On the Dynamics of Long and Ultra-Long Waves in a Baroclinic Zonal Current*

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Abstract

A theoretical study of the dynamics of long and ultra-long waves in a baroclinic zonal current is made by the use of quasi-geostrophic equations. The stability of the zonal current and the characteristics of wave perturbations are investigated numerically with the finite difference approximation of the linearized perturbation equations applied to a multi-layer model. It is found that the wave solutions are classified into three types: (i) a pair of amplifying and decaying waves, (ii) neutral waves without steering levels in the basic current and (iii) neutral waves with a steering level. They are divided into various regimes in accordance with their vertical structure. The examination of the vertical structure of wave disturbances, as well as of the phase velocity, is made in terms of the wavelength and the vertical shear of the zonal current. Some remarks on the validity of the finite difference approximation by a simple 2-layer model are added in relation to the results obtained by the multi-layer model.

The energy conversion processes are discussed by making a comparison between the unstable waves of the first mode and the higher mode. It is found that the maintenance of the unstable wave of the higher mode is only due to the increase of the eddy available potential energy caused by northward heat transport. Finally a comparison between the characteristics of ultra-long waves obtained by the present study and those of the observation is made.

1. Introduction

Since the theoretical study of Charney (1947) and Eady (1949), the stability of a baroclinic zonal current for large-scale wave perturbations has been extensively investigated by many authors, notably, Fjørtoft (1950), Gambo (1950) and Kuo (1952). Charney has shown the existence of large-scale unstable waves in a zonal current with vertical shear and he has further shown the structure of long waves. One of the most important conclusions reached by Charney is that the unstable regime is bounded by a long wave limit due to the \( \beta \)-effect. On the contrary, Eady has presented the short wave limit to instability by neglecting the effect of \( \beta \)-term, and obtained the most preferred scale of the unstable wave. The kinematic characteristics of long waves (or cyclone waves) observed in the westerly current in the atmosphere have been explained in the light of these theoretical studies.

Later, Green (1960) performed a numerical calculation to extend the class of stability problem as posed by Charney and Eady. He examined stability properties of a baroclinic zonal flow bounded above and below by two rigid boundaries, and concluded that an unstable wave exists for nearly all wavelengths. Namely, there is no short wave limit to instability and the longer wave beyond Charney's criterion becomes unstable too. However, this result does not necessarily vitiate the conclusions of previous studies with respect to the class of unstable disturbances presented in their original models.

The results of numerical calculation shown by Green have been established by a more complete examination by the use of analytical methods for a continuous model by Burger (1962) and Miles (1964). Their attention was strongly focused on the singular perturbation problem and their conclusion was that disturbances of all wavelengths, except for a finite...
number of isolated wavelengths, are always unstable. In these theories, however, no physical interpretations of unstable waves such as the wave structure and energy conversion process were offered.

Thus the main purpose of the present study is to discuss the nature of unstable waves, extending the work of Green. Especially an examination is made to discuss unstable waves of longer wavelengths in terms of the phase velocity, the vertical structure and the energy conversion process associated with the waves. Discussions are extended to make some remarks on the validity of the finite difference approximation by a simple two-layer model, in relation to the results obtained by the multi-layer model. A comparison between the characteristics of unstable waves theoretically obtained and those of ultra-long waves in the atmosphere is also made.

2. Model and assumptions

We assume that the motion is quasi-geostrophic with small values of the Rossby number and large values of the Richardson number. It is also assumed that the motion is frictionless and adiabatic. Then appropriate governing equations in the $p$-coordinate system consist of the vorticity equation

$$\frac{\partial \zeta}{\partial t} + V \cdot \nabla \zeta + \beta v = \frac{\partial \omega}{\partial p}$$

(2-1)

and the thermodynamic equation

$$\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial p} \right) + V \cdot \nabla \left( \frac{\partial \phi}{\partial p} \right) + S \omega = 0$$

(2-2)

where the following notations are used:

- $p$: pressure,
- $\zeta$: vertical component of relative vorticity,
- $V$: horizontal velocity,
- $\nabla$: horizontal gradient operator,
- $v$: meridional velocity,
- $\omega$: vertical $p$-velocity,
- $f$: the Coriolis parameter,
- $\beta$: latitudinal variation of the Coriolis parameter $\partial f/\partial \phi$,
- $\phi$: geopotential,
- $S$: measure of the static stability.

Furthermore, in our model the basic flow is assumed to be purely zonal with constant vertical shear and the static stability is assumed to be independent of the pressure. This assumption corresponds to that of Green’s model in the $z$-coordinate.

Linearized equations for small perturbations of $\phi$ and $\omega$ can be written in the following by the use of the geostrophic and the thermal wind relations:

$$\frac{\partial}{\partial t} \phi^{j} + U \frac{\partial}{\partial x} \phi^{j} + \beta \frac{\partial}{\partial x} \phi^{j} = f^2 \frac{\partial \omega}{\partial p}$$

(2-3)

and

$$\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial p} \right) + U \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial p} \right) - \frac{\partial U}{\partial \phi} \frac{\partial \phi}{\partial x} + S \omega = 0$$

(2-4)

where $U$ denotes the zonal velocity of the basic current, and hereafter $\phi$ and $\omega$ denote the perturbation.

The finite difference approximations of the linearized perturbation equations (2-3) and (2-4) corresponding to a multi-layer model shown in Fig. 1 are written as follows:

- \[ \frac{1}{2} \]
- \[ N-1 \]
- \[ N \]
- \[ N+\frac{1}{2} \]

\( \omega = 0, P = 0 \)

\( \omega \)

\( \phi \)

\( \phi \)

\( \phi \)

\( \omega \)

\( \omega \)

\( \omega \)

\( \omega \)

\( \omega \)

\( \omega \)

\( \omega \)

Fig. 1. A schematic representation of the $N$-layer model. $\omega$ is the vertical velocity and $\phi$ is the geopotential.

The vorticity equation applied to the level $j$ is

$$\frac{\partial}{\partial t} \phi^{j} + U_j \frac{\partial}{\partial x} \phi^{j} + \beta \frac{\partial}{\partial x} \phi^{j} = f^2 \Delta p^{-1} (a_{j+1/2} - a_{j-1/2})$$

$$ \ (j = 1, 2, \ldots, N)$$

(2-5)

The thermodynamic equation applied to the level $j+1/2$ is

$$\frac{\partial}{\partial t} (\phi^{j+1} - \phi^{j}) + U_{j+1/2} \frac{\partial}{\partial x} (\phi^{j+1} - \phi^{j})$$

$$+ \frac{1}{2} \frac{\partial}{\partial x} (\phi^{j+1} + \phi^{j}) \Delta p + S \Delta p \omega_{j+1/2} = 0$$

$$ (j = 1, 2, \ldots, N-1)$$

(2-6)
\( \Delta p \) denotes the pressure difference between the levels \( j \) and \( j+1 \), and \( \Lambda \) is the vertical shear \(-\partial U/\partial p\). The equations (2-5) and (2-6), together with the boundary conditions \( \omega_{t/2} \) (top) = 0 and \( \omega_{N+1/2} \) (bottom) = 0, completely determine \( N \) unknown values of \( \phi_j \) and \((N-1)\) unknown values of \( \omega_j \). Eliminating \( \omega_j \) between the equation (2-5) and (2-6), we have \( N \) equations for \( \phi_j \) as

\[
\sum_{j=1}^{N} \left( \frac{\partial}{\partial t} \Gamma \phi_{j+1} + U_j \frac{\partial}{\partial x} \Gamma \phi_{j} + \beta \frac{\partial}{\partial x} \phi_{j} \right) = 0
\]

\[
\frac{\partial}{\partial t} (\phi_{j+1} - \phi_{j}) + U_j \frac{\partial}{\partial x} (\phi_{j+1} - \phi_{j}) + \frac{1}{2} \Lambda \Delta p \frac{\partial}{\partial x} (\phi_{j+1} + \phi_{j}) + S \cdot (\Delta p)^2 \phi_j - 2 i \Gamma \phi_j - \frac{1}{2} \Lambda \Delta p \frac{\partial}{\partial x} (\phi_{j+1} + \phi_{j}) + S \cdot (\Delta p)^2 \phi_j = 0 (2-7)
\]

\( j = 1, 2, \ldots, N-1 \)

We assume that the geopotential perturbation \( \phi_j \) is independent of the meridional direction and is given by

\[
\phi_j = \hat{\phi}_j \exp\{ik(x-ct)\}
\]

where \( \hat{\phi}_j \) is the amplitude, \( k \) is the zonal wave number and \( c \) denotes the phase velocity and the growth rate. Using \( \partial/\partial t = -ik \), \( \partial/\partial x = ik \) and \( \Gamma = -k^2 \), we obtain a set of homogeneous linear algebraic equations for \( \phi_j \) from (2-7). The necessary condition for the existence of non-trivial solutions gives a "frequency equation" which determines the eigenvalues \( \epsilon \). The set of homogeneous equations for \( \phi_j \) can be written symbolically as

\[
(B - cD) \hat{\phi} = 0 \quad (2-8)
\]

where \( B \) and \( D \) are square matrices involving \( U, \Lambda, k \) etc. We operate the inverse matrix \( D^{-1} \) on the left hand side of (2-8) and obtain

\[
(D^{-1}B - cE) \hat{\phi} = 0 \quad (2-9)
\]

where \( E \) is the unit matrix. Thus the frequency equation is given by

\[
\text{Det} (D^{-1}B - cE) = 0 \quad (2-10)
\]

which means that the phase velocities and the growth rates are given as the eigenvalues of the matrix \( D^{-1}B \).

After solving the frequency equation for \( c \), we obtain the eigenfunction

\[
\phi = \phi = \hat{\phi}_j \exp\{ik(x-ct)\}
\]

Since the eigenvalue \( c \) is divided into the real part \( c_r \) and the imaginary part \( c_i \), i.e., \( c = c_r + ic_i \), the eigenfunction \( \phi \) can be written as

\[
\phi = \exp(\triangle \bphi) \exp(ik(x-c_r t)) = \exp(kc_t \cdot \bphi, \cos k(x-c_r t) - \bphi, \sin k(x-c_r t)) + i \exp(kc_t \cdot \bphi, \sin k(x-c_r t) + \bphi, \cos k(x-c_r t))
\]

(2-11)

where \( \bphi \) and \( \hat{\bphi} \) are the real and imaginary parts of the eigenfunction \( \phi \) respectively. Thus the real part of the eigenfunction can be written as

\[
\phi_r = A \exp(\lambda t) \cdot \cos (X+\delta) \quad (2-12)
\]

by the use of the notations \( A = (\hat{\phi}_j + i\hat{\phi}_j)^{1/2} \), \( \lambda = kc \), \( X = k(x-c_r t) \) and \( \delta = \tan^{-1}(\hat{\phi}_j/\hat{\phi}_j) \). When the eigenvalue \( c \) is real, the wave is neutral and the eigenfunction \( \phi \) turns out to be a simple sinusoidal wave such as

\[
\phi_r = \phi \cos k(x-ct). \quad (2-13)
\]

3. Results of the numerical calculation

The numerical computations for solving the frequency equation (2-10) are performed in several cases by varying the number of layers \( N \) and the vertical shear \( \Lambda \). The parameters adopted in the computation are:

\[
f = 10^{-4} \cdot \text{sec}^{-1}, \quad \beta = 1.6 \times 10^{-11} \cdot \text{m}^{-1} \cdot \text{sec}^{-1}
\]

and

\[
S = 2 \times 10^{-2} \cdot \text{m}^2 \cdot \text{sec}^{-2} \cdot \text{mb}^{-1}.
\]

Results obtained in our computations are summarized in the following subsections:

3.1. Baroclinic instability properties

First of all, the instability diagram (\( \lambda-L \) diagram) in the case of 20-layer model is shown in Fig. 2, where the abscissa indicates the wavelength measured in 10\(^3\) km and the ordinate the vertical wind shear of the basic zonal current measured in m \cdot \text{sec}^{-1} \ per 100 mb. In Fig. 2 the growth rate of the unstable waves, \( \lambda = kc \), is shown by converting its value to the e-folding time measured in days. The figure shows that:

(i) For almost all wavelengths and vertical
shears, there exists an unstable (amplifying) wave.

(ii) The unstable regions are divided by "critical curves", on which the growth rate is zero, though in Fig. 2 only three curves are shown.

(iii) In the first unstable region, which is on the upper left in Fig. 2, the most unstable waves are found for the wavelength of about 5,000 km and the e-folding time is a few days. This region is bounded by the so-called Charney's criterion for longer waves, whereas there is no short wave limit.

(iv) In the second unstable region, which is on the lower right side of Charney's critical curve, another kind of unstable waves exist. One of the most interesting features of these waves is that there exists a maximum growth rate for the suitable vertical shear, i.e., \( \Lambda = 2 \text{ m} \cdot \text{sec}^{-1}/100 \text{ mb} \), whereas the growth rate of the unstable wave in the first region has no limited value with increasing vertical shear. The wavelength of the most unstable wave in the second region is about 8,000 km and its growth rate is about one third or less compared with that of the first unstable wave for the same value of \( \Lambda \).

(v) The third unstable region appears on a very weak shear condition, where the growth rate is one order or more smaller than that of the most unstable wave in the second region.

These results confirm the previous computations shown by Green. The instability of the second and the third regions was obtained also by Murray (1960) in connection with the stability problem of the stratospheric circulation, although his attention was not focused on these unstable solutions. The vertical structure and other dynamical characteristics of the second and third unstable waves will be discussed later.

3.2. Phase velocities

Since we have assumed that the wave disturbance in the perturbation equation is in the form of \( \phi = \phi \exp(ik(x-ct)) \), the real part of the eigenvalue \( c \) means the horizontal phase velocity of the wave disturbance. The behaviour of the real part of the numerical solutions of the frequency equation is discussed in this subsection.

In Fig. 3 five eigenvalues corresponding to the 5-layer model in the case of \( \Lambda = 0 \) are shown, where the abscissa indicates the wavelength and the ordinate the phase velocity relative to the mean basic flow \( U \) measured in m\cdot sec\(^{-1}\). For \( \Lambda = 0 \), all waves are neutral and their relative phase velocities are negative, that is, they are retrogressive. One of these
waves is the Rossby wave whose phase velocity is very sensitive to the wavelength as shown by \( c = U - \beta L^2 / 4 \pi^2 \), while the others are almost independent of the wavelength. When the vertical wind shear exists, however, the \( c-L \) diagram looks quite different. Fig. 4 shows a \( c-L \) diagram in the case of \( A = 2 \text{ m} \cdot \text{sec}^{-1}/100 \text{ mb} \) for the 10-layer model. In this figure the neutral waves are illustrated by full lines and the real part of the complex conjugate eigenvalues is drawn by double broken lines.

Looking at the figure, we find that:
(i) The first unstable wave, whose wavelength is shorter than 5,000 or 6,000 km, occurs as the conjunction of the Rossby wave with one of the other neutral waves. The phase velocity of the first unstable wave is dependent upon the wavelength.
(ii) On the contrary, the second unstable wave appears when the first one disappears and its phase velocity is almost independent of the wavelength.
(iii) When the wavelength \( L \) is large, there exist two types of neutral wave; one is the Rossby wave whose phase velocity is less than the zonal velocity at the surface \( U_s (= \bar{U} - 10 \text{ m} \cdot \text{sec}^{-1}) \) in the case of Fig. 4), that is,
\[
c \approx \bar{U} - \beta L^2 / 4 \pi^2 < U_s,
\]
and other neutral waves satisfy
\[
U_t < c < U,
\]
where \( U_t \) is the zonal velocity at the top \((= \bar{U} + 10 \text{ m} \cdot \text{sec}^{-1})\). Accordingly each of these neutral waves possesses a “steering level” in the zonal current whose velocity is increasing upward.

The phase velocity of neutral waves will be discussed again later in connection with their vertical structure. The phase velocities of the first and the second unstable waves are shown in Fig. 5 in terms of the wavelength and the vertical shear. This \( A - L \) diagram for \( c_r \) is in the case of 20-layer model. The inspection of Fig. 5 shows that the phase velocity on the critical curve is nearly equal to the zonal velocity at the surface which is determined as \( U_s = \bar{U} - A \cdot \Delta p \) \((\Delta p = 500 \text{ mb})\). Therefore the critical curve may be defined as the limit on which the wave has no “steering levels”. It is also noteworthy that the relative phase velocity of the unstable wave is always negative, i.e., \( c - \bar{U} < 0 \).

Before we leave this section, we summarize the results in a schematic diagram shown in Fig. 6. As was shown in Fig. 2, the characteristic features of the solutions are divided into various regimes by the critical values of the vertical shear, when the wavelength is fixed, and then the composition of the \( N \) solutions of the \( N \)-layer model is characteriz-
Fig. 6. Schematic representation of the phase velocities of the various waves as a function of the vertical shear. Heavy full lines denote the neutral waves without a steering level, thin full lines the neutral waves with a steering level and the broken lines the unstable waves.

ed only by the value of the vertical shear $A$. Since the velocity of the zonal current $U$ is bounded by $U_s$ and $U_t$, the wave disturbance whose phase velocity lies between $U_s$ and $U_t$ has a "steering level", whereas the others have no steering levels.

Fig. 6 shows that:

(i) In the first region (when $A$ is sufficiently large), $N$ solutions consist of a pair of complex conjugates, which denote amplifying and decaying waves (heavy broken line in Fig. 6), and $(N-2)$ neutral solutions whose phase velocities are $c=U$ at some point in the basic current (thin full lines).

(ii) In the Mth region (when $A$ is small), there exist a pair of complex conjugates, $(M-1)$ neutral solutions without steering levels (heavy full lines) and the remaining $(N-M-1)$ neutral solutions with steering levels. The $(M-1)$ neutral solutions, whose phase velocities are less than $U_u$, are denoted as the neutral waves of the first mode (Rossby wave), the second mode and so forth.

(ii) When the vertical shear $A$ tends to zero, all waves continuously become neutral ones without steering levels as was shown in Fig. 3.

4. The structure of wave disturbances

In this section the structure of neutral and amplifying waves is examined by the use of the eigenfunction presented in the section 2. The vertical distributions of the amplitude functions $\phi$ are shown in Fig. 7 on a strong and a weak shear condition. Fig. 7(a) shows the solutions on the strong shear condition ($A=3 \text{ m} \cdot \text{sec}^{-1}/100 \text{ mb}$) for three different wavelengths $L=5,000, 10,000, \text{ and } 15,000 \text{ km}$. As is seen in Fig. 2, the wavelength 5,000 km in this case corresponds to the first unstable region, whereas the others correspond to the second one. On the contrary, in Fig. 7(b), the case of a weak shear condition ($A=0.4 \text{ m} \cdot \text{sec}^{-1}/100 \text{ mb}$) is shown for the wavelength $L=10,000 \text{ km}$ corresponding to the third unstable region in Fig. 2. In Fig. 7, the amplitude of the wave at the highest level is arranged to be unity and the steering level of each wave is indicated by the black circle on the curve. Now we will discuss the characteristics of each wave in the following.

4.1. Neutral waves

As was discussed in the previous section, there exist two types of neutral waves; one is the wave without a steering level in the zonal current and the other is the wave with a steering level. The latter wave possesses a singularity at the steering level. Then we divide the sub-section into two parts.

(i) Neutral waves without steering level

In the case of a strong shear condition (Fig. 7a), there is only one neutral wave without a steering level, while in the case of a weak shear condition (Fig. 7b), we have two neutral waves of this type. The mode of these waves in vertical is of some interest. The first mode is the Rossby wave whose amplitude is almost constant in vertical. The wave of the second mode has a nodal surface in the middle layer, and the vertical distribution of the amplitude is nearly antisymmetric about the mid-level. In the case of no vertical shear, all waves become neutral waves of this
type with various numbers of nodal surfaces.

In order to show the relation between the vertical mode of neutral waves and their phase velocities, we derive the perturbation equation for $\phi$ by eliminating $\omega$ from the equations (2-3) and (2-4). It turns out to be as

$$ (U-\omega) \left( \frac{\partial^2 \phi}{\partial p^2} - \frac{S}{f^2} k^2 \phi \right) + \frac{S}{f^2} \delta \phi = 0 \quad (4-1) $$

Now we assume that the term $\partial^2 \phi / \partial p^2$ can be replaced by $- (m^2 / p_0^2) \phi$, where $m$ denotes the vertical mode and $p_0$ is a measure of pressure.
Then the phase velocity \(c\) is given by

\[ c = U - \frac{\beta}{k^2 + qm^2} \left( q \equiv \frac{f^2}{Sp_0^2} \right) \tag{4-2} \]

The first mode \(m=0\) gives the phase velocity of the Rossby wave \(c = U - \frac{\beta}{k^2}\), and the remaining retrogressive waves are given by \(m=1, 2, \) and so on as is shown in Fig. 3. It is evident from the expression (4-2) that the phase velocities for \(m \neq 0\) tend to finite values as the wave number \(k\) approaches zero.

(ii) Neutral waves with steering levels

In order to show the features of the wave of this type, we examine the perturbation equation (4-1) again. Since the term \((U-c)\) becomes infinitesimally small changing its sign in the vicinity of the singular point \(p^*\) \((U(p^*) = c)\), the second derivative \(\partial^2 \phi / \partial p^2\) must become \(\pm \infty\) at the point in such a way that the product \((U-c) \partial^2 \phi / \partial p^2\) remains finite to balance with a finite value of \(S \beta/f^2\). Integration of this discontinuous distribution of \(\partial^2 \phi / \partial p^2\) may give finite but discontinuous values of \(\partial \phi / \partial p\) and then a continuous distribution of \(\phi\) with a sharp pointed head at the singular point. This aspect is clearly shown in Fig. 7.

The number of neutral solutions of this type increases with the increase of the number of layers for the finite difference approximation. The corresponding solutions in a continuous model may have a continuous spectrum of phase velocities such that \(U = c\) at some point of the basic current. Moreover it is noted that, among these neutral waves, one whose steering level lies lower has the vertical structure similar to that of the unstable wave in amplitude.

4.2. Unstable waves

Three kinds of unstable waves are presented on the left side of Fig. 7. The first and the second unstable wave appear on the strong shear condition (Fig. 7a) and the third on the weak shear condition (Fig. 7b). The phase angle, which means the slope of the trough axis, is indicated by dotted lines.

The first unstable wave \((L=5,000 \text{ km}, \text{ for } \Lambda=3 \text{ m} \cdot \text{sec}^{-1}/100 \text{ mb})\) shows the well-known characteristics of the "long wave" studied by Charney and others such that the trough axis is tilted westward especially at lower levels with increasing height and the amplitude of the geopotential is nearly constant in vertical. On the other hand, the second unstable wave \((L=10,000 \text{ km and } 15,000 \text{ km}, \text{ for } \Lambda=3 \text{ m} \cdot \text{sec}^{-1}/100 \text{ mb})\) has somewhat different features. There exists a nodal surface in the middle layer. Owing to the existence of the node in the middle layer, the westward tilt of the trough axis is striking in the middle layer, and then the upper half and the lower half of the wave are out of phase with each other. The third unstable wave for \(\Lambda=0.4 \text{ m} \cdot \text{sec}^{-1}/100 \text{ mb}\) has two nodal surfaces.

More detailed studies of the structure of unstable waves in terms of the phase difference of \(\omega\) and \(\alpha\) are shown later in connection with energy conversion processes associated with the unstable waves.

5. Some remarks on the solutions by the 2-layer model

We have hitherto discussed the baroclinic instability properties by the use of a multi-layer model. Then it may be interesting to compare the results presented above with those given by 2-layer models studied by many other authors.

The examination of the baroclinic instability property in a zonal current by the use of a 2-layer model was first attempted by Phillips (1951), and later by Thompson (1953). They have shown that the so-called Eady's and Charney's criteria can be approximately obtained as the vanishing of the discriminant of the quadratic equation, that is the frequency equation for the 2-layer model, in terms of the relation between the vertical shear of the basic current and the wave number of the disturbances.

In fact, the frequency equation in the case of two-layer model is written as

\[
\begin{vmatrix}
    c-U_1 + \frac{\beta}{k^2} & c-U_2 + \frac{\beta}{k^2} \\
    -c + U_2 & -c + U_1 \\
    -2k^2 & \left( c-U_1 + \frac{\beta}{k^2} \right) \\
    q & c-U_1 \\
\end{vmatrix} = 0 \tag{5-1}
\]

where \(q = 2f^2/Sd^2\) with \(d^2 = p_0/2\) \((=500 \text{ mb})\). The equation (5-1) can be rewritten as

\[
\left\{ c - \left( \bar{U} - \frac{\beta}{k^2} \right) \right\} \left\{ c - \left( \bar{U} - \frac{\beta}{1+k^2/q} \right) \right\} + \ldots
\]
by the use of the notation \( \bar{U} = 1/2(U_1 + U_2) \) and \( \Delta U = U_1 - U_2 \). When the zonal current has no vertical shear, i.e., \( \Delta U = 0 \) and \( \bar{U} = U_1 = U_2 \), two solutions of (5-2) are easily obtained as

\[
\begin{align*}
    c_1 &= \bar{U} - \frac{\beta}{k^2} \\
    c_2 &= \bar{U} + \frac{\beta}{1 + k^2/q}
\end{align*}
\]  

This expression may be considered as a special case of (4-2). Concerning the amplitudes of the eigenfunctions, it is shown that:

\[
\begin{align*}
    \phi_1 - \phi_2 &= 0 & \text{for } c_1 \\
    \phi_1 + \phi_2 &= 0 & \text{for } c_2
\end{align*}
\]

This indicates that the amplitude of the first wave with \( c_1 \) (Rossby wave) is symmetric in the vertical and that the amplitude of the second wave with \( c_2 \) is antisymmetric and has a node at the middle level.

Let us consider the physical meaning of the solution given by \( c_2 = \bar{U} - \frac{\beta}{1 + k^2/q} \). Although the unstable wave of the second mode cannot be represented in the two-layer model, the solution \( c_2 \) must correspond to either the unstable wave of the second mode or the (retrogressive) neutral wave of the second mode. Therefore the neutral solution \( c_2 \) is valid at least qualitatively, except for the truncation errors, when \( \Delta U = 0 \) or \( \Delta U \) is sufficiently small so that the phase velocity is smaller than the zonal velocity at the ground surface. We assume that we can use \( c_2 = \bar{U} - \frac{\beta}{1 + k^2/q} \), even in a baroclinic case \( \Delta U \neq 0 \) when \( \Delta U \) is not so large, while retaining the effect of the vertical shear only in the term of \( \bar{U} \). Then the condition that the phase velocity \( c_2 \) is smaller than the zonal velocity at the surface \( U_s \) (the condition that the wave with \( c_2 \) has no steering level) becomes

\[
c_2 - U_s = A \cdot \Delta p - \frac{\beta}{1 + k^2/q} \leq 0
\]

because \( \bar{U} = U_s + A \cdot \Delta p \). The inequality (5-5) gives “critical shear” \( A^* \) for the existence of the neutral wave without a steering level such as

\[
A^* = \frac{\beta}{1 + k^2/q} \cdot \Delta p^{-1}
\]

The criterion given by (5-6) should be interpreted as a kind of approximation to the critical curve which is the lower boundary of the second unstable region shown in Fig. 2. Thus it should be remarked that the solution given by the simple 2-layer model determines the second critical curve as the limit to the existence of a retrogressive wave, although the second unstable wave itself cannot be represented in the model. In a similar way, roughly speaking, the so-called Charney’s criterion is also obtained from the viewpoint that the phase velocity of the Rossby wave must be smaller than \( U_s \) as

\[
c_1 - U_s = A \cdot \Delta p - \frac{\beta}{k^2} \leq 0
\]

then the critical shear \( A^* \) is given by

\[
A^* = \frac{\beta}{k^2} \cdot \Delta p^{-1}
\]

As two critical curves are obtained in the 2-layer model, four curves are determined approximately by four solutions in the 4-layer model. The example of the “approximate critical curves” thus obtained from the neutral solutions for \( A = 0 \) in the case of 4-layer model is shown by broken lines in Fig. 8. Com-
pared with the critical curves obtained in the multi-layer model (full lines in Fig. 8), it is found that the approximation discussed above is very accurate especially when $A$ is not so large. The determination of critical wavelength from the viewpoint that the phase velocity of the neutral wave is equal to the minimum velocity of the basic flow has been presented by Kuo (1949) in the barotropic instability problem. In our problem of baroclinic instability, the critical shear can be determined from the same viewpoint.

In an attempt to describe the dynamics of ultra-long waves, Burger (1958) has shown from the standpoint of the scale consideration that the vorticity equation becomes not prognostic but diagnostic so far as the planetary-scale motion is concerned. Namely, the vorticity equation is written by neglecting the time derivative term and the advection term as

$$\beta = f \frac{\partial \omega}{\partial p} \quad (5-9)$$

By deriving a linearized perturbation equation from Burger's vorticity equation in a two-layer model, Wiin-Nielsen (1961) obtained the neutral wave of the second mode, where the wave of the first mode (Rossby wave) is filtered out. The solution is in the form of

$$c^* = \bar{U} - \frac{\beta}{q}. \quad (5-10)$$

Compared with the solution (5-3), i.e., $c_2 = \bar{U} - \frac{\langle\beta/q\rangle}{1 + k^2/q}$ this solution $c^*$ can be regarded as a kind of approximation to $c_2$ when the wave number $k$ is sufficiently small. It is also shown that the solution $c^*$ is valid only when the vertical shear of the zonal current $A$ is below the critical shear $A^* = \langle\beta/q\rangle \cdot \Delta p^{-1}$.

Another problem is the short wave limit to instability. When the wavelength is sufficiently small the discriminant of the quadratic equation mentioned above becomes positive, indicating that the wave is neutral. This short wave limit has been considered to be identical with Eady's criterion. It should be noted that Eady's criterion is obtained in a continuous model for $\beta = 0$, whereas this short wave limit comes from the two-layer model for $\beta \neq 0$. The result obtained in a multi-layer model for $\beta \neq 0$ shows, however, that there is no short wave limit to instability as is illustrated in Fig. 2. It is shown that the short neutral waves given in the two-layer model are different from the long neutral waves in the same model in that the phase velocities of the former waves are equal to the zonal velocity at the upper and lower levels respectively, while the latter waves are retrogressive waves, namely the Rossby wave and the neutral wave of the second mode. Therefore the short wave limit obtained in a simple layer model for $\beta = 0$ may be spurious and may not be related with Eady's criterion. In the above discussion, of course, the validity of the quasi-geostrophic approximation to very short waves is doubtful.

6. Energy conversions

Theoretical consideration of the energy conversion processes of the unstable long wave in connection with the maintenance of the general circulation was given by Lorenz (1955). He divided the energy of the atmosphere into four forms, i.e., the zonal and the eddy kinetic energy and the zonal and the eddy available potential energy, and then he showed that the baroclinic unstable wave is maintained in such a way as the increase of the eddy available potential energy due to the northward transport of the sensible heat and its conversion into the eddy kinetic energy in the form of ascending warm air and descending cold air.

This mechanism is, of course, related with the three-dimensional structure of the unstable wave, and therefore different mechanism can be expected for the disturbance with different structure. Murakami (1962) investigated the energetics of the various large-scale disturbances from the viewpoint of the scale analysis. According to his results, only the interaction between zonal and eddy available potential energy is predominant for ultra-long waves, and other terms are two orders of magnitude smaller than that. Since we have obtained several unstable waves which have different vertical structure and different scales, an attempt is made in this section to compare the energy conversion processes between the first and the second unstable waves.

Now the perturbation energy equations corresponding to the linear theory of adiabatic, frictionless motion should be derived. The
linearized vorticity equation and thermodynamic equation are rewritten as

\[
\frac{\partial}{\partial t} P^2 \phi + U \frac{\partial}{\partial x} P^2 \phi + \beta \frac{\partial}{\partial x} \phi = f^2 \frac{\partial \omega}{\partial \rho} \tag{6-1}
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial \rho} \right) + U \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial \rho} \right) - \frac{\partial U}{\partial \rho} \frac{\partial \phi}{\partial x} + S \omega = 0 \tag{6-2}
\]

Multiplying the equation (6-1) by \(-f^2 \phi\) and carrying out the integration in horizontal with the cyclic boundary condition, we have

\[
\frac{\partial}{\partial t} K = - \frac{\partial}{\partial \rho} \left( \omega \phi \right) - \omega \phi \tag{6-3}
\]

where the notation \(\bar{\ldots}\) denotes the horizontal integration over one wavelength, i.e.,

\[
\bar{X} = \frac{1}{L} \int_0^L X \, dx, \quad K = \frac{1}{2} \left( \frac{1}{f} P \phi \right)^2, \quad \alpha = - \frac{\partial \phi}{\partial \rho},
\]

and the following relations are used:

\[
-\phi \frac{\partial}{\partial t} P^2 \phi = \frac{1}{2} \frac{\partial}{\partial t} (P \phi)^2,
\]

\[
\phi \frac{\partial}{\partial x} P^2 \phi = 0, \quad \phi \frac{\partial}{\partial x} \phi = 0,
\]

and

\[
-\phi \frac{\partial \omega}{\partial \rho} = - \frac{\partial}{\partial \rho} \left( \omega \phi \right) + \omega \frac{\partial \phi}{\partial \rho}.
\]

After carrying out the vertical integration of (6-3) with the boundary conditions \(\omega = 0\) at \(\rho = 0\) and \(p = p_0\) we have

\[
\frac{\partial}{\partial t} \langle K \rangle = - \langle \omega \phi \rangle \tag{6-4}
\]

where the notation \(\langle \ldots \rangle\) denotes the vertical average, i.e., \(\langle X \rangle = \int_0^{p_0} X \, dp\).

Next, multiplying the equation (6-2) by \(\partial \phi / \partial \rho\) and carrying out the integration, we have

\[
\frac{\partial}{\partial t} P = \frac{f A}{S} \omega \phi - \omega \phi = 0 \tag{6-5}
\]

where

\[
P = \frac{1}{2S} \left( \frac{\partial \phi}{\partial \rho} \right)^2.
\]

By integrating (6-5) in vertical, we have

\[
\frac{\partial}{\partial \rho} \langle \omega \phi \rangle + \omega \phi = 0 \tag{6-6}
\]

The equations (6-4) and (6-6) show that the time changes of the eddy kinetic energy \(\langle K \rangle\) and eddy available potential energy \(\langle P \rangle\) are characterized by the release of potential energy \(-\langle \omega \phi \rangle\) and the meridional heat transport \(\langle \omega \phi \rangle\). We shall evaluate these two quantities in terms of the structure and the scale of the unstable wave disturbances.

Numerical calculations for evaluating \(\langle \omega \phi \rangle\) and \(\langle \omega \phi \rangle\) are performed in the case of 4-layer model and \(A = 4 \, m \cdot \text{sec}^{-1} / 100 \, \text{mb}\), the result of which is presented in Fig. 9. The abscissa indicates the wavelength and the ordinate \(\langle \omega \phi \rangle\) and \(\langle \omega \phi \rangle\) in an arbitrary unit. It is found that the absolute values of \(\langle \omega \phi \rangle\) and \(\langle \omega \phi \rangle\) are proportional to each other for the long wave (unstable wave of the first mode), but for the ultra-long wave (unstable wave of the second mode) only \(\langle \omega \phi \rangle\) remains and \(\langle \omega \phi \rangle\) vanishes. This conclusion is fairly coincident with that reached by Murakami (1962).

It is noteworthy that the aspect of the vanishing of \(\langle \omega \phi \rangle\) for ultra-long waves is approximately shown by the use of Burger's vorticity equation. By multiplying both sides of the equation \(\beta (f) (\partial \phi / \partial x) = f (\partial \omega / \partial \rho)\) by \(\phi\), we have

\[
\frac{\beta}{2f} \frac{\partial}{\partial x} \phi = f \frac{\partial}{\partial \rho} \left( \omega \phi \right) - f \omega \frac{\partial \phi}{\partial \rho} \tag{6-7}
\]

Carrying out the integration of (6-7) in horizontal and in vertical, we have

\[
\frac{\partial}{\partial \rho} \langle \omega \phi \rangle + \omega \phi = 0 \tag{6-8}
\]
As was mentioned above, these characteristic features of the energy conversion must be related with the structure of the unstable wave, and then we show the fine structure of the unstable waves of two different modes in Fig. 10. In both cases, the westward tilting of the trough axis with increasing height causes a northward sensible heat transport $\langle \omega \alpha \rangle$. Concerning the release of potential energy, however, there exist quite different features in these two waves: The wave of the first mode (Fig. 10a) shows the upward motion to the east of the trough and the downward motion to the west of the trough. Accordingly there is a negative correlation between the vertical $p$-velocity $\omega$ and the specific volume $\alpha$. On the contrary, the wave of the second mode (Fig. 10b) shows that the upward

Fig. 10. Structure of the unstable wave of the first mode (upper) and the second mode (lower) in the 4-layer model. Unit in $10^{-3}$ mb/sec for $\omega$ and $m^2/mb\cdot sec^2$ for $\alpha$. 
motion is to the west of the trough in the upper layer and to the east of the trough in the lower layer. Since the phase lag of $\omega$ behind $\alpha$ is about $\pi/2$ (a quarter of one wavelength) the $\alpha-\omega$ correlation is nearly zero.

Finally the vertical distributions of the conversion terms are represented in Fig. 11 in the case of 4-layer model and $L=4 \text{ m} \cdot \text{sec}^{-1}/100 \text{ mb}$. The four quantities $\bar{\omega} \alpha$, $\omega \alpha$, $\frac{\partial}{\partial \rho} (\omega \phi)$ and $\bar{\omega} \frac{\partial}{\partial \rho} (\omega \phi)$ are shown there for the three values of wavelength $L=4,000, 12,000$ and $20,000 \text{ km}$. These results show that:

(i) In the wave of the first mode (Fig. 11a), $\omega \alpha$ is negative at all levels and the sum $\bar{\omega} \frac{\partial}{\partial \rho} (\omega \phi) + \omega \alpha$ does not always vanish at each level.

(ii) On the contrary, in the wave of the second mode (Fig. 11b, c), $\omega \alpha$ has opposite signs at the upper and at the lower level so that $<\omega \alpha>=0$. Moreover $\omega \alpha$ is almost canceled with $\bar{\omega} \frac{\partial}{\partial \rho} (\omega \phi)$ at each level.

(iii) Concerning the sensible heat transport $\bar{\omega} \alpha$, it is always positive at all levels.

Accordingly, the reason why the increase of eddy kinetic energy can be ignored for the unstable wave of the second mode may be explained that the conversion from the eddy available potential energy into the eddy kinetic energy is always balanced by the redistribution of the eddy kinetic energy.

7. Discussions

In this section, a comparison is made between the characteristic features of unstable waves obtained theoretically and those observed in the actual atmosphere.

First, concerning the vertical structure of ultra-long waves, synoptic descriptions have not been so clearly given as those of long waves. Eliasen (1958) has shown the statistical properties of the structure of ultra-long waves on the basis of zonal harmonic analyses of hemispherical weather charts, although the data used in his study are confined to the lower atmosphere below the 500 mb level. The results show that the westward tilting of the trough axis with increasing altitude is observed in wintertime. Muench (1965) has presented the vertical structure of monthly-mean ultra-long waves from the surface up to the stratosphere at 50°N for January 1958 as shown in Fig. 12. Compared with the result of the theoretical study (Fig. 10b), the observed pattern shows several characteristics similar to those of theoretical waves in the following:

(i) The westward tilt of the trough and ridge with increasing height is striking in the middle layer.

(ii) In the upper layer the downward motion is observed to the east of the trough and the upward motion to the east of the ridge.
Fig. 12. Vertical structure of monthly-mean ultra-long waves at 50°N for January 1958. Wave number one, upper; number two, lower. Figures along ridge are amplitude in meters. W; thermal ridge, double shafted arrows; maxima and monthly mean vertical motions. (After Muench)

(iii) The phase line of temperature is also tilted westward with increasing height, and the phase lag behind the vertical motion is about π/2 so that \( \omega \) may be very small.

The ultra-long wave shown by Muench may be, of course, regarded not as a purely baroclinic unstable wave but as a wave modified by the effect of large-scale external forces due to the topography or the heating and cooling. But the structure of the ultra-long waves shown in Fig. 12 is still interesting, because it is the first description of the existence of the ultra-long wave of the "second mode" in the actual atmosphere.

Statistical studies of the motion of planetary-scale disturbances in an atmospheric zonal flow have been made by many authors; notably, Kubota and Iida (1954), Eliassen (1958), Deland (1964, 1965) Eliassen and Machenhauer (1964) and Arai (1965). Recently Anderssen (1965) examined the planetary-scale motions in the Southern Hemisphere. All of these studies are based on the zonal or spherical harmonic analysis of the observed patterns of geopotential height in the troposphere for the purpose of determining the phase angle. The results of these studies, however, have shown various and rather complicated features. Generally speaking, it is suggested from these results that there are two kinds of ultra-long waves in the middle latitude westerlies. One is the quasi-stationary wave and the other is the travelling wave. The motion of the latter is retrogressive and might be regarded as that of Rossby-Haurwitz type wave. However, it should be noted that the method of statistical analysis mentioned above represents only the behavior of waves of the "first mode" so far as the level of analysis is chosen to be the 500 mb, where the amplitude of waves of the "second mode" may be very small.

On the contrary, theoretical studies of the quasi-stationary planetary waves in the atmosphere have been made from various viewpoints in connection with the problem of the retrogression of ultra-long waves in the westerlies. This problem first arises in carrying out the numerical weather prediction of hemispheric scale. Charney and Eliassen (1949), and others dealt with them as the forced waves caused by the topographic effect of the earth's surface, and Smagorinsky (1953) investigated the effect of heating and cooling on standing waves.

Accordingly, planetary waves in the atmosphere may be considered to consist of the forced (standing) wave and the free wave. The result of the present study also suggests the existence of quasi-stationary (unstable) waves and the retrogressive (neutral) waves in accordance with the condition of the basic flow. Indeed, which wave is predominant in the actual atmosphere might depend upon the basic state of the zonal circulation. Then it may be suggested that the method of statistical analysis for the purpose of detecting planetary-scale motions should be modified so that the waves of various types are clearly separated.
8. Conclusions

A theoretical investigation of the dynamical features of long and ultra-long waves in a baroclinic zonal current was made as an eigenvalue problem. Numerical computations were performed for the finite difference approximation to the linearized perturbation equations applied to a multi-layer model.

Results obtained in the present study are:

1. The stability diagram \( (A-L) \) diagram and the phase velocity diagram \( (c-L) \) diagram show that the wave solutions are classified into three types: One is a pair of amplifying and decaying waves, and the others are the neutral waves with or without steering levels in the basic zonal current.

2. The stability diagram is divided into various regions by critical curves. The growth rates of the unstable waves, except that in the first region, are bounded by the critical shear of the zonal current.

3. The vertical structure of the wave disturbances of three types is shown in terms of the wavelength and the vertical shear of the basic current. Unstable waves and neutral waves without steering levels are characterized by their vertical mode with various numbers of nodal surfaces. Concerning neutral waves with a steering level, the amplitude has a sharp pointed head at the steering level.

4. Some remarks on the validity of the finite difference approximation by a simple 2-layer model are added. It is shown that one of the solutions given by the 2-layer model is valid only when the vertical shear is sufficiently small. The limit of the validity gives the approximate "critical shear curve".

5. The energy conversion processes are discussed for the unstable wave of the second mode in comparison with that of the first mode. It is found that the unstable wave of the higher mode is maintained by the increase of the eddy available potential energy due to the northward sensible heat transport caused by the westward tilting structure of the disturbance, whereas the increase of the eddy kinetic energy can be ignored since the term \( \frac{\partial}{\partial \phi} \frac{\partial \theta}{\partial \phi} \) is always canceled with the term \( \frac{\partial}{\partial \phi} (\omega \phi) \). This conclusion confirms the result of scale consideration shown by Murakami.

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頸圧帯状流中の長波及び超長波の力学

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頸圧帯状流中の長波及び超長波の力学的諸特性を、準地衡風近似の方程式系を用いて理論的に調べた。線型化した動方程式の多層モデルに対する差分近似が、固有数をきめる振動数方程式となるが、それを数値的に解いた結果、固有解として(i)発震及び減衰する一組の解、(ii)帯状流中を逆進する中立解、及び(iii)帯状流に流される中立解の三種が存在することがわかる。これらは、帯状流の垂直ツアーと擾乱の波長により規定されると同時に、擾乱の振幅の垂直分布(vertical mode)によって特徴づけられている。これら各波の振幅の垂直構造と位相速度をくわしく調べた。従来、長波の特性が、2層モデルの解で近似的に現われることが知られているが、その適用限界を指摘し、更に3層モデルの中立解の位相速度から、第2 modeの境界曲線が求められることを示した。

次に、不安定超長波のエネルギー変換の過程を調べ、長波のそれと比較してみると、前者においては、北向き熱輸送に起因する有効位置エネルギーの増加のみが卓越し、運動エネルギーへの変換は無視して良いことがわかる。最後に実際の長波の特徴と、理論的に得られたものとの異同についても考察した。