Numerical Forecasting over the Far East by the Double Fourier Series Method

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Abstract

The results of the numerical forecasting over the Far East is presented. In this paper, we use double Fourier series in order to calculate the Jacobian of vorticity ζ and the height of isobaric surface Z and also to solve the Poisson equation. As the preliminary report we explain mainly the detailed technique of our method how to treat and apply. It is also shown that the results of numerical forecasting will be improved, if we take shorter time interval of each step. In connection with this problem, the outline of the relay-computer (FACOM-100) which is available is written briefly.

§ 1. Introduction

The purpose of this article is to present one method to calculate the Jacobian of vorticity ζ and the height of isobaric surface Z and also to integrate it with respect to time by using harmonic functions.

Charney et al. [2] [3] used the extrapolated Liebman procedure to integrate the vorticity equation. On the other hand, Fjørtoft [4] proposed a graphical method which is convenient from practical point of view, when a high speed computer is not available.

If we consider the Fjørtoft's method from the point of view of harmonic function, we may say as follows: The field of Z is divided into two parts,

$$Z = Z_1 + Z_2 \quad (1)$$

where Z_1 and Z_2 are harmonic functions such as being connected with their wave numbers μ_1 and μ_2 in the following way

$$\nabla^2 Z_1 = -\mu_1^2 Z_1, \quad \nabla^2 Z_2 = -\mu_2^2 Z_2. \quad (2)$$

The instantaneous change of the absolute vorticity in non-divergent and barotropic atmosphere is written

$$\frac{\partial \eta}{\partial t} = \frac{g}{f} J(\eta, Z) \quad (3)$$

where

$$\eta = \frac{g}{f} \nabla^2 Z + f.$$

Substituting (1) and (2) into (3), we have

$$\frac{\partial \eta}{\partial t} = \frac{g}{f} (\mu_1^2 - \mu_2^2) J(Z_1, Z_2). \quad (4)$$

The equation (4) is interpreted in the following two ways. First, we rewrite (4) as follows

$$\frac{\partial \eta}{\partial t} = \mu_1^2 \left(1 - \frac{\mu_2^2}{\mu_1^2} \right) V_2 \cdot \nabla \eta_1 \quad (5)$$

where

$$V_2 = \frac{g}{f} \mathbf{K} \times \nabla Z_2, \quad \eta_1 = \frac{g}{f} \nabla^2 Z_1$$

The equation (5) means that the instantaneous change of η is caused by the advection of η_1 in the field of Z_2 . On the other hand, we have another equation from (4) i.e.,

$$\frac{\partial \eta}{\partial t} = \mu_2^2 \left(\frac{\mu_1^2}{\mu_2^2} - 1 \right) V_1 \cdot \nabla \eta_2 \quad (6)$$

where

$$V_1 = \frac{g}{f} \mathbf{K} \times \nabla Z_1, \quad \eta_2 = \frac{g}{f} \nabla^2 Z_2$$

The equation (6) means that the instantaneous change of η is caused by the advection of η_2 in the field of Z_1 . Both (5) and (6) represent the same thing concerning the instantaneous change of η while the standpoints of understanding of the role of Z_1 and Z_2 are different. The choice of Z_1 or Z_2 as the fundamental velocity field depends upon which field has the more stable character with respect to time. Suppose, for instance, the field Z_1 is more stable than the field Z_2 , it would give us better estimation to use (4) in order to

calculate η at $t=t_0+\Delta t$ from η at $t=t_0$ by the following simple extrapolation formula,

$$(\eta)_{t=t_0+\Delta t}=(\eta)_{t=t_0}+\left(\frac{\partial\eta}{\partial t}\right)_{t=t_0}\Delta t \quad (7)$$

Although Fjørtoft did not use harmonic functions, he considered in the same way. As the more stable pattern of Z , he used $\bar{Z}$ where bar means the spatial mean value of $Z(x, y)$ which is defined by $\frac{1}{4}\{Z(x+d, y)+Z(x-d, y)+Z(x, y+d)+Z(x, y-d)\}$. In connection with this field he divided Z into two parts

$$Z=(Z-\bar{Z})+\bar{Z} \quad (8)$$

The corresponding vorticity η is

$$\eta=\frac{g}{f}\nabla^2 Z\approx-\frac{4g}{fd^2}(Z-\bar{Z}).$$

Thus the vorticity equation becomes

$$\frac{\partial\eta}{\partial t}=\frac{g}{f}J(\eta, (Z-\bar{Z})+\bar{Z})=\frac{g}{f}J(\eta, \bar{Z}). \quad (9)$$

Another equation which corresponds to this equation, just like relation between (5) and (6), is

$$\frac{\partial\eta}{\partial t}=\frac{g}{f}J(\eta, Z). \quad (10)$$

Since $\bar{Z}$ is usually more stable than Z with respect to time, (9) gives better results than (10), when we use the extrapolation formula like (7) in the numerical forecasting of η . In the present article, we expand Z into double Fourier series with finite components. In this case there are two problems; how many components should be adopted and how to select appropriate harmonic functions among many others. The former problem may be determined from the point of view of the relation between the mesh-size and the scale of disturbances. The latter problem depends upon the character of disturbances, i.e., the stableness of disturbances with respect to time. Concerning the latter problem, we would like to add some comment that the physical consideration mentioned above gives us some clue how to select the appropriate harmonic function. As may be understood from the foregoing discussion, the choice of Z_1 is better than that of Z_2 if Z_1 is more stable than Z_2 with respect to time. As the

fundamental assumption of the problem of this choice, we assume that the large scale patterns do not change so quickly compared with the small scale patterns. Such a consideration enables us to make a choice to select large scale patterns instead of small scale patterns as the selection of the finite number of disturbances. The detailed procedures of the selection of the number and the scale of disturbances will be written in the following sections.

§ 2. Numerical Forecasting by Double Fourier Series Method

The vorticity equation in non-divergent and barotropic atmosphere is written as follows

$$\frac{\partial}{\partial t}\nabla^2 Z=J(\eta, Z). \quad (11)$$

Here Z and η are the height of isobaric surface of non-divergent level and the absolute vertical vorticity respectively. Using the usual notation, η is written in the following way

$$\eta=h\nabla^2 Z+f, \quad \left(h=\frac{g}{f}m^2\right) \quad (12)$$

where m and f are the map scale factor and coriolis parameter respectively.

Suppose that the height of the isobaric surface Z may be expanded into harmonic functions, i.e.,

$$Z=\sum_n a_n Z_n, \quad (13)$$

where Z_n is the harmonic function of order n . Since Z_n satisfies the following relation

$$\nabla^2 Z_n=-\mu_n^2 Z_n, \quad (14)$$

where μ_n is the wave number of Z_n , the substitution of (14) into (11) leads to the following equation

$$\begin{aligned} \frac{\partial}{\partial t}\nabla^2 Z &= \nabla^2 \left(\frac{\partial Z}{\partial t} \right) \\ &= J(f - h \sum_n \mu_n^2 a_n Z_n, \sum_n a_n Z_n) \\ &= -h \sum_{i=2} (\mu_1^2 - \mu_i^2) a_1 a_i J(Z_1, Z_i) \\ &\quad - h \sum_{j=3} (\mu_2^2 - \mu_j^2) a_2 a_j J(Z_2, Z_j) \\ &\quad - \dots \dots \dots \\ &\quad - \beta \sum_n a_n \frac{\partial Z_n}{\partial x}. \end{aligned} \quad (15)$$

The term of $J(Z_1, Z_i)$ in (15) is in proportion to $V_1 \cdot \nabla \zeta_i$, where V and ζ_i are the velocity

and the vorticity respectively, which are connected with $Z_1, Z_2, \dots$ as follows:

$$V_1 = \frac{g}{f} \mathbf{K} \times \nabla Z_1, \quad \zeta_1 = \frac{g}{f} \nabla^2 Z_1. \quad (16)$$

Thus the first part on the right hand side in (15) means that the field of Z_1 makes the vorticity of other patterns of $Z_2, Z_3, \dots$ flow away. In the expression of $J(Z_2, Z_j)$ in (15) we see that the field of Z_2 makes the vorticity of other patterns of $Z_3, Z_4, \dots$ flow away.

Similar relations may also be expected with the fields of $Z_3, Z_4, Z_5, Z_6, \dots$. In general, the field of Z_n makes the vorticities of $Z_{n+1}, Z_{n+2}, \dots$ flow away.

Since Z_i is the harmonic function, $J(Z_i, Z_j)$ is also the combination of harmonic functions. Such a circumstance makes it easy to integrate (15) with respect to space. Therefore, obtaining $\frac{\partial Z}{\partial t}$ from (15), we make the forecasting of Z as follows

$$(Z)_{t-t_0+\Delta t} = (Z)_{t-t_0} + \left(\frac{\partial Z}{\partial t} \right)_{t-t_0} \Delta t. \quad (17)$$

As the practical procedure, we expand into the following double Fourier series,

$$\begin{aligned} Z = & \sum_{m=0}^6 \sum_{n=0}^6 A_{mn} \cos mx \cos ny \\ & + \sum_{m=0}^6 \sum_{n=1}^5 B_{mn} \cos mx \sin ny \\ & + \sum_{m=1}^5 \sum_{n=0}^6 C_{mn} \sin mx \cos ny \\ & + \sum_{m=1}^5 \sum_{n=1}^5 D_{mn} \sin mx \sin ny. \end{aligned} \quad (18)$$

The domain where we used for the expansion into the double Fourier series is shown in Fig. 1. From the consideration mentioned above concerning the scale-effect, we put $Z_1, Z_2, Z_3, \dots$ as follows,

$$\begin{aligned} a_0 Z_0 &= A_{00} \quad a_1 Z_1 = A_{01} \cos y, \quad a_2 Z_2 = B_{01} \sin y \\ a_3 Z_3 &= A_{10} \cos x, \quad a_4 Z_4 = C_{10} \sin x \\ a_5 Z_5 &= A_{11} \cos x \cos y, \dots \dots \dots \end{aligned} \quad (19)$$

Since $J(Z_i, Z_j)$ is a function of sine and cosine, we can write the vorticity equation in the following way

$$\nabla^2 \frac{\partial Z}{\partial t} = \nabla^2 \frac{\partial}{\partial t} \sum_n a_n Z_n = \sum_n b_n Z_n, \quad (20)$$

where b_n is a function of $(A_{01} A_{02}, \dots), (B_{01} B_{02}, \dots), (C_{10} C_{11}, \dots)$ and $(D_{11} D_{12}, \dots)$.

Integration of this equation gives the following relation

$$\frac{\partial a_n Z_n}{\partial t} = - \frac{b_n}{\mu_n^2} Z_n + F_n, \quad (21)$$

where F_n is the integration function, which satisfies the Laplace equation, i.e.,

$$\nabla^2 F_n = 0. \quad (22)$$

If we write $F = \sum_n F_n$, the constants which are involved in F are determined from the boundary conditions of Z . In order to make the discussion simple, we do not consider the effect from boundaries, so that we neglect F . In such a situation, (21) may be written as follows

$$Z \frac{\partial a_n}{\partial t} = - \frac{b_n}{\mu_n^2} Z_n. \quad (23)$$

Since $a_n Z_n$ is one component of Z and similar relations hold for all n , we can extrapolate Z with respect to time in the following way

$$Z(t_0 + \Delta t) = \sum_n a_n(t_0 + \Delta t) Z_n, \quad (24)$$

where

$$a_n(t_0 + \Delta t) Z_n = a_n(t_0) Z_n + \left(\frac{\partial a_n}{\partial t} \right)_{t=t_0} Z_n \Delta t. \quad (25)$$

For example, we may write $\frac{\partial a_n}{\partial t}$ as follows,

$$\frac{\partial A_{01}}{\partial t} = R \times \begin{bmatrix} 2A_{10}D_{11} + 4A_{20}D_{21} + 3A_{11}D_{12} \\ + 3A_{12}D_{11} - 2B_{11}C_{10} - 4B_{21}C_{20} \\ - 3B_{11}C_{12} - 3B_{12}C_{11} \end{bmatrix} \quad (26)$$

where $R = h \frac{\pi^2}{LD}$; L and D are the longitudinal and latitudinal lengths of the domain respectively, and A_{ij}, B_{ij}, C_{ij} and D_{ij} are the coefficients of the double Fourier series just as may be seen in (18). Here we neglect the terms of wave number larger than 2. If we take into account up to the wave number 3, the following equation may be rewritten:

$$\frac{\partial A_{01}}{\partial t} = R \times \begin{bmatrix} 2(A_{10}D_{11} - B_{11}C_{10}) \\ + 3(A_{11}D_{12} + A_{12}D_{11} - B_{11}C_{12} \\ - B_{12}C_{11}) \\ + 4(A_{20}D_{21} - B_{21}C_{20}) \\ + 5(A_{12}D_{13} + A_{13}D_{12} - B_{12}C_{13} \\ - B_{13}C_{12}) \\ + 6(A_{30}D_{31} - B_{31}C_{30}) \\ + 6(A_{21}D_{22} + A_{22}D_{21} - B_{21}C_{22} \\ - B_{22}C_{21}) \end{bmatrix} \quad (27)$$

Although it is a complicated calculation to make the schema of the height change for each component of A_{ij} , B_{ij} , C_{ij} and D_{ij} , such formula will be useful in order to calculate $\frac{\partial Z}{\partial t}$ at $t=t_0$ automatically from the coefficient of Z at $t=t_0$. During the process of calculation of $\frac{\partial Z}{\partial t}$ from the value of Z , it is not necessary to use the relaxation method.

§ 3. Discussion of Results

i) Practical method.

In order to verify the above method, we made 12 hour forecasts ($\Delta t=12$ hour) at 500-mb level for 1500 GCT 9 Jan., 1953 and for 0300 GCT 11 Jan., 1953. The forecasts for these

Fig. 1. The map used for calculation. Larger and smaller rectangle in the figure show the area for the calculation and the forecasted area respectively.

Fig. 2. Initial 500-mb contours for 1500 GCT 9 Jan., 1953.

two days were already made by N.W.P. group in Tokyo [5]. The reason why we select these days is that we can verify our method comparing our results with those which were obtained by N.W.P. group by other methods

(by Fjørtoft & Bolin-Charney). In Fig. 1, the rectangle enclosed by solid line shows the area which was selected for the calculation ($6000 \text{ km} \times 4500 \text{ km}$), and the smaller rectangle in the figure shows the forecast area. Fig. 2 shows the weather map for 1500 GCT 9 Jan., which was carefully analysed by N. W.P. group in Tokyo. Analysing these data into double Fourier series of 12 components, barotropic changes of contour height at 500 mb were calculated using double Fourier series method.

ii) Neglection of the waves which have the wave number larger than 2.

At first we neglected the small scale waves which have the wave number larger than 2, and calculated the height changes which were caused by the coupling of large scale waves only. Table 1 shows the wave components which were adopted in this case. In other words, we neglected the components which

Table 1. Wave components analysed by double Fourier series from the height pattern on 500 mb, 1500GCT 9 Jan. (Unit in 100 feet). The gothic figures are the selected components used for calculation.

		A_{mn}				
$m \backslash n$		0	1	2	3	4
0		175.38	5.96	1.42	1.48	1.52
1		2.55	-0.17	-0.09	-0.03	-0.18
2		0.53	-0.91	0.17	-0.08	0.10

		B_{mn}			
$m \backslash n$		1	2	3	4
0		-7.08	-1.78	-1.67	-0.77
1		1.04	-0.07	0.18	0.28
2		-0.37	+0.38	-0.03	-0.26

		C_{mn}				
$m \backslash n$		0	1	2	3	4
1		0.65	-1.66	0.30	0.19	0.21
2		1.98	-1.71	-0.26	-0.13	-0.14

		D_{mn}			
$m \backslash n$		1	2	3	4
1		-0.67	-0.67	-0.07	-0.44
2		1.21	-0.09	0.07	0.07

have the wave length shorter than 2250 km in latitudinal direction and 3000 km in longitudinal direction. Fig. 3 shows the initial contour height of 1500 GCT 9 Jan. being filtered small scale wave components. The results of this forecast ($\Delta t=12$ hour) for 1500 GCT 9 Jan. may be rather satisfactory. The correlation coefficient is $r=0.66$, which is better than the result calculated by Bolin-Charney's method ($r=0.41$), and worse than the one by Fjörtoft's method ($r=0.76$). Fig. 4 shows the calculated and observed change. Adopting 6 hr time interval, nearly same result with Fig. 4 is

Fig. 3. Initial 500-mb contours for 1500 GCT 9 Jan., 1953, from which small scale waves are filtered.

Fig. 4. Forecast of 1500 GCT 11 Jan., 1953. Solid lines show the observed height change for 12 hour of 500-mb contours, and dashed lines the forecasted height change.

obtained. This continuous tendency during 12 hour shows the slow change in larger scale flow patterns, which have the wave number $m, n \leq 2$.

Next, the 12 hour forecast for 0300 GCT 11 Jan. was made using 6 hour time interval. However, the result was poor. The correlation between the observed and the calculated height change was 0.43¹⁾. In Fig. 5 solid and dashed lines show the observed and calculated height changes respectively. In this case, the initial height pattern and accordingly the pattern of the change of contour height is more complicated than those of 1500 GCT 9 Jan.. Therefore, it may be reasonable to conclude that we must consider the smaller scale wave in order to follow the change of actual height pattern.

Fig. 5. Forecast of 0300 GCT 11 Jan., 1953. Solid lines show the observed height change for 12 hour of 500-mb contours, and dashed lines the forecasted height change.

iii) Insertion of smaller scale waves.

Finding that it is necessary to consider the contribution of the smaller scale waves, we took into account of the wave number 3; in other words, we neglected the waves which have the wave length shorter than 1500 km in latitudinal direction and 2000 km in longitudinal direction. Table 2 shows the wave components which were used for calculation. Owing to the insertion of these waves, the

1) N.W.P. group in Tokyo forecasted the 12 hour height change for this case using Fjörtoft's method. However, their results were not good. The correlation between observed and calculated height changes was 0.13 when they used 12 hour time interval and 0.53 for 6 hour time interval.

Table 2. Wave components used for the revised forecast are shown by the gothic figures (Unit in 100 feet)

		A_{mn}				B_{mn}		
n	m	0	1	2	3	1	2	3
0	0	177.18	2.22	-1.14	-1.10	7.31	1.63	1.29
1	1	3.28	-1.63	0.94	0.17	-0.28	-0.19	-0.46
2	2	0.15	-0.44	0.18	-0.05	1.07	-0.18	0.10
3	3	-0.21	0.39	-0.25	0.02	0.03	-0.07	-0.09

		C_{mn}				D_{mn}		
n	m	0	1	2	3	1	2	3
1	1	-1.04	-2.08	-0.69	-0.11	0.32	1.37	0.76
2	2	1.28	0.32	-0.03	0.42	-1.66	-0.12	0.08
3	3	0.97	-0.65	0.19	0.06	-0.64	0	0.02

result was improved remarkably. The correlation between the observed and calculated height change for 12 hour was 0.67²⁾, which is better than the result ($r=0.53$) obtained by N.W.P. group using the graphical method. Fig. 6 shows the result of this calculation. In this calculation, the advection of small scale vorticity is neglected in order to prevent the accumulation of errors.

Fig. 6. Revised forecast of 0300 GCT 11 Jan. 1953. Solid lines show the observed height change for 12 hour of 500-mb contours, and dashed lines the forecasted height change.

The equation for the change of contour height of (15) is to be written briefly as follows

$$r^2 \frac{\partial Z}{\partial t} = -R \sum_i \sum_j N_{ij} a_i a_j J(Z_i, Z_j) + \beta \sum_n a_n \frac{\partial Z_n}{\partial x} \quad (N_{ij} \equiv \mu_i^2 - \mu_j^2) \quad (28)$$

2) Here, we adopted the 6 hour time interval.

where R is the coefficient which is defined in § 2. In the schema for the calculation of height change one example of which is shown in § 2, we can find out many terms which have large N_{ij} . These terms may amplify the small errors which are contained in a_i . Therefore, after testing various filtration, we neglected the terms which have $N_{ij} \mu_i \mu_j$ larger than 20. This corresponds to the neglect of the advection of small scale vorticity, i.e. roughly to the neglect of latter two terms in the following equation,

$$V \cdot r \zeta = \bar{V} \cdot r \bar{\zeta} + V' \cdot r \bar{\zeta} + \bar{V} \cdot r \zeta' + V \cdot r \zeta'$$

where

(29)

$$V = \bar{V} + V', \quad \zeta = \bar{\zeta} + \zeta'.$$

This filtration was very effective and improved remarkably the result of calculation. Without this neglect, computational instability would occur.

So far, we have realized that our method by double Fourier series was useful and gave us better result than the graphical method.

§ 4. Discussion about the Influence of Boundaries.

In the calculation of height change following two assumption was adopted for the sake of simplicity.

i) Periodicity of height pattern.

When we analyse the weather map into double Fourier series, we must assume the periodic repetition of height pattern, which is quite different from the actual situation outside the boundary.

ii) Free boundary condition.

Same as the graphical solution of relaxation method proposed by Fjørtoft, we adopted the free boundary condition for the sake of simplicity.

Owing to these assumption, the effect of erroneous boundary condition penetrates inward. But it is emphasized by many authors that we can avoid this influence by taking the forecast area sufficiently smaller than the region for calculation, because the effect of boundary is propagated by the group velocity which is comparable with the velocity of trough. When the forecast period is shorter

than 24 hours, we can use the present area shown in Fig. 5 for the calculation of height change, while we must, of course, extend this area for the forecast of longer range.

§ 5. Notes on the Relay-computer.

Since the relay-computer, FACOM-100, which was made by Fuji Communication Apparatus Manufacturing Company, became to be available, we made the preliminary test of our method by machine. According to our results, a 12-hour forecast by using our method has required the following time:

Expansion of Z into the double Fourier series	20 minutes
Calculation of Δa , each 1 step ..	30 minutes
Composition of the double Fourier series	20 minutes

As the reference of the characteristics of FACOM-100, the short outline will be written as follows:

Storage:—

(a) 20 relay-storages—Among them, 4 storages have a printer and another storage has a luminous indicator.

(b) Tape-storages—Each of them has a reader and a perforator.

Both (a) and (b) have the available number part of 8-decimal digits and the power part of 2-decimal digits.

Instructions:—

- (a) Three address system
- (b) Reading by 2 readers

In the coming year, however, this calculation by relay-computer will be replaced by the electronic computer machine (TAC).

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