On a New Expression for the Diurnal Variation of Air Temperature

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Abstract

A new expression $A_0 + A \sin(\sigma t - \gamma \sin \sigma t + \varepsilon)$ is suggested by the author to show the diurnal variation of air temperature. Method for finding the four constants A_0 , A , γ and ε is given and its applicability and the degree of approximation are discussed by some actual data. Physical remarks and some applications to climatology are also given in this paper. A noticeable success is the discovery of an expression with only the diurnal single period, not containing the semi-diurnal or other higher harmonic periods which are of no physical significance.

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§ 1. Introduction

It is well known that the diurnal characteristic of air temperature is a best indicator of the local climatological conditions. The daily mean temperature together with the maximum and minimum values is listed in almost all climatological tables, and the times of maximum and minimum temperatures are added in some of these tables. The description of these data gives a primary idea of the diurnal characteristic; however, the more complete expressions are desired.

Adopting the method of harmonic analysis, we can express the diurnal variation of air temperature in the form of a Fourier series and its intrinsic nature enables itself to be expressed approximately with sufficient accuracy by the first three terms of the series, namely,

$$\theta \approx a_0 + r_1 \sin(\sigma t + \delta_1) + r_2 \sin(2\sigma t + \delta_2) \quad (1)$$

where $\sigma = 2\pi/T$ and $T = 24$ hours. Thus the temperature variation can be defined by the five harmonic constants a_0 , r_1 , r_2 , δ_1 and δ_2 .

The present writer proposes a new approxi-

mate expression of the type

$$\theta = A_0 + A \sin(\sigma t - \gamma \sin \sigma t + \varepsilon) \quad (2)$$

which can be defined by the four constants A_0 , A , γ and ε . The applicability and utilization of this expression are discussed in the following sections.

§ 2. Method of Reduction

Now we illustrate the working method to derive the four constants from the hourly data of the diurnal variation of air temperature. The illustration is carried out by employing an example from N. K. Johnson's data at Leaffield [1]*.

The hourly values (θ_t) are listed in Column II of Table 1.

(i) The constants A_0 and A can be derived easily from the maximum (M) and minimum (m) values by the aid of the relations

$$\begin{aligned} A_0 + A &= M, & A_0 - A &= m \\ A_0 &= \frac{M+m}{2}, & A &= \frac{M-m}{2}. \end{aligned} \quad (3)$$

* The numbers in brackets refer to the references listed at the end of the paper.

For this example, $M=11.6^{\circ}\text{C}$, $m=6.2^{\circ}\text{C}$ and hence $A_0=8.9^{\circ}\text{C}$, $A=2.7^{\circ}\text{C}$.

(ii) Calculate the values of $\theta_i - A_0$ and $\theta_i - A_0/A$ which are listed in Cols. III and IV respectively.

(iii) Find $\sin^{-1} \frac{\theta_i - A_0}{A}$ (Col. V) which must be equal to $\sigma t - \gamma \sin \sigma t + \varepsilon$.

(iv) Referring the values of σt (Col. VI), calculate $\sigma t - \sin^{-1} \frac{\theta_i - A_0}{A}$ which must be equal to $\gamma \sin \sigma t - \varepsilon$ (Col. VII). And, rejecting the average of these values (which is equal to $-\varepsilon$), we have the values corresponding to those of $\gamma \sin \sigma t$.

(v) The final test whether the given data

can be represented approximately by the formula in question is dependent on the reducibility of the values listed in Col. VIII into the form $\gamma \sin \sigma t$. The graphical representation of these values listed in Col. VIII (Fig. 1) shows that those are approximately representable by the form $28^{\circ} \sin \sigma t$.

(vi) By the procedure mentioned above, we can determine the four constants A_0 , A , γ and ε of the approximate expression $A_0 + A \sin(\sigma t - \gamma \sin \sigma t + \varepsilon)$. For this example, $A_0=8.9^{\circ}\text{C}$, $A=2.7^{\circ}\text{C}$, $\gamma=28^{\circ}$ (about 0.5 radian) and $\varepsilon=228^{\circ}$.

The degree of approximation can be judged by Table 2 which shows the deviation of the computed values from the observed data. In

Table 1. Diurnal variation of air temperature at 1.2 m height above the ground,
Leaffield ($51^{\circ}50\frac{1}{2}'\text{N}$. $1^{\circ}34\frac{1}{2}'\text{W}$.).
(Annual mean)

(I)	(II)	(III)	(IV)	(V)	(VI)	(VII)	(VIII)
Hour i G.M.T.	θ_i	$\theta_i - A_0$	$\frac{\theta_i - A_0}{A}$	$\sin^{-1} \frac{\theta_i - A_0}{A}$	σt	$\sigma t - \sin^{-1} \frac{\theta_i - A_0}{A}$	$\gamma \sin \sigma t$
0	6.9°C	-2.0°C	-0.742	$-132^{\circ} 5'$	0°	$132^{\circ} 5'$	$-0^{\circ} 5'$
1	6.7	-2.2	-0.816	$-125^{\circ} 20'$	15	$140^{\circ} 20'$	$+8^{\circ} 10'$
2	6.5	-2.4	-0.890	$-117^{\circ} 10'$	30	$147^{\circ} 10'$	$15^{\circ} 0'$
3	6.4	-2.5	-0.926	$-112^{\circ} 10'$	45	$157^{\circ} 10'$	$25^{\circ} 0'$
4	6.2	-2.7	-1.000	$-90^{\circ} 0'$	60	$150^{\circ} 0'$	$17^{\circ} 50'$
5	6.2	-2.7	-1.000	$-90^{\circ} 0'$	75	$165^{\circ} 0'$	$32^{\circ} 50'$
6	6.5	-2.4	-0.890	$-62^{\circ} 50'$	90	$152^{\circ} 50'$	$20^{\circ} 40'$
7	7.1	-1.8	-0.667	$-41^{\circ} 50'$	105	$146^{\circ} 50'$	$14^{\circ} 40'$
8	7.8	-1.1	-0.408	$-24^{\circ} 5'$	120	$144^{\circ} 5'$	$11^{\circ} 55'$
9	8.6	-0.3	-0.111	$-6^{\circ} 20'$	135	$141^{\circ} 20'$	$9^{\circ} 10'$
10	9.5	+0.6	+0.222	$+12^{\circ} 50'$	150	$137^{\circ} 10'$	$5^{\circ} 0'$
11	10.3	1.4	0.519	$31^{\circ} 15'$	165	$133^{\circ} 45'$	$+1^{\circ} 35'$
12	10.9	2.0	0.742	$47^{\circ} 55'$	180	$132^{\circ} 5'$	$-0^{\circ} 5'$
13	11.3	2.4	0.890	$62^{\circ} 50'$	195	$132^{\circ} 10'$	$0^{\circ} 0'$
14	11.6	2.7	1.000	$90^{\circ} 0'$	210	$120^{\circ} 0'$	$-12^{\circ} 10'$
15	11.5	2.6	0.963	$105^{\circ} 40'$	225	$119^{\circ} 20'$	$-12^{\circ} 50'$
16	11.2	2.3	0.852	$121^{\circ} 35'$	240	$118^{\circ} 25'$	$-13^{\circ} 45'$
17	10.6	1.7	0.630	$140^{\circ} 55'$	255	$114^{\circ} 5'$	$-18^{\circ} 5'$
18	9.9	1.0	0.371	$158^{\circ} 15'$	270	$111^{\circ} 45'$	$-20^{\circ} 25'$
19	9.1	+0.2	+0.074	$175^{\circ} 45'$	285	$109^{\circ} 15'$	$-22^{\circ} 55'$
20	8.4	-0.5	-0.186	$190^{\circ} 45'$	300	$109^{\circ} 15'$	$-22^{\circ} 55'$
21	7.9	-1.0	-0.371	$201^{\circ} 45'$	315	$113^{\circ} 15'$	$-18^{\circ} 55'$
22	7.5	-1.4	-0.519	$211^{\circ} 15'$	330	$118^{\circ} 45'$	$-13^{\circ} 25'$
23	7.2	-1.7	-0.630	$+219^{\circ} 5'$	345	$125^{\circ} 55'$	$-6^{\circ} 15'$

$A_0=8.9^{\circ}\text{C}$, $A=2.7^{\circ}\text{C}$

$\varepsilon=-132^{\circ}10'$, $\gamma=27^{\circ}55'$
 $+227^{\circ}50'$, $=0.49$ rad.

Table 2. Comparison of the computed values with the observed data.

$$\begin{aligned}\theta_{1.2} &= 8.9 + 2.7 \sin(\sigma t - 28^\circ \sin \sigma t + 228^\circ) \\ \theta_{30.5} &= 9.0 + 2.1 \sin(\sigma t - 29^\circ \sin \sigma t + 215^\circ) \\ \theta_{87.7} &= 8.9 + 1.8 \sin(\sigma t - 29^\circ \sin \sigma t + 213^\circ) \\ &\quad (t=0 \text{ at midnight})\end{aligned}$$

(1) Computed value

(2) Observed value

(2) Deviation

Hour	1.2 m			30.5 m			87.7 m		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
0	6.9°C	6.9°C	0.0°C	7.8°C	7.7°C	0.1°C	7.9°C	7.8°C	0.1°C
1	6.7	6.7	0.0	7.6	7.5	0.1	7.7	7.6	0.1
2	6.5	6.5	0.0	7.4	7.3	0.1	7.6	7.4	0.2
3	6.3	6.4	-0.1	7.2	7.1	0.1	7.4	7.3	0.1
4	6.2	6.2	0.0	7.0	7.0	0.0	7.2	7.1	0.1
5	6.2	6.2	0.0	6.9	6.9	0.0	7.1	7.1	0.0
6	6.4	6.5	-0.1	6.9	6.9	0.0	7.1	7.1	-0.0
7	6.7	7.1	-0.4	7.1	7.2	-0.1	7.2	7.2	0.0
8	7.3	7.8	-0.5	7.4	7.6	-0.2	7.5	7.5	0.0
9	8.1	8.6	-0.5	7.9	8.2	-0.3	7.9	8.0	-0.1
10	9.1	9.5	-0.4	8.7	8.9	-0.2	8.5	8.5	0.0
11	10.1	10.3	-0.2	9.5	9.6	-0.1	9.2	9.1	0.1
12	10.9	10.9	0.0	10.2	10.2	0.0	9.9	9.6	0.3
13	11.4	11.3	0.1	10.8	10.6	0.2	10.4	10.1	0.3
14	11.6	11.6	0.0	11.1	11.0	0.1	10.7	10.4	0.3
15	11.4	11.5	-0.1	11.1	11.1	0.0	10.7	10.6	0.1
16	10.9	11.2	-0.3	10.8	11.0	-0.2	10.5	10.5	0.0
17	10.3	10.6	-0.3	10.4	10.6	-0.2	10.2	10.2	0.0
18	9.6	9.9	-0.3	9.9	10.2	-0.3	9.7	9.8	-0.1
19	8.9	9.1	-0.2	9.4	9.6	-0.2	9.3	9.4	-0.1
20	8.3	8.4	-0.1	9.0	9.1	-0.1	9.0	8.9	0.1
21	7.9	7.9	0.0	8.6	8.6	0.0	8.6	8.6	0.0
22	7.5	7.5	0.0	8.3	8.2	0.1	8.4	8.2	0.2
23	7.2	7.2	0.0	8.0	7.9	0.1	8.1	8.0	0.1

this example, the deviation appears within $\pm 0.5^\circ\text{C}$.

Some more examples taken from Johnson's observation at Leafield are treated by the same process, and we get

$$\theta = 9.0 + 2.1 \sin(\sigma t - 29^\circ \sin \sigma t + 215^\circ)$$

for 30.5 m height

$$\theta = 8.9 + 1.8 \sin(\sigma t - 29^\circ \sin \sigma t + 213^\circ)$$

for 87.7 m height.

The test (v) mentioned above and the comparison with the observed data are shown in Fig. 1 and Table 2 respectively.

The temperature variations at different le-

vels for every season are also treated, and the results are summarized in Table 3. The graphs of the test (v) are appended in Fig. 2.

§3. Physical and Meteorological Remarks

(a) It is clear that the ideal expression which represents the diurnal variation of air temperature is a periodic function with a single period of a day, that is, $\theta = f(t) = f(t+T)$ where $T=24$ hours. However, it is very difficult to find the functional form which represents rigorously the observed data, or to derive it theoretically.

There is no reason that the periodic part

Table 3. Four constants for the temperature variations at Leaffield.

$$\theta=A_0+A \sin (\sigma t-\gamma \sin \sigma t+\varepsilon) \quad (t=0 \text { at midnight})$$

	Height	A_0	A	γ (radian)	ε
Year	1.2 m	8.9°C	2.7°C	28° (0.49)	228°
	30.5	9.0	2.1	29 (0.51)	215
	87.7	8.9	1.8	29 (0.51)	213
Spring	1.2	7.9	3.5	26 (0.45)	225
	30.5	7.9	2.8	23 (0.40)	212
	87.7	7.7	2.3	26 (0.45)	210
Summer	1.2	14.4	3.7	16 (0.28)	229
	30.5	14.4	3.0	18 (0.31)	216
	87.7	14.2	2.6	25 (0.44)	212
Autumn	1.2	9.8	2.5	37 (0.65)	230
	30.5	10.0	1.9	44 (0.77)	218
	87.7	9.8	1.5	43 (0.75)	214
Winter	1.2	3.6	1.4	58 (1.01)	218
	30.5	3.7	1.0	51 (0.89)	207
	87.7	3.7	0.8	51 (0.89)	207

Fig. 1.

can be represented by a simplest form of periodic function, *i.e.*, $r \sin (\sigma t+\delta)$. In fact, the temperature variation is mainly governed by the receiving solar radiation during the daytime and the emission of terrestrial radiation throughout day and night at the ground surface. And, if we consider a model suggested by D. Brunt [2] that

$$\begin{aligned} Ra(t) &= -I \sin \sigma t - R_N & \text{for } -\pi < \sigma t < 0 \\ &= -R_N & \text{for } 0 < \sigma t < \pi \end{aligned}$$

$(t=0 \text { at sunset})$

and

$$\int_{-\pi / \sigma}^{+\pi / \sigma} Ra(t) d t=0,$$

which is expressed by the form of a Fourier series

$$\begin{aligned} Ra(t) &= -\frac{1}{2} I \sin \sigma t \\ &\quad -\frac{2}{\pi} I\left\{\frac{1}{3} \cos 2 \sigma t+\frac{1}{15} \cos 4 \sigma t+\cdots\right. \\ &\quad \left.+\frac{1}{n^2-1} \cos n \sigma t+\cdots\right\} \end{aligned}$$

$(n, \text { even integer}), \quad (4)$

then we see the appearance of the higher harmonic terms on the controlling agency of the diurnal temperature variation. However,

Fig. 2.

it must be borne in mind that appearance of the semi-diurnal and other harmonic shorter periods is merely mathematical and has no physical reality.

Moreover, if we assume that the heat transfer is governed by the eddy diffusion equation $\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial z^2}$ ($\kappa = \text{const.}$) in the air layer and by the conduction equation $\frac{\partial \theta_B}{\partial t} = \kappa_B \frac{\partial^2 \theta_B}{\partial z^2}$ ($\kappa_B = \text{const.}$) through the ground, we have a solution for the air layer

$$\begin{aligned} \theta = \bar{\theta} + & \frac{2I}{\pi(\sqrt{\kappa \rho c} + \sqrt{\kappa_B \rho_B c_B})\sqrt{\sigma}} \\ & \times \left\{ \frac{\pi}{4} e^{-\sqrt{\frac{\sigma}{2\kappa}}z} \sin\left(\sigma t - \sqrt{\frac{\sigma}{2\kappa}}z + \frac{5}{4}\pi\right) \right. \\ & + \frac{1}{3\sqrt{2}} e^{-\sqrt{\frac{\sigma}{\kappa}}z} \sin\left(2\sigma t - \sqrt{\frac{\sigma}{\kappa}}z + \frac{1}{4}\pi\right) \\ & + \frac{1}{30} e^{-\sqrt{\frac{2\sigma}{\kappa}}z} \sin\left(4\sigma t - \sqrt{\frac{2\sigma}{\kappa}}z + \frac{5}{4}\pi\right) \\ & + \dots \\ & + \frac{(-1)^{\frac{n}{2}+1}}{(n^2-1)\sqrt{n}} e^{-\sqrt{\frac{n\sigma}{2\kappa}}z} \sin\left(n\sigma t - \sqrt{\frac{n\sigma}{2\kappa}}z + \frac{1}{4}\pi\right) \\ & + \dots \left. \right\} \quad (5) \end{aligned}$$

($t=0$ at mid night; n , even integer)

which satisfies the conditions

$$\theta|_{z=0} = \theta_B|_{z=0}, \quad \theta|_{z \rightarrow \infty} = \bar{\theta}, \quad \theta_B|_{z \rightarrow -\infty} = \bar{\theta}_B$$

and

$$Ra(t) = -\kappa \rho c \frac{\partial \theta}{\partial z} \Big|_{z=0} + \kappa_B \rho_B c_B \frac{\partial \theta_B}{\partial z} \Big|_{z=0}. \quad (6)$$

Then, the amplitude ratios of the semi-diurnal and higher harmonic terms to the diurnal one are, at the ground surface,

$$\begin{aligned} \frac{r_2}{r_1} &= \frac{2\sqrt{2}}{3\pi} = 0.30, & \frac{r_3}{r_1} &= 0, & \frac{r_4}{r_1} &= \frac{2}{15\pi} = 0.04, \\ & \dots, & \frac{r_n}{r_1} &= \frac{4}{(n^2-1)\sqrt{n\pi}}. \end{aligned}$$

This result shows that the diurnal temperature variation can be approximately expressed by the diurnal and semi-diurnal terms, neglecting the higher harmonic terms.

Referring the result of the discussion by a model, we can generally speak that the observed temperature variation may be approxi-

mately expressed by the diurnal and semi-diurnal terms only. In fact, the results of actual analysis of the diurnal variations verify this conclusion.

(b) The assumption of the equation $\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial z^2}$ which governs the heat transfer in the air layer must be modified from various points of view. We may suppose that the existing diurnal variation of the eddy diffusivity (κ) may have much influence upon the temperature variation itself. In fact, κ is larger during the daytime than at night. If we take

$$\kappa = \kappa_0(1 - \gamma \cos \sigma t) \quad (t=0 \text{ at midnight}) \quad (7)$$

as a simple model which contains the idea mentioned above, we have a periodic solution of the eddy diffusion equation in the following form

$$\begin{aligned} \theta = \bar{\theta} + \sum_{n=1}^{\infty} R_n e^{-\sqrt{\frac{n\sigma}{2\kappa_0}}z} \sin \left\{ n(\sigma t - \gamma \sin \sigma t) \right. \\ \left. - \sqrt{\frac{n\sigma}{2\kappa_0}}z + \epsilon_n \right\}. \quad (8) \end{aligned}$$

The functional form of author's new expression is suggested by the above expression.

Any expression given in the form of the ordinary Fourier series can be rearranged into the new series by the aid of the following relations. If we express in our series

$$\begin{aligned} \theta &= A_0 + R_1 \sin(x + \epsilon_1) + R_2 \sin(2x + \epsilon_2) + \dots \\ &= A_0 + A_1 \cos x + A_2 \cos 2x + \dots \\ &\quad + B_1 \sin x + B_2 \sin 2x + \dots \quad (9) \end{aligned}$$

where

$$x = \sigma t - \gamma \sin \sigma t,$$

and in the ordinary Fourier series

$$\begin{aligned} \theta &= a_0 + r_1 \sin(\sigma t + \delta_1) + r_2 \sin(2\sigma t + \delta_2) \\ &\quad + r_3 \sin(3\sigma t + \delta_3) + \dots \\ &= a_0 + a_1 \cos \sigma t + a_2 \cos 2\sigma t + a_3 \cos 3\sigma t + \dots \\ &\quad + b_1 \sin \sigma t + b_2 \sin 2\sigma t + b_3 \sin 3\sigma t + \dots, \quad (10) \end{aligned}$$

then we have the relations

$$\begin{aligned} A_0 &= a_0 - \frac{\gamma}{2} a_1 \\ A_1 &= a_1 \{J_0(\gamma) - J_2(\gamma)\} + 2a_2 \{J_{-1}(\gamma) - J_3(\gamma)\} \\ &\quad + 3a_3 \{J_{-2}(\gamma) - J_4(\gamma)\} \dots \end{aligned}$$

$$\begin{aligned}
B_1 &= b_1\{J_0(r) + J_2(r)\} + 2b_2\{J_{-1}(r) + J_3(r)\} \\
&\quad + 3b_3\{J_{-2}(r) + J_4(r)\} + \dots \\
A_2 &= \frac{1}{2} [a_1\{J_1(2r) - J_3(2r)\} + 2a_2\{J_0(2r) - J_4(2r)\} \\
&\quad + 3a_3\{J_{-1}(2r) - J_5(2r)\} + \dots] \\
B_2 &= \frac{1}{2} [b_1\{J_1(2r) + J_3(2r)\} + 2b_2\{J_0(2r) + J_4(2r)\} \\
&\quad + 3b_3\{J_{-1}(2r) + J_5(2r)\} + \dots] \quad (11)
\end{aligned}$$

by the aid of the mathematical relations

$$\begin{aligned}
\cos m\sigma t &= p_0 + p_1 \cos x + p_2 \cos 2x + \dots \\
&\quad + p_n \cos nx + \dots \\
\sin m\sigma t &= q_1 \sin x + q_2 \sin 2x + \dots \\
&\quad + q_n \sin nx + \dots \quad (12)
\end{aligned}$$

where

$$\begin{aligned}
p_0 &= \begin{cases} -\frac{\gamma}{2} & m=1 \\ 0 & m \neq 1 \end{cases} \\
p_n &= \frac{m}{n} \{J_{n-m}(nr) - J_{n+m}(nr)\} \\
q_n &= \frac{m}{n} \{J_{n-m}(nr) + J_{n+m}(nr)\}
\end{aligned}$$

and $J_N(x)$ indicates the Bessel function of N th order with argument x . Accordingly, if we can specify the value of γ , the variations ex-

pressed by the ordinary Fourier series (10) can be rearranged into our new series (9).

Johnson's observations are treated along the above line, and the results by giving $\gamma=0.5$ (radian) are shown in Table 4. This table shows that the ratio R_2/R_1 is reduced to less than 0.1 without exception. Accordingly, we can suppose that almost all of the variations in question may be expressed approximately by the diurnal term only, that is, $R_1 \sin(\sigma t - \gamma \sin \sigma t + \varepsilon_1)$. On the other hand, the amplitude of the semi-diurnal term of the ordinary Fourier series (r_2) is rather great and the ratio r_2/r_1 is nearly 0.2 and hence can not be neglected even in the sense of approximation. The accuracy of approximation of the expression $a_0 + r_1 \sin(\sigma t + \delta_1) + r_2 \sin(2\sigma t + \delta_2)$ may be better to a certain extent than that of the expression $A_0 + R_1 \sin(\sigma t - \gamma \sin \sigma t + \varepsilon_1)$. But, it must be remembered that the former expression involves five constants, while the latter is defined by four constants and yet does not contain the semi-diurnal term which has no physical significance. Therefore, the latter expression seems to be better than the former if the accuracy of the latter is the same order or even when it is dropping in small degree.

Table 4. Diurnal variation of air temperature at heights 1.2 m and 30.5 m, (annual mean), Leafield.

(1) 1.2 m height.

Harmonic constants.

$a_0 = 8.57^\circ\text{C}$		
$a_1 = -2.04$ $b_1 = -1.62$	$r_1 = 2.60^\circ\text{C}$ $\delta_1 = 231^\circ 35'$	$\frac{r_2}{r_1} = 0.19$
$a_2 = 0.35$ $b_2 = 0.34$	$r_2 = 0.49^\circ\text{C}$ $\delta_2 = 46^\circ 10'$	
$a_3 = 0.07$ $b_3 = 0.04$	$r_3 = 0.07^\circ\text{C}$ $\delta_3 = 61^\circ 45'$	$\frac{r_3}{r_1} = 0.03$

Constants of the series (9).

$A_0 = 9.08^\circ\text{C}$		
$A_1 = -2.01$ $B_1 = -1.73$	$R_1 = 2.66^\circ\text{C}$ $\varepsilon_1 = 229^\circ 15'$	$\frac{R_2}{R_1} = 0.09$
$A_2 = -0.20$ $B_2 = -0.13$	$R_2 = 0.24^\circ\text{C}$ $\varepsilon_2 = 237^\circ 5'$	

(2) 30.5 m height.

Harmonic constants.

$a_0 = 8.74^\circ\text{C}$		
$a_1 = -1.26$ $b_1 = -1.58$	$r_1 = 2.02^\circ\text{C}$ $\delta_1 = 218^\circ 25'$	$\frac{r_2}{r_1} = 0.19$
$a_2 = 0.18$ $b_2 = 0.35$	$r_2 = 0.39^\circ\text{C}$ $\delta_2 = 27^\circ 50'$	
$a_3 = 0.02$ $b_3 = 0.03$	$r_3 = 0.03^\circ\text{C}$ $\delta_3 = 36^\circ 55'$	$\frac{r_3}{r_1} = 0.01$

Constants of the series (9).

$A_0 = 9.06^\circ\text{C}$		
$A_1 = -1.24$ $B_1 = -1.70$	$R_1 = 2.10^\circ\text{C}$ $\varepsilon_1 = 216^\circ 10'$	$\frac{R_2}{R_1} = 0.09$
$A_2 = -0.15$ $B_2 = -0.12$	$R_2 = 0.19^\circ\text{C}$ $\varepsilon_2 = 231^\circ 20'$	

A theoretical discussion to illustrate the smallness of the ratio R_2/R_1 can be done along the same line as mentioned previously. But, it is very troublesome and the rigorous solution can not be easily found. Omitting to write this discussion, the following note is added to supply our purpose.

Assuming that the theoretical expression obtained by putting $z=0$ in the solution (5), namely,

$$\begin{aligned} \theta = \bar{\theta} + I' \left\{ \frac{\pi}{4} \sin \left(\sigma t + \frac{5}{4} \pi \right) \right. \\ \left. + \frac{1}{3\sqrt{2}} \sin \left(2\sigma t + \frac{1}{4} \pi \right) \right. \\ \left. + \frac{1}{30} \sin \left(4\sigma t + \frac{5}{4} \pi \right) + \dots \right\} \quad (13) \end{aligned}$$

represents roughly the temperature variation in question, and transforming into our series, we have, putting $\gamma=0.5$,

$$\begin{aligned} \theta = \bar{\theta} + 0.439I' + 0.850I' \sin(\sigma t - 28^\circ \sin \sigma t + 223^\circ) \\ + 0.007I' \sin(2\sigma t - 56^\circ \sin \sigma t + 129^\circ) \\ + \dots \quad (14) \end{aligned}$$

Then, $R_2/R_1=0.008$, which is the negligible order compared with unity. Consequently, we may suppose that almost all of the variations in question can be represented approximately by the new expression $A_0 + A \sin(\sigma t - \gamma \sin \sigma t + \varepsilon)$ with sufficient accuracy.

§ 4. Application to Climatology

The new expression which the writer proposed can be utilized for the shorter description to define the diurnal temperature variations at the localities.

When the hourly data or bi-hourly data of these variations are given, the four constants which define these variations can be found by the method written in § 2. These constants are listed in Tables 5 and 6, which give the values for 11 selected Japanese stations including a Formosan one and for the world-wide 15 selected observatories and other special regions. The author wants such list to be added in the climatological table.

When we know the maximum and minimum values (M and m) and their occurring times (t_M and t_m), we can also find the four constants, by the aid of the relations

$$A_0 = \frac{M+m}{2}, \quad A = \frac{M-m}{2}$$

and by solving the simultaneous equations

$$\left. \begin{aligned} \sigma t_M - \gamma \sin \sigma t_M + \varepsilon &= 90^\circ \\ \sigma t_m - \gamma \sin \sigma t_m + \varepsilon &= -90^\circ \end{aligned} \right\} \quad (15)$$

Inspecting the listed values of γ in Tables 3, 5 and 6, we see:

(i) γ appears to be roughly independent of height, but varies from about 20° in summer to about 55° in winter at Leafield (Table 3).

(ii) For Japanese stations (Table 5), γ lies between 31° and 46° for the year, 45° and 65° for January and 25° and 41° for July. A marked seasonal variation, larger in winter and smaller in summer, can be found at every station.

(iii) Similar remarkable seasonal variation can be found at almost all observatories in the world (Table 6), despite of being less remarkable in the equatorial region (for example, Batavia).

(iv) It appears that γ decreases generally with latitude, ranging from 21° for high latitude to 40° for low latitude for the annual mean, but the very small value of γ is seen in the polar regions.

(v) γ is larger at high lever stations than at low levels without exception in the present three examples.

If we consider physically, γ means the relative amplitude of the diurnal variation of the eddy diffusivity, that is, $\kappa = \kappa_0(1 - \gamma \cos \sigma t)$. Accordingly, the value of γ greater than 1 radian (about 57°) gives a negative of κ which has no physical reality. In fact, there is no reason that the actual change of κ undergoes such simple model as above. And, remembering that the observed temperature variation is empirically representable in the form $A_0 + A \sin(\sigma t - \gamma \sin \sigma t + \varepsilon)$, we may suppose that γ , even when it exceeds 1 radian, plays the role of an index which shows the diurnal variation of eddy diffusivity or an index of the diurnal changeability of atmospheric stirring at the locality. Therefore, adding the values of γ to define the diurnal variation will bring some advantages for the specification of the climatic state from the point of view mentioned above.

Table 5. Four constants for the temperature variations at 11 selected Japanese stations.

Station		A_0	A	γ (radian)	ε
Nemuro	Jan.	- 4.8°C	1.5°C	65° (1.13)	229°
43°20'N	July	14.2	2.0	41 (0.72)	256
145°35'E	Year	5.8	1.9	46 (0.80)	250
Sapporo	Jan.	- 5.7	2.1	59 (1.03)	235
43° 4'N	July	19.0	3.2	31 (0.54)	244
141°20'E	Year	7.3	3.1	38 (0.66)	243
Niigata	Jan.	1.6	1.0	63 (1.10)	223
37°55'N	July	25.0	3.2	35 (0.61)	239
139° 3'E	Year	13.1	2.4	38 (0.66)	236
Tokyo	Jan.	3.5	3.8	49 (0.86)	220
35°41'N	July	24.7	3.1	31 (0.54)	231
139°46'E	Year	14.3	3.4	34 (0.59)	227
Nagoya	Jan.	3.9	3.8	49 (0.86)	218
35°10'N	July	26.5	4.0	29 (0.51)	226
136°58'E	Year	15.0	4.0	31 (0.54)	226
Kyoto*	Jan.	3.6	4.1	50 (0.87)	224
35° 1'N	July	25.7	4.4	28 (0.49)	231
135°44'E	Year	14.5	4.5	33 (0.58)	231
Kobe*	Jan.	5.4	2.7	54 (0.94)	224
34°41'N	July	26.8	3.2	33 (0.58)	229
135°11'E	Year	15.6	3.0	42 (0.73)	227
Hirosima	Jan.	4.7	3.5	45 (0.79)	217
34°22'N	July	25.8	3.2	25 (0.44)	222
132°26'E	Year	15.1	3.7	34 (0.59)	221
Nagasaki	Jan.	5.9	1.8	57 (0.99)	224
32°44'N	July	25.7	2.5	33 (0.58)	230
129°53'E	Year	15.8	2.3	43 (0.75)	229
Naha	Jan.	16.1	1.9	53 (0.93)	232
26°12'N	July	28.4	2.6	39 (0.68)	233
127°39'E	Year	22.5	2.2	43 (0.75)	234
Taihoku**	Jan.	15.3	2.4	57 (0.99)	233
25° 2'N	July	28.6	3.9	37 (0.65)	237
121°31'E	Year	22.2	3.0	39 (0.68)	238

** Station in Formosa.

Data were taken from Reference [3], except two marked with asterisk *.

It must be noticed that A_0 in the new expression is not the daily mean temperature (a_0), but the average of the maximum and minimum temperatures ($M+m/2$). From the relations (11), we have

$$\begin{aligned}
 A_0 &= a_0 - \frac{\gamma}{2} a_1 \\
 &= a_0 - \frac{\gamma}{2} r_1 \cdot \sin \delta_1
 \end{aligned} \quad (16)$$

where r_1 and δ_1 are the amplitude and phase

Table 6. Four constants for the temperature variations at a few observatories in the world.

Observatory		A_0	A	γ (radian)	ε
in the Arctic	Year	-19.2°C	0.3°C	7° (0.12)	239°
Aberdeen 57°10'N 2° 6'W	Jan. July Year	3.7 13.2 7.9	0.8 2.0 1.5	51 (0.89) 25 (0.44) 25 (0.44)	221 234 231
Fort William 56°48'N 5° 5'W	Jan. July Year	3.9 14.0 8.5	0.5 2.4 1.7	66 (1.15) 3 (0.05) 21 (0.37)	213 220 220
Ben Nevis 56°48'N 5° 0'W (1343 m)	Jan. July Year	- 4.7 5.4 - 0.1	0.2 1.1 0.7	85 (1.48) 14 (0.24) 31 (0.54)	229 221 226
Parc St Maur (Paris) 48°49'N 2°29'E	Jan. July Year	2.7 18.2 10.4	1.6 4.6 3.4	57 (0.99) 16 (0.28) 22 (0.38)	215 228 227
Clermont Ferrand 45°46'N 3° 5'E (388 m)	Jan. July Year	2.2 18.4 10.4	2.7 5.6 4.4	58 (1.01) 18 (0.31) 23 (0.40)	223 231 231
Puy de Dome 45°46'N 2°58'E (1467 m)	Jan. July Year	- 2.6 11.2 3.8	0.4 1.6 1.0	75 (1.31) 23 (0.40) 35 (0.61)	208 213 216
Tiflis* 41°43'N 44°48'E	Dec.-Feb. June-Aug.	3.9 22.2	3.4 4.7	46 (0.80) 20 (0.35)	213 220
Lahore 31°34'N 74°20'E (214 m)	Jan. July Year	13.4 32.3 24.5	6.2 3.8 6.1	41 (0.72) 31 (0.54) 35 (0.61)	221 221 222
Leh 34°10'N 77°42'E (3506 m)	Jan. July Year	-7.0 17.4 5.5	4.9 6.0 5.8	42 (0.73) 35 (0.61) 38 (0.66)	223 222 224
Calcutta 22°32'N 88°20'E	Jan. July Year	19.1 28.5 25.8	4.7 1.6 3.2	46 (0.80) 41 (0.72) 40 (0.70)	216 235 224
San José* de Costarica 10° 0'N 84°20'W	Jan., Feb. Sept., Oct.	19.8 20.7	4.6 3.7	47 (0.82) 60 (1.05)	233 248
Batavia 6°11'S 106°50'E	Jan. July Year	25.7 26.1 26.3	2.3 3.4 3.0	36 (0.63) 37 (0.65) 38 (0.66)	225 221 226
Cape of Good Hope 33°56'S 18°29'E	Jan. July Year	20.1 12.7 16.3	2.9 2.5 2.7	37 (0.65) 50 (0.87) 36 (0.63)	238 220 232
Melbourne* 37°50'S 144°59'E	Year	15.2	4.0	30 (0.52)	245
Hut Point Cape Evans 77°30'S 167° E	Year	-18.0	0.5	9 (0.16)	230
Atlantic* Ocean 12½°N-4°S		26.4	0.5	27 (0.47)	232

Data were taken from Reference [4], with exception of some marked with asterisk * and taken from References [5] and [6].

angle of the diurnal harmonic term of the temperature variation. If we allow the rough approximation that $r_1 \approx A$ where A is the periodic daily amplitude $(M-m/2)$ and that $\delta_1 \approx 225^\circ$ for all of these variations, we have a practical rough formula

$$A_0 \approx a_0 + sA \quad (17)$$

where s is a constant which varies with r . This practical formula is tested by the data of selected Japanese stations and we have the following values for a constant s .

Season	Mean	Range
Year	0.14	0.08-0.21
Jan.	0.18	0.13-0.28
July	0.07	0.00-0.20

Moreover, if we allow a further approximation that $A \approx A'$ where A' is the non-periodic daily amplitude derived from the daily observation of the maximum and minimum thermometers, then we have a practical formula

$$A_0 \approx a_0 + sA' \quad (18)$$

However, $A' \geq A$, therefore a smaller value of s may be expected. This relation shows that the difference between the daily mean temperature (a_0) and the average of the maximum and minimum temperatures (A_0) is roughly proportional to the daily temperature range.

§ 5. Summary

(i) The diurnal variation of air tempera-

ture can be expressed by the new approximate form $A_0 + A \sin(\sigma t - \gamma \sin \sigma t + \epsilon)$ with sufficient accuracy.

(ii) A constant r in the new expression seems to play the rôle of an index of the diurnal changeability of atmospheric stirring at the locality.

(iii) Listing the four constants A_0 , A , r and ϵ advantages for the specification of the local climatological conditions.

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