On the Calculation of the Vertical $p$-velocity Concerning
the Wet Adiabatic Process

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Abstract

The vertical $p$-velocity in a state of saturation is derived from the equation of vertical
$p$-vorticity and the conservation of the equivalent potential temperature. The difference
of vertical $p$-velocities in the dry and wet adiabatic processes are estimated. In the last
section of this paper, the comment of the effects of condensation on the hight change are
explained.

§ 1. Introduction

Recently, the importance of calculation of
the vertical $p$-velocity is realized in regard
to the integration of vorticity equation or the
numerical forecasting of rainfall amounts and
the strictly solution of the equation for vertical
motion was treated in the paper of the
title “Numerical prediction of the develop-
ment of the cyclone over the Far East and
the associated precipitation” (Gambo et. al
1956)

Hitherto, many papers concerning the
numerical forecasting have assumed the dry
adiabatic process. Therefore, the treatments
on the vertical $p$-velocity have been also cal-
culated assuming the same process.

As for the vertical $p$-velocity assumed the
wet adiabatic process, one comment was pre-
sented in the paper above mentioned.

In this paper the author presents more de-
tailed discussion and example for the vertical
$p$-velocity on the assumption of the wet
adiabatic process.

2. Equation of vertical $p$-velocity

The field of vertical motion is not an
observable quantity but the field is deduced
from the equation of motion and the thermo-
dynamics. (Smagorinsky 1955 and Gambo et.
al 1956)

In general we used the vorticity equation
as equation of motion and the concervation of
potential temperature as one of thermody-
namics, but in the case of wet adiabatic process,
equivalent potential temperature is conserved,
so that the differential equation of vertical $p$-
velocity is deduced from the following two
equations.

$$\frac{\partial \eta}{\partial t} + v \cdot \nabla \eta + \frac{\partial \eta}{\partial p} = \eta \frac{\partial \omega}{\partial p}$$ (2.1)

$$\frac{d \ln \theta_e}{dt} = 0$$ (2.2)

where, $\eta$, $v$ and $\theta_e$ are the absolute vorticity,
the horizontal wind vector and the equivalent
potential temperature respectively.

$\theta_e$ is rewritten as

$$\ln \theta_e = \ln \theta + \frac{L_e}{C_p T_e} q^*$$ (2.3)

where $\theta$ is potential temperature and $C_p$, $L_e$, $T_e$ and $q^*$ are the specific heat of dry air at
constant pressure, the latent heat of condensa-
tion, temperature of the condensation level
and mixing ratio respectively. But in practi-
ce, we apply specific humidity as $q^*$ instead
of mixing ratio.

Then if we define vertical $p$-velocity in a
state of saturation as $\omega^*$, (2.3) becomes

$$\left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \ln \theta + \omega^* \frac{\partial \ln \theta}{\partial p} - C m^* = 0$$ (2.4)

where

$$C = \frac{L_e}{C_p T_e}$$

The quantity defined by $m^*$ is equivalent to
the amount of precipitation.

We put following relations (Gambo et. al
1956)

$$-m^* = F^* \cdot \omega^*$$ (2.5)

$$\omega = -K \left[ \frac{\partial}{\partial t} \left( \frac{\partial}{\partial p} \right) + v \cdot \nabla \left( \frac{\partial z}{\partial p} \right) \right]$$ (2.6)
where
\[ K = \frac{-g}{\alpha \ln \theta}. \]

By using these expressions, (2.1) and (2.2) are rewritten as follows
\[ \frac{\partial \gamma}{\partial t} + v \cdot \nabla \gamma + \omega^* \frac{\partial \gamma}{\partial p} = \frac{\partial \omega^*}{\partial p}. \]  
(2.7)

\[ \omega^* = -K \left[ \frac{\partial (\partial \gamma)}{\partial t} + v \cdot \nabla \left( \frac{\partial \gamma}{\partial p} \right) \right] = \frac{\omega}{1 - \tau}. \]  
(2.8)

where \( \tau \) is \( \frac{CK \alpha}{g} F^* \).

From (2.7) and (2.8) we derive the differential equation of \( \omega^* \) and follow.
\[ (1 - \tau) \frac{p^2 \omega^* + \alpha^2 \frac{\partial^2 \omega^*}{\partial p^2}}{g} = F(x, y, \theta). \]  
(2.9)

where
\[ F = \frac{FK}{g} \left( \frac{\partial (v \cdot \nabla \gamma)}{\partial \theta} \right) + \frac{g}{f} \left( \omega \left( \frac{\partial \gamma}{\partial p} \right) \right). \]

Here we assume
\[ K \frac{\partial \omega}{\partial p} > \omega \frac{\partial K}{\partial p} \] and \( v \cdot \nabla \gamma > \omega \frac{\partial \gamma}{\partial p} \).

The field of \( \omega^* \) is obtained from (2.9) under the appropriate boundary condition.

Now we divide \( \omega^* \) into two parts
\[ \omega^* = \omega + \omega' \]  
(2.10)

where \( \omega \) and \( \omega^* \) are assumed to be vertical \( p \)-velocity in a state of unsaturation and the one in a state of saturation respectively.

Put (2.10) into (2.9), following equation is derived
\[ (1 - \tau) \frac{p^2 \omega + (1 - \tau) \frac{p^2 \omega'}{\partial p^2} + \alpha^2 \frac{\partial^2 \omega}{\partial p^2} + \alpha^2 \frac{\partial^2 \omega'}{\partial p^2}}{g} = F(x, y, \theta). \]  
(2.11)

where \( \alpha^2 = f^2 K / g \).

Because \( \omega \) is vertical \( p \)-velocity in a state of unsaturation, it must satisfy following relation (Gambo et. al. 1956).
\[ \frac{p^2 \omega + \alpha^2 \frac{\partial^2 \omega}{\partial p^2}}{g} = F(x, y, \theta). \]  
(2.12)

and \( \omega \) is known from (2.12) by three dimensional relaxation method.

From (2.11) and (2.12), following equation is derived,
\[ (1 - \tau) \frac{p^2 \omega'}{g} + \alpha^2 \frac{\partial^2 \omega'}{\partial p^2} = \gamma \frac{p^2 \omega}{g}. \]  
(2.13)

where \( \gamma = 0 \) on the domain in a state of unsaturation
\[ \omega' = 0 \] on the domain in a state of saturation.

The equation (2.12) is rewritten as follow,
\[ \frac{p^2 \omega'}{g} + \alpha^2 \frac{\partial^2 \omega'}{\partial p^2} = \frac{\gamma}{1 - \tau} \frac{p^2 \omega}{g}. \]  
(2.14)

Under the boundary condition \( \omega' = \omega = 0 \) (where \( p_s \) is surface pressure), \( \omega' \) is obtained from (2.14) by means of relaxation method.

Adding \( \omega \) and \( \omega' \), the vertical \( p \)-velocity in a state of saturation \( \omega^* \) is obtained.

§ 3. Example

The author treated the numerical prediction of rainfall amount which included orographic effect and this result is published in this journal under the title "Topographic effects upon the numerical weather prediction in the lower atmosphere".

In this paper, using the data of above mentioned report, we calculated \( \omega^* \) by means of relaxation method with grid point increment of 75 km. This small size of grid point increment is used for the calculation of vertical motion caused by mountain range. The grid point is shown in Fig. 1. The initial map we used was taken at 0300 GCT, September 29, 1955. The distribution of \( \omega \) which is calculated from (2.12) under the boundary condition which takes account of kinematical condition of air flow along the topography of the

![Fig. 1. Location of grids of point used in computation of vertical motion caused by mountains.](image-url)
mesh-point are expressed by $d$ and $\Delta p$ respectively.

In this example, value of $C$, $K$ and $F^*$ are follows,

\[
(C)_{r=17^*\circ} = \left( \frac{L_0}{C_p T} \right)_{r=17^*\circ} = \frac{600}{24 \times 10^{-2} \times 290} = 8.3,
\]

\[- \left( 1 \times \frac{\partial \ln \theta}{\partial \Delta p} \right)_{r=17^*\circ} = \frac{2.5 \times 10^{-3}}{100} = 2.5 \times 10^{-3},
\]

\[
(F^*)_{p=850} = \frac{2.5 \times 10^{-3}}{100} = 2.5 \times 10^{-5}
\]

then, $r=0.95$ and

\[
K = 1 \left/ \frac{\partial \ln \theta}{\partial \Delta p} \right. = 5.2 \times 10^2 (mb^2/m)
\]

In adopting coefficient of equation (2.15) in the case of $d=75$ km and $\Delta p=150$ mb, (3.1) become,

\[
\overline{\omega_{k}}_{i,j,k} - 9\omega_{k,i,j,k} + 2.5(\overline{\omega_{k}}_{i,j}) = 19G \tag{3.2}
\]

Because $\omega_{85}$, $\omega_{70}$ and $\omega_{65}$ are known the solution of equation (3.2) are easily obtained under the following boundary condition,

\[
\begin{cases}
\omega' = 0 : P = 1000 \text{ mb} \\
\omega' = 0 : P = 400 \text{ mb}
\end{cases}
\]

The result is shown in Fig. 3. The domain in a state of saturation is assumed as black points in the Fig. 3.

Since the coefficient of $\omega'_{i,k}$ in equation (3.2) is large, the effect of condensation does not so extend in the horizontal direction as in the vertical direction.

From this calculation, $\omega^*$ is much larger
The prognostic equation of height change is

$$\rho \frac{\partial z}{\partial t} = \frac{f^2}{g} \frac{\partial \omega}{\partial p} + \frac{f^2}{g} \frac{\partial \omega^*}{\partial p} \tag{3.3}$$

In the state of saturation, this equation is rewritten as follows

$$\rho \frac{\partial z^*}{\partial t} = \frac{f^2}{g} \frac{\partial \omega^*}{\partial p} \tag{3.4}$$

The effect of condensation is estimated by difference of equation (3.3) and (3.4). That is

$$\rho \left( \frac{\partial z}{\partial t} - \frac{\partial z^*}{\partial t} \right) = \frac{f^2}{g} \frac{\partial (\omega - \omega^*)}{\partial p} = \frac{f^2}{g} \frac{\partial \Delta \omega}{\partial p} \tag{3.5}$$

The value of $\partial \Delta z/\partial t$ is obtained from the solution which is derived by relaxation method from (3.5).

$$\frac{\partial \Delta z}{\partial t} = \rho^{-1} \left( \frac{f^2}{g} \frac{\partial \Delta \omega}{\partial p} \right) \tag{3.6}$$

where

$$\rho^{-1} \cdot f^2 = 1$$

Making use of above mentioned initial condition, we estimate the value of $\partial \Delta z/\partial t$ which is approximately seemed as the deviation of isobaric surface under the wet adiabatic process from the one under the dry adiabatic change. The value of $\partial \Delta z/\partial t$ is calculated on

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Fig. 3. Fields of $\omega$ (mb/hr) on the surfaces 850 mb, 700 mb and 500 mb corresponding to the vertical velocity $\omega$ indicated in fig. 2. Black points indicate the domain in a state of saturation.

than $\omega$ but the difference of $\omega^*$ and $\omega$ is dependent on the value of $\gamma$ that is condition of saturation and the scale of domain in state of saturation.

Next, we estimate the effect of condensation on the height change of isobaric surface.

In this calculation we use $\omega^*$ instead of $\omega$. This problem was remarked from another view point by S. Manabe (1955).

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Fig. 4. Solid lines are 12 hr, height tendency of 850 mb-surface caused from baroclinic term $\rho^{-1} \left( \frac{f^2}{g} \frac{\partial \omega}{\partial p} \right)$ (feet/12 hr) under the assumption of dry adiabatic process and dashed lines show the field of $\partial \Delta z/\partial t$ that is 12 hr height tendency on the 850 mb surface corresponding to $\rho^{-1} \left( \frac{f^2}{g} \frac{\partial \Delta \omega}{\partial p} \right)$ for 0300 GCT, June 29, 1954.
the grid points and grid increment \(d\) is taken as 300 km as usually. The result of calculation and the domain in state of saturation are shown in Fig. 4.

The order of \(\partial \Delta z / \partial t\) is about half of which is calculated under the assumption of dry adiabatic process.

How does the deviation of height tendency \(\partial \Delta z / \partial t\) changes the prognostic with the increasing time? This is interest problem and has the same difficult point with the problem of effect of ageostrophic wind on the height tendency.

Ageostrophic wind is related to the vertical motion by following equation

\[
p \partial \omega* = p \partial (\omega + \Delta \omega) = -\partial (\omega + \Delta \omega) / \partial p
\]

where

\[
\frac{\partial \omega}{\partial x} = u'
\]

and

\[
\frac{\partial \omega}{\partial y} = v'
\]

The detailed discussion on this problem is set forth in future paper.

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**References**

2. ———, (1956): Topographical effects upon the numerical weather prediction in the lower atmosphere (to be published in Journ. Met. Soc. Japan)