &= \frac{2}{r} \sum_{\rho=0}^{r-1} y_{\rho} \begin{cases} \cos \nu \rho \frac{2\pi}{r} \\ \sin \nu \rho \frac{2\pi}{r} \end{cases} \\ \nu &= 1, 2, \dots, \frac{r}{2}-1. \quad (r; \text{偶数}) \\ &\quad , \frac{r-1}{2}. \quad (r; \text{奇数})\end{aligned}$$

$$\begin{aligned}a_0 &= \frac{1}{r} \sum_{\rho=0}^{r-1} y_{\rho} \\ a_n &= \frac{1}{r} \sum_{\rho=0}^{r-1} (-1)^{\rho} y_{\rho} \quad n = \frac{r}{2} \quad (r; \text{偶数})\end{aligned}$$

.....(1)

の計算をなせば良いのであるが、 r が大きくなるとその手数は頗る繁雑であつて、その計算の方法については簡単化のために色々の工夫がなされている。

例えば、P. Terebesi⁽¹⁾, A. Hußman⁽²⁾, K. Stumpff⁽³⁾ らによつて、 r として 36 までの各整数及び 72 までの幾つかの整数についての便利な計算様式が示されている。

更に又 K. Stumpff は、 p 項解析を 2 回或は 3 回繰返して、夫々 $2p$, $3p$ 項の解析を行う方法を示し、この方法を適当に重ねることによつて 360 項までの解析法を挙げている。

一例として 60 項の調和解析は、30 項の解析を 2 度行うことによつて次の如く求められる。

$$(R') \quad y_0, y_2, y_4, \dots, y_{58} \quad (30 \text{ 項})$$

$$(R'') \quad y_m, y_{m+2}, \dots, y_{59}, \dots, y_1, y_{m-2} \quad (30 \text{ 項})$$

但し, m は奇数, $y_{61} \equiv y_1, y_{63} \equiv y_3$

$$\left. \begin{aligned} \sum_{\rho=0}^{59} y_{\rho} \cos \nu \rho \alpha &= \sum_{\rho=0}^{29} y_{2\rho} \cos 2\nu \rho \alpha + \sum_{\rho=0}^{29} y_{2\rho+m} \cos(2\rho+m)\nu \alpha = \sum_{\rho=0}^{29} y_{2\rho} \cos 2\nu \rho \alpha \\ &+ \cos \nu m \alpha \sum_{\rho=0}^{29} y_{2\rho+m} \cos 2\nu \rho \alpha - \sin \nu m \alpha \sum_{\rho=0}^{29} y_{2\rho+m} \sin 2\nu \rho \alpha \\ \sum_{\rho=0}^{59} y_{\rho} \sin \nu \rho \alpha &= \sum_{\rho=0}^{29} y_{2\rho} \sin 2\nu \rho \alpha + \cos \nu m \alpha \sum_{\rho=0}^{29} y_{2\rho+m} \sin 2\nu \rho \alpha \\ &+ \sin \nu m \alpha \sum_{\rho=0}^{29} y_{2\rho+m} \cos 2\nu \rho \alpha \\ &\left(\alpha = \frac{2\pi}{60} \right) \end{aligned} \right\} \dots\dots\dots (2)$$

従つて, 系列 R', R'' についてのフーリエ係数を夫々 $A_{\nu}', B_{\nu}', A_{\nu}'', B_{\nu}''$ としておけば,

$$\left. \begin{aligned} \sum_{\rho=0}^{59} y_{\rho} \cos \nu \rho \alpha &= A_{\nu}' + A_{\nu}'' \cos \nu m \alpha - B_{\nu}'' \sin \nu m \alpha \\ \sum_{\rho=0}^{59} y_{\rho} \sin \nu \rho \alpha &= B_{\nu}' + B_{\nu}'' \cos \nu m \alpha + A_{\nu}'' \sin \nu m \alpha \end{aligned} \right\} \dots\dots\dots (3)$$

であつて, m は $\cos m\alpha, \sin m\alpha$ が簡単な数値をとる様に最初に決めて置けばよいのである。

§ 2. “重調和解析法”

この K. Stumpff の示した方法を拡張し定式化したものを, “重調和解析法”と呼んでこれを次に述べよう。

m, n が互に素なる場合, mn 項を週期とする時系列

$$y_0, y_1, y_2, \dots, y_{\rho-1}, y_{\rho}, \dots, y_{mn-1}$$

$$\text{を} \quad \left(\begin{array}{ccccccc} y_0, & y_n, & y_{2n}, & \dots, & y_{(m-1)n}, & & \\ y_m, & y_{n+m}, & y_{2n+m}, & \dots, & y_{(m-1)n+m}, & & \\ y_{2m}, & y_{n+2m}, & & \dots, & y_{(m-1)n+2m}, & & \\ & & & \dots & & & \\ & & & \dots & & & \\ & & & \dots & & & \\ y_{(n-1)m}, & & & & & & y_{(m-1)n+(n-1)m} \end{array} \right) \dots\dots\dots (4)$$

と並べる。その一般項は

$$y_{kn+jm} \quad \begin{cases} k=0, 1, 2, \dots, m-1 \\ j=0, 1, 2, \dots, n-1 \end{cases}$$

又 mn なる週期を持つことから、例えば

$$y_{(m-1)n+(n-1)m} \equiv y_{mn-(m+n)}$$

で表わされる。

(i) 先ず (4) の第 j 行のフーリエ係数を求める。

$$\alpha_j(p) = \sum_{k=0}^{m-1} y_{kn+jm} \cos\left(p \cdot k \cdot \frac{2\pi}{m}\right) \quad \dots\dots\dots (5)$$

$$\beta_j(p) = \sum_{k=0}^{m-1} y_{kn+jm} \sin\left(p \cdot k \cdot \frac{2\pi}{m}\right)$$

$$\begin{cases} j=0, 1, 2, \dots\dots\dots, n-1 \\ p=0, 1, 2, \dots\dots\dots, \frac{m}{2} & (m: \text{偶数}) \\ \phantom{p=0, 1, 2, \dots\dots\dots, \frac{m}{2}} \cdot \frac{m-1}{2} & (m: \text{奇数}) \end{cases}$$

(ii) 次に p を ν と書き換えて、 $\nu=0, 1, 2, \dots\dots, \frac{mn}{2}$, (m, n 共に奇数の場合は $\frac{mn-1}{2}$) について形式的に拡張すると、

$$\alpha_j(\nu) = \sum_{k=0}^{m-1} y_{kn+jm} \cos\left(\nu \cdot k \cdot \frac{2\pi}{m}\right) \quad \dots\dots\dots (6)$$

$$\beta_j(\nu) = \sum_{k=0}^{m-1} y_{kn+jm} \sin\left(\nu \cdot k \cdot \frac{2\pi}{m}\right)$$

は夫々 ν について m を週期とする週明函数である。

(iii) 今 $y_0, y_1, y_2, \dots\dots\dots, y_{mn-1}$ なる全系列 mn' 項のフーリエ係数を $P_{(\nu)}, Q_{(\nu)}$ とすると、

$$\begin{aligned} \frac{mn}{2} P_{(\nu)} &= \sum_{p=0}^{mn-1} y_p \cos \nu p \frac{2\pi}{mn} \\ &= \sum_{k=0}^{m-1} y_{kn} \cos \nu kn \frac{2\pi}{mn} + \sum_{k=0}^{m-1} y_{kn+m} \cos \nu (kn+m) \frac{2\pi}{mn} + \dots\dots \\ &\quad \dots\dots + \sum_{k=0}^{m-1} y_{kn+jn} \cos \nu (kn+jm) \frac{2\pi}{mn} + \dots\dots \\ &\quad \dots\dots + \sum_{k=0}^{m-1} y_{kn+(n-1)m} \cos \nu \{kn+(n-1)m\} \frac{2\pi}{mn} \\ &= \sum_{k=0}^{m-1} y_{kn} \cos \nu \cdot k \frac{2\pi}{m} \\ &\quad + \sum_{k=0}^{m-1} y_{kn+m} \left(\cos \nu \cdot k \frac{2\pi}{m} \cos \nu \frac{2\pi}{n} - \sin \nu k \frac{2\pi}{m} \sin \nu \frac{2\pi}{n} \right) + \dots\dots \\ &\quad \dots\dots + \sum_{k=0}^{m-1} y_{kn+jm} \left(\cos \nu k \frac{2\pi}{m} \cos \nu j \frac{2\pi}{n} - \sin \nu k \frac{2\pi}{m} \sin \nu j \frac{2\pi}{n} \right) + \dots\dots \end{aligned}$$

$$\begin{aligned}
& \cdots + \sum_{k=0}^{m-1} y_{kn+(n-1)m} \left\{ \cos \nu k \frac{2\pi}{m} \cos \nu(n-1) \frac{2\pi}{n} - \sin \nu k \frac{2\pi}{m} \sin \nu(n-1) \frac{2\pi}{n} \right\} \\
& = \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} y_{kn+jm} \cos \nu k \frac{2\pi}{m} \cos \nu j \frac{2\pi}{n} \\
& \quad - \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} y_{kn+jm} \sin \nu k \frac{2\pi}{m} \sin \nu j \frac{2\pi}{n} \\
& = \sum_{j=0}^{n-1} \alpha_j(\nu) \cos \nu j \frac{2\pi}{n} - \sum_{j=0}^{n-1} \beta_j(\nu) \sin \nu j \frac{2\pi}{n} \\
& \quad \nu: 0, 1, 2, \dots, \frac{mn}{2} \quad (m \text{ 又は } n: \text{ 偶数}) \\
& \quad \quad \quad \frac{mn-1}{2} \quad (m, n: \text{ 共に奇数}) \\
& \quad \quad \quad \dots\dots\dots(7)
\end{aligned}$$

同様にして

$$\begin{aligned}
\frac{mn}{2} Q(\nu) &= \sum_{j=0}^{n-1} \beta_j(\nu) \cos \nu j \frac{2\pi}{n} + \sum_{j=0}^{n-1} \alpha_j(\nu) \sin \nu j \frac{2\pi}{n} \\
& \quad \nu: 1, 2, \dots, \frac{mn}{2} - 1. \quad (m \text{ 又は } n: \text{ 偶数}) \\
& \quad \quad \quad \frac{mn-1}{2}. \quad (m, n: \text{ 共に奇数}) \\
& \quad \quad \quad \dots\dots\dots(8)
\end{aligned}$$

以上に明らかな通り mn 項の調和解析には、先ず (4) の各行について m 項解析を n 回行つて $\alpha_j(p)$, $\beta_j(p)$ を求め、次に $\alpha_j(p)$, $\beta_j(p)$ の夫々の p について n 項解析を m 回行い、その結果を (7) (8) によつて組合わせればよいことがわかる。但しこの際、 $\alpha_j(\nu)$, $\beta_j(\nu)$ は ν に関し m を週期とする函数であり、 $\frac{\cos \nu j \frac{2\pi}{n}}{\sin \nu j \frac{2\pi}{n}}$ は ν に関し n を週期とする函数であることから、その符号に注意することが必要である。

この方法によれば、適当な項数の調和解析の scheme を種々組合わせることによつて、可成り大きな項数の調和解析を行うことも可能となり、又 mn 項の解析を置接行う代りに m 項、 n 項の解析を数多くしなければならなくなるにしても、項数の大きな解析を行うよりは時間的にも経済であり、それと同時に、 m 項、 n 項の解析に当つて夫に検証が可能であるので途中に誤算の入る危険をも減少することが出来る。

§ 3. 計 算 例

前節に述べた方法に基いて実際の計算を行うために、幾つかの例を挙げて schematically に示さう。4 項、5 項、8 項、9 項及び 15 項の調和解析の scheme を本稿の末尾に掲げたが、(第 1 表～第 5 表) これらを用いての 36 項、60 項、72 項、120 項、180 項の計算例を次

に掲げる。

a) 36 (4×9) 項重調和解析

36 項を週期とする系列 y_p を次の如く並べる。

(0)	y_0	y_9	y_{18}	y_{27}	
(1)	4	13	22	31	
(2)	8	17	26	35	
(3)	12	21	30	3	
(4)	16	25	34	7	(I・36)
(5)	20	29	2	11	(第2行以下は添字のみを示す)
(6)	24	33	6	15	
(7)	28	1	10	19	
(8)	32	5	14	23	

(I) の各横行について 4 項調和解析を行つて、そのフーリエ係数を次の通り表わす。

	(0)	(i)	(ii)	(iii)	
(0)	α_{00}	α_{01}	α_{02}	β_{01}	
(1)	α_{10}	α_{11}	α_{12}	β_{11}	
(2)	α_{20}	α_{21}	α_{22}	β_{21}	
(3)	α_{30}	α_{31}	α_{32}	β_{31}	(II・36)
(4)	α_{40}	α_{41}	α_{42}	β_{41}	
(5)	α_{50}	α_{51}	α_{52}	β_{51}	
(6)	α_{60}	α_{61}	α_{62}	β_{61}	
(7)	α_{70}	α_{71}	α_{72}	β_{71}	
(8)	α_{80}	α_{81}	α_{82}	β_{81}	

次に (II) の各縦列について 9 項調和解析を行つて、そのフーリエ係数を次の通り表わす。

(0)	A_{00}	A_{01}	A_{02}	A_{03}	A_{04}	B_{01}	B_{02}	B_{03}	B_{04}	
(i)	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	B_{11}	B_{12}	B_{13}	B_{14}	
(ii)	A_{20}	A_{21}	A_{22}	A_{23}	A_{24}	B_{21}	B_{22}	B_{23}	B_{24}	(III・36)
(iii)	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	b_{11}	b_{12}	b_{13}	b_{14}	

最後に $P_{(\nu)}$, $Q_{(\nu)}$ は次の第 IV 表より求められる。

ν	$P_{(\nu)}$	$Q_{(\nu)}$	ν	$P_{(\nu)}$	$Q_{(\nu)}$
0	A_{00}				
	$\frac{1}{2}(A_{11}-b_{11})$	$\frac{1}{2}(a_{11}+B_{11})$	4	A_{04}	B_{04}
2	A_{22}	B_{22}	5	$\frac{1}{2}(A_{14}+b_{14})$	$\frac{1}{2}(a_{14}-B_{14})$
3	$\frac{1}{2}(A_{13}+b_{13})$	$-\frac{1}{2}(a_{13}-B_{13})$	6	A_{23}	$-B_{23}$

7	$\frac{1}{2}(A_{12}-b_{12})$	$-\frac{1}{2}(a_{12}+B_{12})$	13	$\frac{1}{2}(A_{14}-b_{14})$	$\frac{1}{2}(a_{14}+B_{14})$
8	A_{01}	$-B_{01}$	14	A_{24}	$-B_{24}$
9	A_{10}	a_{10}	15	$\frac{1}{2}(A_{13}-b_{13})$	$-\frac{1}{2}(a_{13}+B_{13})$
10	A_{21}	B_{21}	16	A_{02}	$-B_{02}$
11	$\frac{1}{2}(A_{12}+b_{12})$	$-\frac{1}{2}(a_{12}-B_{12})$	17	$\frac{1}{2}(A_{11}+b_{11})$	$\frac{1}{2}(a_{11}-B_{11})$
12	A_{03}	B_{03}	18	A_{20}	

.....(IV・36)

b. 60 (4×15) 項重調和解析

60 項の系列 y_0 を次の如く並べる。

(0)	y_0	y_{15}	y_{30}	y_{45}	
(1)	4	19	34	49	
(2)	8	23	38	53	
(3)	12	27	42	57	
(4)	16	31	46	1	
(5)	20	35	50	5	
(6)	24	39	54	9	 (I・60)
(7)	28	43	58	13	
(8)	32	47	2	17	(第2行以下は添 字のみを示す)
(9)	36	51	6	21	
(10)	40	55	10	25	
(11)	44	59	14	29	
(12)	48	3	18	33	
(13)	52	7	22	37	
(14)	56	11	26	41	

(I) の各横行について 4 項調和解析, そのフーリエ係数を:

	(0)	(i)	(ii)	(iii)
(0)	α_{00}	α_{01}	α_{02}	β_{01}
(1)	α_{10}	α_{11}	α_{12}	β_{11}
(2)	α_{20}	α_{21}	...	...
(3)	α_{30}	...	...	...
(4)	...	...	...	...
(5)	...	...		
(6)	...			 (II・60)
⋮	...	...	...	...
(11)	$\alpha_{11.0}$	$\alpha_{11.1}$	$\alpha_{11.2}$	$\beta_{11.1}$
(12)	$\alpha_{12.0}$	$\alpha_{12.1}$	$\alpha_{12.2}$	$\beta_{12.1}$
(13)	$\alpha_{13.0}$	$\alpha_{13.1}$	$\alpha_{13.2}$	$\beta_{13.1}$
(14)	$\alpha_{14.0}$	$\alpha_{14.1}$	$\alpha_{14.2}$	$\beta_{14.1}$

(II) の各縦列について 15 項調和解析, そのフーリエ係数を:

(0) $A_{00} A_{01} A_{02} A_{03} A_{04} A_{05} A_{06} A_{07} B_{01} B_{02} B_{03} B_{04} B_{05} B_{06} B_{07}$

(i) $A_{10} A_{11} A_{12} A_{13} A_{14} A_{15} A_{16} A_{17} B_{11} B_{12} B_{13} B_{14} B_{15} B_{16} B_{17}$

(ii) $A_{20} A_{21} A_{22} A_{23} A_{24} A_{25} A_{26} A_{27} B_{21} B_{22} B_{23} B_{24} B_{25} B_{26} B_{27} \dots \dots \dots$ (III・60)

(iii) $a_{10} a_{11} a_{12} a_{13} a_{14} a_{15} a_{16} a_{17} b_{11} b_{12} b_{13} b_{14} b_{15} b_{16} b_{17}$

ν	$P(\nu)$	$Q(\nu)$	ν	$P(\nu)$	$Q(\nu)$
0	A_{00}		16	A_{01}	B_{01}
1	$\frac{1}{2}(A_{11}-b_{11})$	$\frac{1}{2}(a_{11}+B_{11})$	17	$\frac{1}{2}(A_{12}-b_{12})$	$\frac{1}{2}(a_{12}+B_{12})$
2	A_{22}	B_{22}	18	A_{23}	B_{23}
3	$\frac{1}{2}(A_{13}+b_{13})$	$-\frac{1}{2}(a_{13}-B_{13})$	19	$\frac{1}{2}(A_{14}+b_{14})$	$-\frac{1}{2}(a_{14}-B_{14})$
4	A_{04}	B_{04}	20	A_{05}	B_{05}
5	$\frac{1}{2}(A_{15}-b_{15})$	$\frac{1}{2}(a_{15}+B_{15})$	21	$\frac{1}{2}(A_{16}-b_{16})$	$\frac{1}{2}(a_{16}+B_{16})$
6	A_{26}	B_{26}	22	A_{27}	B_{27}
7	$\frac{1}{2}(A_{17}+b_{17})$	$-\frac{1}{2}(a_{17}-B_{17})$	23	$\frac{1}{2}(A_{17}-b_{17})$	$-\frac{1}{2}(a_{17}+B_{17})$
8	A_{07}	$-B_{07}$	24	A_{06}	B_{06}
9	$\frac{1}{2}(A_{16}+b_{16})$	$\frac{1}{2}(a_{16}-B_{16})$	25	$\frac{1}{2}(A_{15}+b_{15})$	$\frac{1}{2}(a_{15}-B_{15})$
10	A_{25}	$-B_{25}$	26	A_{24}	B_{24}
11	$\frac{1}{2}(A_{14}-b_{14})$	$-\frac{1}{2}(a_{14}+B_{14})$	27	$\frac{1}{2}(A_{13}-b_{13})$	$-\frac{1}{2}(a_{13}+B_{13})$
12	A_{03}	$-B_{03}$	28	A_{02}	B_{02}
13	$\frac{1}{2}(A_{12}+b_{12})$	$\frac{1}{2}(a_{12}-B_{12})$	29	$\frac{1}{2}(A_{11}+b_{11})$	$\frac{1}{2}(a_{11}-B_{11})$
14	A_{21}	$-B_{21}$	30	A_{20}	
15	A_{10}	$-a_{10}$			

.....(IV・60)

C. 72 (8×9) 項重調和解析

72 項の系列 y_p を次の如く並べる。

(0)	y_0	y_9	y_{18}	y_{27}	y_{36}	y_{45}	y_{54}	y_{63}
(1)	8	17	26	35	44	53	62	71
(2)	16	25	34	43	52	61	70	7
(3)	24	33	42	51	60	69	6	15
(4)	32	41	50	59	68	5	14	23
(5)	40	49	58	67	4	13	22	31
(6)	48	57	66	3	12	21	30	39
(7)	56	65	2	11	20	29	38	47
(8)	64	1	10	19	28	37	46	55

(第2行以下は添字のみを示す)

(I) の各横行について 8 項調和解析, そのフーリエ係数を:

	(0)	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	
(0)	α_{00}	α_{01}	α_{02}	α_{03}	α_{04}	β_{01}	β_{02}	β_{03}	
(1)	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	β_{11}	β_{12}	β_{13}	
(2)	α_{20}	α_{21}	...	...	...	...	...	...	
(3)	α_{30}	...	...	...	...	...	...	...	 (II・72)
(4)	...	...	...	...				...	
(5)	...	...	...	...				...	
(6)	...	...						...	
(7)	...	...						...	
(8)	α_{80}	α_{81}	α_{82}	α_{83}	α_{84}	β_{81}	β_{82}	β_{83}	

(II) の各縦列について 9 項調和解析, そのフーリエ係数を:

(0)	A_{00}	A_{01}	A_{02}	A_{03}	A_{04}	B_{01}	B_{02}	B_{03}	B_{04}	
(I)	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	B_{11}	B_{12}	B_{13}	B_{14}	
(II)	A_{20}	A_{21}	...	...	...	...	...	...	...	
(III)	...	...	...			...	...	...	...	
(IV)	A_{40}	A_{41}	A_{42}	A_{43}	A_{44}	B_{41}	B_{42}	B_{43}	B_{44}	 (III・72)
(V)	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	b_{11}	b_{12}	b_{13}	b_{14}	
(VI)	a_{20}	a_{21}	...	...	...	...	...	...	...	
(VII)	a_{30}	a_{31}	a_{32}	a_{33}	a_{34}	b_{31}	b_{32}	b_{33}	b_{34}	

ν	$P(\nu)$	$Q(\nu)$	ν	$P(\nu)$	$Q(\nu)$
0	A_{00}		13	$\frac{1}{2}(A_{34}+b_{34})$	$-\frac{1}{2}(a_{34}-B_{34})$
1	$\frac{1}{2}(A_{11}-b_{11})$	$\frac{1}{2}(a_{11}+\hat{B}_{11})$	14	$\frac{1}{2}(A_{24}-b_{24})$	$-\frac{1}{2}(a_{24}+B_{24})$
2	$\frac{1}{2}(A_{22}-b_{22})$	$\frac{1}{2}(a_{11}+B_{22})$	15	$\frac{1}{2}(A_{13}-b_{13})$	$-\frac{1}{2}(a_{13}+B_{13})$
3	$\frac{1}{2}(A_{33}-b_{33})$	$\frac{1}{2}(a_{33}+B_{33})$	16	A_{02}	$-B_{02}$
4	A_{44}	B_{44}	17	$\frac{1}{2}(A_{11}+b_{11})$	$\frac{1}{2}(a_{11}-B_{11})$
5	$\frac{1}{2}(A_{34}-b_{34})$	$-\frac{1}{2}(a_{34}+B_{34})$	18	A_{20}	a_{20}
6	$\frac{1}{2}(A_{23}-b_{23})$	$-\frac{1}{2}(a_{23}+B_{23})$	19	$\frac{1}{2}(A_{31}-b_{31})$	$\frac{1}{2}(a_{31}+B_{31})$
7	$\frac{1}{2}(A_{12}-b_{12})$	$-\frac{1}{2}(a_{12}+B_{12})$	20	A_{42}	B_{42}
8	A_{01}	$-B_{01}$	21	$\frac{1}{2}(A_{33}+b_{33})$	$-\frac{1}{2}(a_{33}-B_{33})$
9	A_{10}	a_{10}	22	$\frac{1}{2}(A_{24}+b_{24})$	$-\frac{1}{2}(a_{24}-B_{24})$
10	$\frac{1}{2}(A_{21}-b_{21})$	$\frac{1}{2}(a_{21}+B_{21})$	23	$\frac{1}{2}(A_{14}-b_{14})$	$-\frac{1}{2}(a_{14}+B_{14})$
11	$\frac{1}{2}(A_{32}-b_{32})$	$\frac{1}{2}(a_{32}+B_{32})$	24	A_{03}	$-B_{03}$
12	A_{43}	B_{43}			

ν	$P(\nu)$	$Q(\nu)$	ν	$P(\nu)$	$Q(\nu)$
25	$\frac{1}{2}(A_{12}+b_{12})$	$\frac{1}{2}(a_{12}-B_{12})$	31	$\frac{1}{2}(A_{14}+b_{14})$	$-\frac{1}{2}(a_{14}-B_{14})$
26	$\frac{1}{2}(A_{21}+b_{21})$	$\frac{1}{2}(a_{21}-B_{21})$	32	A_{04}	B_{04}
27	A_{30}	a_{30}	33	$\frac{1}{2}(A_{13}+b_{13})$	$\frac{1}{2}(a_{13}-B_{13})$
28	A_{41}	B_{41}	34	$\frac{1}{2}(A_{22}+b_{22})$	$\frac{1}{2}(a_{22}-B_{22})$
29	$\frac{1}{2}(A_{32}+b_{32})$	$-\frac{1}{2}(a_{32}-B_{32})$	35	$\frac{1}{2}(A_{31}+b_{31})$	$\frac{1}{2}(a_{31}-B_{31})$
30	$\frac{1}{2}(A_{23}+b_{23})$	$-\frac{1}{2}(a_{23}-B_{23})$	36	A_{40}	

.....(IV・72)

d. 120 (8×15) 項重調和解析

120 項の系列 y_p を次の如く並べる。

(0)	y_0	y_{15}	y_{30}	y_{45}	y_{60}	y_{75}	y_{90}	y_{105}
(1)	8	23	38	53	68	83	98	113
(2)	16	31	46	61	76	91	106	1
(3)	24	39	54	69	84	99	114	9
(4)	32	47	62	77	92	107	2	17
(5)	40	55	70	85	100	115	10	23
(6)	48	63	78	93	108	3	18	33
(7)	56	71	86	101	116	11	26	41
(8)	64	79	94	109	4	19	34	49
(9)	72	87	102	117	12	27	42	57
(10)	80	95	110	5	20	35	50	65
(11)	88	103	118	13	28	43	58	73
(12)	96	111	6	21	36	51	66	81
(13)	104	119	14	29	44	59	74	89
(14)	112	7	22	37	52	67	82	97

..... (I・120)

(第2行以下は添字のみを示す)

(I) の各横行について 8 項調和解析, そのフーリエ係数を:

	(0)	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
(0)	α_{00}	α_{01}	α_{02}	α_{03}	α_{04}	β_{01}	β_{02}	β_{03}
(1)	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	β_{11}	β_{12}	β_{13}
(2)	α_{20}	α_{21}	α_{22}	...	...	...	...	...
(3)	α_{30}	α_{31}	...	...			...	...
(4)	...	...	...					
⋮								
(12)	...	...	...	...	...	...	...	...
(13)	$\alpha_{13.0}$	$\alpha_{13.1}$	$\alpha_{13.2}$	$\alpha_{13.3}$	$\alpha_{13.4}$	$\beta_{13.1}$	$\beta_{13.2}$	$\beta_{13.3}$
(14)	$\alpha_{14.0}$	$\alpha_{14.1}$	$\alpha_{14.2}$	$\alpha_{14.3}$	$\alpha_{14.4}$	$\beta_{14.1}$	$\beta_{14.2}$	$\beta_{14.4}$

..... (II・120)

(II) の各縦列について 15 項調和解析, そのフーリエ係数を:

$$\begin{aligned}
 (0) & A_{00} A_{01} A_{02} A_{03} A_{04} A_{05} A_{06} A_{07} B_{01} B_{02} B_{03} B_{04} B_{05} B_{06} B_{07} \\
 (I) & A_{10} A_{11} A_{12} A_{13} A_{14} \cdots \cdots A_{17} B_{11} B_{12} \cdots \cdots \cdots B_{17} \\
 (II) & A_{20} A_{21} \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
 (III)^* & \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
 (IV) & A_{40} A_{41} A_{42} A_{43} \cdots \cdots \cdots A_{47} B_{41} B_{42} \cdots \cdots \cdots B_{47} \cdots \cdots (III \cdot 120) \\
 (V) & a_{10} a_{11} a_{12} a_{13} a_{14} a_{15} a_{16} a_{17} b_{11} b_{12} b_{13} b_{14} b_{15} b_{16} b_{17} \\
 (VI) & a_{20} a_{21} a_{22} \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots b_{27} \\
 (VII) & a_{30} a_{31} a_{32} a_{33} a_{34} a_{35} a_{36} a_{37} b_{31} b_{32} b_{33} b_{34} b_{35} b_{36} b_{37}
 \end{aligned}$$

ν	$P(\nu)$	$Q(\nu)$	ν	$P(\nu)$	$Q(\nu)$
0	A_{00}		23	$\frac{1}{2}(A_{17}-b_{17})$	$-\frac{1}{2}(a_{17}+B_{17})$
1	$\frac{1}{2}(A_{11}-b_{11})$	$\frac{1}{2}(a_{11}+B_{11})$	24	A_{06}	$-B_{06}$
2	$\frac{1}{2}(A_{22}-b_{22})$	$\frac{1}{2}(a_{22}+B_{22})$	25	$\frac{1}{2}(A_{15}+b_{15})$	$\frac{1}{2}(a_{15}-B_{15})$
3	$\frac{1}{2}(A_{33}-b_{33})$	$\frac{1}{2}(a_{33}+B_{33})$	26	$\frac{1}{2}(A_{24}+b_{24})$	$\frac{1}{2}(a_{24}-B_{24})$
4	A_{44}	B_{44}	27	$\frac{1}{2}(A_{33}+b_{33})$	$\frac{1}{2}(a_{33}-B_{33})$
5	$\frac{1}{2}(A_{35}+b_{35})$	$-\frac{1}{2}(a_{35}-B_{35})$	28	A_{42}	$-B_{42}$
6	$\frac{1}{2}(A_{26}+b_{26})$	$-\frac{1}{2}(a_{26}-B_{26})$	29	$\frac{1}{2}(A_{31}-b_{31})$	$-\frac{1}{2}(a_{31}+B_{31})$
7	$\frac{1}{2}(A_{17}+b_{17})$	$-\frac{1}{2}(a_{17}-B_{17})$	30	A_{20}	$-a_{20}$
8	A_{07}	$-B_{07}$	31	$\frac{1}{2}(A_{11}+b_{11})$	$-\frac{1}{2}(a_{11}-B_{11})$
9	$\frac{1}{2}(A_{16}+b_{16})$	$\frac{1}{2}(a_{16}-B_{16})$	32	A_{02}	B_{02}
10	$\frac{1}{2}(A_{25}+b_{25})$	$\frac{1}{2}(a_{25}-B_{25})$	33	$\frac{1}{2}(A_{13}-b_{13})$	$\frac{1}{2}(a_{13}+B_{13})$
11	$\frac{1}{2}(A_{34}+b_{34})$	$\frac{1}{2}(a_{34}-B_{34})$	34	$\frac{1}{2}(A_{24}-b_{24})$	$\frac{1}{2}(a_{24}+B_{24})$
12	A_{43}	$-B_{34}$	35	$\frac{1}{2}(A_{35}-b_{35})$	$\frac{1}{2}(a_{35}+B_{35})$
13	$\frac{1}{2}(A_{32}-b_{32})$	$-\frac{1}{2}(a_{32}+B_{32})$	36	A_{46}	B_{46}
14	$\frac{1}{2}(A_{21}-b_{21})$	$-\frac{1}{2}(a_{21}+B_{21})$	37	$\frac{1}{2}(A_{37}+b_{37})$	$-\frac{1}{2}(a_{37}-B_{37})$
15	A_{10}	$-a_{10}$	38	$\frac{1}{2}(A_{27}-b_{27})$	$-\frac{1}{2}(a_{27}+B_{27})$
16	A_{01}	B_{01}	39	$\frac{1}{2}(A_{16}-b_{16})$	$-\frac{1}{2}(a_{16}+B_{16})$
17	$\frac{1}{2}(A_{12}-b_{12})$	$\frac{1}{2}(a_{12}+B_{12})$	40	A_{05}	$-B_{05}$
18	$\frac{1}{2}(A_{23}-b_{23})$	$\frac{1}{2}(a_{23}+B_{23})$	41	$\frac{1}{2}(A_{14}+b_{14})$	$\frac{1}{2}(a_{14}-B_{14})$
19	$\frac{1}{2}(A_{34}-b_{34})$	$\frac{1}{2}(a_{34}+B_{34})$	42	$\frac{1}{2}(A_{23}+b_{23})$	$\frac{1}{2}(a_{23}-B_{23})$
20	A_{45}	B_{45}	43	$\frac{1}{2}(A_{32}+b_{32})$	$\frac{1}{2}(a_{32}-B_{32})$
21	$\frac{1}{2}(A_{36}+b_{36})$	$-\frac{1}{2}(a_{36}-B_{36})$	44	A_{41}	$-B_{41}$
22	$\frac{1}{2}(A_{27}+b_{27})$	$-\frac{1}{2}(a_{27}-B_{27})$			

ν	$P(\nu)$	$Q(\nu)$	ν	$P(\nu)$	$Q(\nu)$
45	A_{30}	$-a_{30}$	53	$\frac{1}{2}(A_{37}-b_{37})$	$-\frac{1}{2}(a_{37}+B_{37})$
46	$\frac{1}{2}(A_{21}+b_{21})$	$-\frac{1}{2}(a_{21}-B_{21})$	54	$\frac{1}{2}(A_{26}-b_{26})$	$-\frac{1}{2}(a_{26}+B_{26})$
47	$\frac{1}{2}(A_{12}+b_{12})$	$-\frac{1}{2}(a_{12}-B_{12})$	55	$\frac{1}{2}(A_{15}-b_{15})$	$-\frac{1}{2}(a_{15}+B_{15})$
48	A_{03}	B_{03}	56	A_{04}	$-B_{04}$
49	$\frac{1}{2}(A_{14}-b_{14})$	$\frac{1}{2}(a_{14}+B_{14})$	57	$\frac{1}{2}(A_{13}+b_{13})$	$\frac{1}{2}(a_{13}-B_{13})$
50	$\frac{1}{2}(A_{25}-b_{25})$	$\frac{1}{2}(a_{25}+B_{25})$	58	$\frac{1}{2}(A_{22}+b_{22})$	$\frac{1}{2}(a_{22}-B_{22})$
51	$\frac{1}{2}(A_{36}-b_{36})$	$\frac{1}{2}(a_{36}+B_{36})$	69	$\frac{1}{2}(A_{31}+b_{31})$	$\frac{1}{2}(a_{31}-B_{31})$
52	A_{47}	B_{47}	60	A_{40}	

.....(IV・120)

e. 180 (5×36) 項重調和解析

以下簡略して第 IV 表のみを掲げる。

ν	$P(\nu)$	$Q(\nu)$	ν	$P(\nu)$	$Q(\nu)$
0	A_{00}		18	$A_{2,18}$	$-a_{2,18}$
1	$\frac{1}{2}(A_{11}-b_{11})$	$\frac{1}{2}(a_{11}+B_{11})$	19	$\frac{1}{2}(A_{1,17}-b_{1,17})$	$-\frac{1}{2}(a_{1,17}+B_{1,17})$
2	$\frac{1}{2}(A_{22}-b_{22})$	$\frac{1}{2}(a_{22}+B_{22})$	20	$A_{0,16}$	$-B_{0,16}$
3	$\frac{1}{2}(A_{23}+b_{23})$	$-\frac{1}{2}(a_{23}-B_{23})$	21	$\frac{1}{2}(A_{1,15}+b_{1,15})$	$\frac{1}{2}(a_{1,15}-B_{1,15})$
4	$\frac{1}{2}(A_{14}+b_{14})$	$-\frac{1}{2}(a_{14}-B_{14})$	22	$\frac{1}{2}(A_{2,14}+b_{2,14})$	$\frac{1}{2}(a_{2,14}-B_{2,14})$
5	A_{05}	B_{05}	23	$\frac{1}{2}(A_{2,13}-b_{2,13})$	$-\frac{1}{2}(a_{2,13}+B_{2,13})$
6	$\frac{1}{2}(A_{16}-b_{16})$	$\frac{1}{2}(a_{16}+B_{16})$	24	$\frac{1}{2}(A_{1,12}-b_{1,12})$	$-\frac{1}{2}(a_{1,12}+B_{1,12})$
7	$\frac{1}{2}(A_{27}-b_{27})$	$\frac{1}{2}(a_{27}+B_{27})$	25	$A_{0,11}$	$-B_{0,11}$
8	$\frac{1}{2}(A_{28}+b_{28})$	$-\frac{1}{2}(a_{28}-B_{28})$	26	$\frac{1}{2}(A_{1,10}+b_{1,10})$	$\frac{1}{2}(a_{1,10}-B_{1,10})$
9	$\frac{1}{2}(A_{19}+b_{19})$	$-\frac{1}{2}(a_{19}-B_{19})$	27	$\frac{1}{2}(A_{29}+b_{29})$	$\frac{1}{2}(a_{29}-B_{29})$
10	$A_{0,10}$	$B_{0,10}$	28	$\frac{1}{2}(A_{28}-b_{28})$	$-\frac{1}{2}(a_{28}+B_{28})$
11	$\frac{1}{2}(A_{1,11}-b_{1,11})$	$\frac{1}{2}(a_{1,11}+B_{1,11})$	29	$\frac{1}{2}(A_{17}-b_{17})$	$-\frac{1}{2}(a_{17}+B_{17})$
12	$\frac{1}{2}(A_{2,12}-b_{2,12})$	$\frac{1}{2}(a_{2,12}+B_{2,12})$	30	A_{06}	$-B_{06}$
13	$\frac{1}{2}(A_{2,13}+b_{1,13})$	$\frac{1}{2}(a_{2,13}-B_{2,13})$	31	$\frac{1}{2}(A_{15}+b_{15})$	$\frac{1}{2}(a_{15}-B_{15})$
14	$\frac{1}{2}(A_{1,14}+b_{1,14})$	$-\frac{1}{2}(a_{1,14}-B_{1,14})$	32	$\frac{1}{2}(A_{24}+b_{24})$	$\frac{1}{2}(a_{24}-B_{24})$
15	$A_{0,15}$	$B_{0,15}$	33	$\frac{1}{2}(A_{23}-b_{23})$	$-\frac{1}{2}(a_{23}+B_{23})$
16	$\frac{1}{2}(A_{1,16}-b_{1,16})$	$\frac{1}{2}(a_{1,16}+B_{1,16})$	34	$\frac{1}{2}(A_{12}-b_{12})$	$-\frac{1}{2}(a_{12}+B_{12})$
17	$\frac{1}{2}(A_{2,17}-b_{2,17})$	$\frac{1}{2}(a_{2,17}+B_{2,17})$			

ν	$P(\nu)$	$Q(\nu)$	ν	$P(\nu)$	$Q(\nu)$
35	A_{01}	$-B_{01}$	63	$\frac{1}{2}(A_{29}-b_{29})$	$-\frac{1}{2}(a_{29}+B_{29})$
36	A_{10}	a_{10}	64	$\frac{1}{2}(A_{18}-b_{18})$	$-\frac{1}{2}(a_{18}+B_{18})$
37	$\frac{1}{2}(A_{21}-b_{21})$	$\frac{1}{2}(a_{21}+B_{21})$	65	A_{07}	$-B_{07}$
38	$\frac{1}{2}(A_{22}+b_{22})$	$-\frac{1}{2}(a_{22}-B_{22})$	66	$\frac{1}{2}(A_{16}+b_{16})$	$\frac{1}{2}(a_{16}-B_{16})$
39	$\frac{1}{2}(A_{13}+b_{13})$	$-\frac{1}{2}(a_{13}-B_{13})$	67	$\frac{1}{2}(A_{25}+b_{25})$	$\frac{1}{2}(a_{25}-B_{25})$
40	A_{04}	B_{04}	68	$\frac{1}{2}(A_{24}-b_{24})$	$-\frac{1}{2}(a_{24}+B_{24})$
41	$\frac{1}{2}(A_{15}-b_{15})$	$\frac{1}{2}(a_{15}+B_{15})$	69	$\frac{1}{2}(A_{13}-b_{13})$	$-\frac{1}{2}(a_{13}+B_{13})$
42	$\frac{1}{2}(A_{26}-b_{26})$	$\frac{1}{2}(a_{26}+B_{26})$	70	A_{02}	$-B_{02}$
43	$\frac{1}{2}(A_{27}+b_{27})$	$-\frac{1}{2}(a_{27}-B_{27})$	71	$\frac{1}{2}(A_{11}+b_{11})$	$\frac{1}{2}(a_{11}-B_{11})$
44	$\frac{1}{2}(A_{18}+b_{18})$	$-\frac{1}{2}(a_{18}-B_{18})$	72	A_{20}	a_{20}
45	A_{09}	B_{09}	73	$\frac{1}{2}(A_{21}+b_{21})$	$-\frac{1}{2}(a_{21}-B_{21})$
46	$\frac{1}{2}(A_{1,10}-b_{1,10})$	$\frac{1}{2}(a_{1,10}+B_{1,10})$	74	$\frac{1}{2}(A_{12}+b_{12})$	$-\frac{1}{2}(a_{12}-B_{12})$
47	$\frac{1}{2}(A_{2,11}-b_{2,11})$	$\frac{1}{2}(a_{2,11}+B_{2,11})$	75	A_{03}	B_{03}
48	$\frac{1}{2}(A_{2,12}+b_{2,12})$	$-\frac{1}{2}(a_{2,12}-B_{2,12})$	76	$\frac{1}{2}(A_{14}-b_{14})$	$\frac{1}{2}(a_{14}+B_{14})$
49	$\frac{1}{2}(A_{1,13}+b_{1,13})$	$-\frac{1}{2}(a_{1,13}-B_{1,13})$	77	$\frac{1}{2}(A_{25}-b_{25})$	$\frac{1}{2}(a_{25}+B_{25})$
50	$A_{0,14}$	$B_{0,14}$	78	$\frac{1}{2}(A_{26}+b_{26})$	$-\frac{1}{2}(a_{26}-B_{26})$
51	$\frac{1}{2}(A_{1,15}-b_{1,15})$	$\frac{1}{2}(a_{1,15}+B_{1,15})$	79	$\frac{1}{2}(A_{17}+b_{17})$	$-\frac{1}{2}(a_{17}-B_{17})$
52	$\frac{1}{2}(A_{2,16}-b_{2,16})$	$\frac{1}{2}(a_{2,16}+B_{2,16})$	80	A_{08}	B_{08}
53	$\frac{1}{2}(A_{2,17}+b_{2,17})$	$-\frac{1}{2}(a_{2,17}-B_{2,17})$	81	$\frac{1}{2}(A_{19}-b_{19})$	$\frac{1}{2}(a_{19}+B_{19})$
54	$A_{1,18}$	$-a_{1,18}$	82	$\frac{1}{2}(A_{2,10}-b_{2,10})$	$\frac{1}{2}(a_{2,10}+B_{2,10})$
55	$A_{0,17}$	$-B_{0,17}$	83	$\frac{1}{2}(A_{2,11}+b_{2,11})$	$-\frac{1}{2}(a_{2,11}-B_{2,11})$
56	$\frac{1}{2}(A_{1,16}+b_{1,16})$	$\frac{1}{2}(a_{1,16}-B_{1,16})$	84	$\frac{1}{2}(A_{1,12}+b_{1,12})$	$-\frac{1}{2}(a_{1,12}-B_{1,12})$
57	$\frac{1}{2}(A_{2,15}+b_{2,15})$	$\frac{1}{2}(a_{2,15}-B_{2,15})$	85	$A_{0,13}$	$B_{0,13}$
58	$\frac{1}{2}(A_{2,14}-b_{2,14})$	$-\frac{1}{2}(a_{2,14}+B_{2,14})$	86	$\frac{1}{2}(A_{1,14}-b_{1,14})$	$\frac{1}{2}(a_{1,14}+B_{1,14})$
59	$\frac{1}{2}(A_{1,13}-b_{1,13})$	$-\frac{1}{2}(a_{1,13}+B_{1,13})$	87	$\frac{1}{2}(A_{2,15}-b_{2,15})$	$\frac{1}{2}(a_{2,15}-B_{2,15})$
60	$A_{0,12}$	$-B_{0,12}$	88	$\frac{1}{2}(A_{2,16}+b_{2,16})$	$-\frac{1}{2}(a_{2,16}-B_{2,16})$
61	$\frac{1}{2}(A_{1,11}+b_{1,11})$	$\frac{1}{2}(a_{1,11}-B_{1,11})$	89	$\frac{1}{2}(A_{1,17}+b_{1,17})$	$-\frac{1}{2}(a_{1,17}+B_{1,17})$
62	$\frac{1}{2}(A_{2,10}+b_{2,10})$	$\frac{1}{2}(a_{2,10}-B_{2,10})$	90	$A_{0,18}$	

.....(IV・180)

§4. あ と が き

次に m, n が任意の整数の場合について附言して置く。 r を n とは互に素なる整数として

mn 項の系列を次の如くに並べる。

$$\left(\begin{array}{cccccccc} y_0, & y_n, & y_{2n}, & \dots, & y_{(m-1)n} \\ y_r, & y_{r+n}, & y_{r+2n}, & \dots, & y_{r+(m-1)n} \\ y_{2r}, & y_{2r+n}, & & \dots, & y_{2r+(m-1)n} \\ \dots & \dots & & \dots & \dots \\ y_{(n-1)r}, & y_{(n-1)r+n}, & & \dots, & y_{(n-1)r+(m-1)n} \end{array} \right) \dots \quad (9)$$

即ち、一般項は

$$y_{jr+kn} \begin{cases} k=0, 1, 2, \dots, m-1 \\ j=0, 1, 2, \dots, n-1 \end{cases}$$

第 j 行についてのフーリエ係数を求め次の如く置く。

$$\left. \begin{array}{l} a_j(\nu) \\ b_j(\nu) \end{array} \right\} = \sum_{k=0}^{m-1} y_{jr+kn} \cos \left(\nu \cdot k \frac{2\pi}{m} \right) \dots \quad (10)$$

$$\text{但し } \nu=0, 1, 2, \dots, \frac{mn}{2}$$

この場合、前と全く同様にして

$$\left. \begin{array}{l} \sum_{\rho=0}^{mn-1} y_{\rho} \cos \nu \rho \frac{2\pi}{mn} = \sum_{j=0}^{n-1} a_j(\nu) \cos \nu jr \frac{2\pi}{mn} + \sum_{j=0}^{n-1} b_j(\nu) \sin \nu jr \frac{2\pi}{mn} \\ \sum_{\rho=0}^{mn-1} y_{\rho} \sin \nu \rho \frac{2\pi}{mn} = \sum_{j=0}^{n-1} b_j(\nu) \cos \nu jr \frac{2\pi}{mn} + \sum_{j=0}^{n-1} a_j(\nu) \sin \nu jr \frac{2\pi}{mn} \end{array} \right\} \dots (11)$$

となるが、これは (3) の形式的な一般化であつて、(11) の右辺の各項の計算には調和解析の schema を適用することは出来ない。従つて実際の計算を行うには、 n の比較的小さい場合の外は極めて困難であろう。

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第 1 表 4 項調和解析

A	y_0	y_1	A	c_0
B	y_2	y_3	B	c_1
$A+B$	c_0	c_1	$A+B$	$4 \cdot a_0$
$A-B$	$2 \cdot a_1$	$2 \cdot b_1$	$B+A$	$4 \cdot a_2$

第 2 表 5 項調和解析

A	y_0	y_1	y_2
B		y_4	y_3
$A+B$	c_0	c_1	c_1
$A-B$		d_1	d_2
Σc_μ	s_0		

ν	$\frac{\cos \nu \alpha}{2.5}$	0	1	2	Probe	
					A_μ	
0	0.4000	$\frac{1}{2}s_0$	c_0	c_0	$3.0c_0$	
1	0.1236		c_1	c_2	$0.5c_1$	
2	0.3236		$-c_2$	$-c_1$	$0.5c_2$	
		a_0	a_1	a_2	ΣA_μ	$5a_0 + 2.5 \Sigma a_\mu$

ν	$\frac{\sin \nu \alpha}{2.5}$	1	2	Probe	
				B_μ	
1	0.3804	d_1	$-d_2$	$0.615d_1$	
2	0.2352	d_2	d_1	$-0.145d_2$	
		b_1	b_2	ΣB_μ	Σb_μ

第 3 表 8 項調和解析

A	y_0	y_1	y_2	y_3
B	y_4	y_5	y_6	y_7
$A+B$	c_0	c_1	c_2	c_3
$A-B$	d_0	d_1	d_2	d_3

A	c_0	c_1
B	c_2	c_3
$A+B$	e_0	e_1
$A-B$	f_0	f_1

A	d_0	d_1	d_2
B		d_3	
$A+B$	g_0	g_1	g_2
$A-B$		h_1	

A	e_0
B	e_1
$A+B$	i_0
$A-B$	j_0

A	y_0	y_1	y_2	y_3	y_4
B		y_7	y_6	y_5	
$A+B$	l_0		l_2		l_4
$A-B$		m_1		m_3	

ν	$\frac{\cos \nu \alpha}{4}$	0	1	2	3	4
0	0.2500	$\frac{1}{2}i_0$	g_0	f_0	g_0	$\frac{1}{2}j_0$
1	0.1768		h_1		$-h_1$	
	Σ	a_0	a_1	a_2	a_3	a_4

Probe

ν	$\frac{\sin \nu \alpha}{4}$	1	2	3
0	0.1768	g_1		g_1
1	0.2500	g_2	f_1	$-g_2$
	Σ	b_1	b_2	b_3

B_μ	A_μ
	$5.0l_0$
$0.6035m_1$	$1.0l_2$
$0.1035m_3$	$1.0l_4$
ΣB_μ	ΣA_μ
Σb_μ	$8 \cdot a_0 + 4 \Sigma a_\mu + 8 \cdot a_4$

第 4 表 9 項調和解析

A	y_0	y_1	y_2	y_3	y_4
B		y_8	y_7	y_6	y_5
$A+B$	c_0	c_1	c_2	c_3	c_4
$A-B$		d_1	d_2	d_3	d_4
Σc_μ	s_0				

A	y_0	y_1	y_2
B	y_3	y_4	y_5
C	y_6	y_7	y_8
$A+B+C$	v_0	u_1	u_2
		u_1+u_2	v_1
		u_1-u_2	w_1

ν	$\frac{\cos \nu \alpha}{4.5}$	0	1	2	3	4	A_μ	Probe
0	0.2222	$\frac{1}{2}s_0$	c_0	c_0	v_0	c_0	$5.0c_0$	
1	0.1702		c_1	c_4		c_2	$0.5c_1$	
2	0.0387		c_2	c_1		c_4	$0.5c_2$	
3	0.1111		$-c_3$	$-c_3$	$-v_1$	$-c_3$	$0.5c_3$	
4	0.2089		$-c_4$	$-c_2$		$-c_1$	$0.5c_4$	
	Σ	a_0	a_1	a_2	a_3	a_4	ΣA_μ	$9 \cdot a_0 + 4.5 \Sigma a_\mu$

ν	$\frac{\sin \nu \alpha}{4.5}$	1	2	3	4	B_μ	Probe
1	0.1429	d_1	$-d_4$		$-d_2$	$0.6302d_1$	
2	0.2189	d_2	d_1		$-d_4$	$-0.0404d_2$	
3	0.1924	d_3	$-d_3$	w_1	d_3	$0.1924d_3$	
4	0.0760	d_4	d_2		d_1	$-0.0933d_4$	
	Σ	b_1	b_2	b_3	b_4	ΣB_μ	Σb_μ

第 5 表 15 項調和解析

A	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7
B		y_{14}	y_{13}	y_{12}	y_{11}	y_{10}	y_9	y_8
$A+B$	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7
$A-B$		d_1	d_2	d_3	d_4	d_5	d_6	d_7

<i>A</i>	c_0	c_1	c_2
<i>B</i>		c_4	c_3
<i>C</i>	c_5	c_6	c_7
<i>A+B+C</i>	e_0	e_1	e_2

	c_0	c_1	d_1
		c_2	$-d_2$
	c_3	c_4	d_4
		c_5	$-d_5$
	c_6	c_7	d_7
Σ	g_0	g_1	h_1

<i>A</i>	d_1	d_2
<i>B</i>	d_4	d_3
<i>C</i>	d_6	d_7
<i>A-B+C</i>	f_1	f_2

<i>A</i>	e_0
<i>B</i>	e_1
<i>C</i>	e_2
<i>A+B+C</i>	i_0

ν	$\frac{\cos \nu \alpha}{7.5}$	0	1	2	3	4	5	6	7	Probe	
0	0.1333	$\frac{1}{2}i_0$	c_0	c_0	e_0	c_0	g_0	e_0	c_0	8.0	c_0
1	0.1219		c_1	c_7		c_4			c_2	0.5	c_1
2	0.0892		c_2	c_1		c_7			c_4	0.5	c_2
3	0.0412		c_3	c_6	e_1	c_3		e_2	c_6	0.5	c_3
4	0.0140		$-c_4$	$-c_2$		$-c_1$			$-c_7$	0.5	c_4
5	0.0667		$-c_5$	$-c_5$		$-c_5$	$-g_1$		$-c_5$	0.5	c_5
6	0.1079		$-c_6$	$-c_3$	$-e_2$	$-c_6$		$-e_1$	$-c_3$	0.5	c_6
7	0.1304		$-c_7$	$-c_4$		$-c_2$			$-c_1$	0.5	c_7
	Σ	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	Σ	$15a_0 + 7.5 \Sigma a_{\mu}$

ν	$\frac{\sin \nu \alpha}{7.5}$	1	2	3	4	5	6	7	Probe	
1	0.0543	d_1	$-d_7$		d_4			$-d_2$	$0.6343d_1$	
2	0.0991	d_2	d_1		$-d_7$			$-d_4$	$-0.0141d_2$	
3	0.1268	d_3	$-d_6$	f_1	$-d_3$		$-f_2$	$-d_6$	$0.2052d_3$	
4	0.1327	d_4	d_2		d_1			d_7	$-0.0297d_4$	
5	0.1155	d_5	$-d_5$		d_5	h_1		d_5	$0.1155d_5$	
6	0.0784	d_6	d_3	f_2	$-d_6$		f_1	d_3	$-0.0484d_6$	
7	0.0277	d_7	$-d_4$		$-d_2$			d_1	$0.0740d_7$	
	Σ	b_1	b_2	b_3	b_4	b_5	b_6	b_7	Σ	Σb_μ