A Discrimination Theory Based on Categorical Variables and Its Application to Meteorological Problems

By Eiichi Suzuki

Meteorological Research Institute, Tokyo

(Manuscript received 14 October 1968, in revised form 1 April 1969)

Abstract

The statistical discrimination theory was first proposed by C. R. Rao (1948) and has been widely applied to the several domains of scientific classification. Its theoretical development has been due to the work of C. R. Rao, S. S. Wilks, J. G. Bryan and others. It was recently applied to meteorological classificatory prediction by R. G. Miller (1962-1964) and by the present author (1963-1965) almost concurrently.

In this paper, a simple optimum linear discrimination function is firstly investigated for the case containing several qualitative or categorical variables only, based on the joint probability distribution (i. e., multi-variate binomial distribution) defined by A. S. Krishnamoorthy (1951) and by the author (1966).

Secondly the quantification procedure maximizing the correlation ratio between classified predictand variable Y and categorical set of predictor variable X used for classification (of Y), which has already been shown by C. Hayashi and S. Chino, is taken into account through combining it with a linear discriminant function.

Lastly, several explanatory examples are shown for qualitative prediction of the rain state and the occurrence or non-occurrence of snowslide by using the effective presaged predictors.

1. Introduction

Hitherto, the statistical linear discrimination function has been computed by using continuous stochastic variables only, and widely applied to scientific discriminatory problems. Some applications of it to meteorological prediction were recently attempted through technical contrivances by R. G. Miller and the author. However, in the case of mixed population (both discrete and continuous variables are to be taken into consideration at the same time) or of multi-variate binomial population, the problem of discrimination has not yet been studied, though a certain type of conditional probability has conventionally been used by the author.

The likelihood ratio based on multi-variate binomial distribution will be quite suggestive for discrimination fundamentally, as has been well known. Therefore, an approximately linearized form of logarithmic likelihood ratio is induced for this case to serve the routine operation of forecasting in the sense of the least square procedure, and two special cases of the linearized discrimination index are practically applied in cate-

gorical prediction of the local heavy rain state at the Hokuriku district in Japan.

In this approach, two states only of multiple presaged factors are to be taken into consideration since prediction of either of the two contingencies of the predictand is in question, but when several states or a more detailed classification of predictors has been actually given, the linear discrimination function is constituted through optimum quantifying categorical stochastic variables.

One optimum quantification has been proposed by C. Hayashi in direction of maximizing correlation ratio, instead of $\text{tr}[(\text{inverse matrix of covariance within}) \times (\text{matrix of covariance between})]$.

After all, the ordinary linear discrimination function can be induced by this quantification and applied to discrimination concerning the occurrence or non-occurrence of snowslide by use of vegetational and geomorphological factors, and in this case, several originally quantitative variables (i. e., altitude and dimensions of the slope, etc.) are used by proper classifications because the population type containing categorical predictor variables only is the main object of this study.

2. Explicit form of likelihood ratio and its linearization in multi-variate binomial distribution

Let $(X_1, X_2, \dots, X_s)$ and Y be s -variate discrimination predictor and a predictand respectively, and the following notations and assumptions are necessary to describe statistical relations:

$\pi_i (i=1, 2) : \text{population occurrence probability of a predictand } Y \text{ specified by}$

$$P(Y \in A) = \pi_1, \quad P(Y \in B) = \pi_2, \quad \pi_1 + \pi_2 = 1 \quad (2.1)$$

where A and B are mutually exclusive and exhaustive cases of Y .

$p_i, q_i (i=1, 2, \dots, s) : \text{population occurrence probability of predictor } X_i (i=1, 2, \dots, s) \text{ specified by}$

$$P(X_i=1) = p_i, \quad P(X_i=0) = q_i = 1 - p_i \quad (2.2)$$

$p_{Ai}, p_{Bi}, q_{Ai}, q_{Bi} : \text{conditional occurrence probability of predictor } X_i (i=1, 2, \dots, s) \text{ specified by}$

$$\begin{aligned} P(X_i=1/Y \in A) &= p_{Ai}, & P(X_i=0/Y \in A) &= q_{Ai} \\ P(X_i=1/Y \in B) &= p_{Bi}, & P(X_i=0/Y \in B) &= q_{Bi} \end{aligned} \quad (2.3)$$

By these specifications, we have

$$\begin{aligned} (p_{Ai} + q_{Ai})\pi_1 &= \pi_1, & (p_{Bi} + q_{Bi})\pi_2 &= \pi_2 \\ p_{Ai}\pi_1 + p_{Bi}\pi_2 &= p_i, & q_{Ai}\pi_1 + q_{Bi}\pi_2 &= q_i \end{aligned} \quad (2.4)$$

and then, the corresponding relation between X_i and Y can be shown by the contingency Table 1.

Table 1. Schematic contingency table showing the relation between X_i and Y .

$Y \backslash X_i$	1	0	Total
A	$p_{Ai}\pi_1$	$q_{Ai}\pi_1$	π_1
B	$p_{Bi}\pi_2$	$q_{Bi}\pi_2$	π_2
Total	p_i	q_i	1

For the purpose of prediction for Y , we can assume the following inequality without loss of generality:

$$p_{Ai} > p_{Bi}, \quad q_{Ai} < q_{Bi} \quad (i=1, 2, \dots, s)$$

since we can put X_i as 1 when the state of X_i is more favorable for the case $Y \in A$ than for the other exclusive case $Y \in B$.

Multi-variate binomial distribution was independently investigated by A.S. Krishnamoorthy (1951) and E. Suzuki (1966) and could be expressed after all as follows:

$$\begin{aligned} F_A(\mathbf{X}) &\equiv P(X_i = x_i, i=1, 2, \dots, s/A) \\ &= \prod_{i=1}^s p_{Ai}^{x_i} q_{Ai}^{1-x_i} G_A(\mathbf{X}) \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} G_A(\mathbf{X}) &= 1 + \sum_{i < j} \rho_{ij}(A) \frac{(x_i - p_{Ai})(x_j - p_{Aj})}{\sqrt{p_{Ai}q_{Ai}p_{Aj}q_{Aj}}} \\ &+ \sum_{i < j < k} E\{(x_i - p_{Ai}) \cdot (x_j - p_{Aj})(x_k - p_{Ak})\} \\ &\frac{(x_i - p_{Ai})(x_j - p_{Aj})(x_k - p_{Ak})}{p_{Ai}q_{Ai}p_{Aj}q_{Aj}p_{Ak}q_{Ak}} + \dots + \\ &E\left\{\prod_{i=1}^s (x_i - p_{Ai})\right\} \frac{\prod_{i=1}^s (x_i - p_{Ai})}{\prod_{i=1}^s p_{Ai}q_{Ai}} \\ &(x_i, x_j, x_k = 0 \text{ or } 1, i=1, 2, \dots, s) \end{aligned} \quad (2.6)$$

$$P_B(\mathbf{X}) \equiv P(X_i = x_i, i=1, 2, \dots, s/B)$$

$$= \prod_{i=1}^s p_{Bi}^{x_i} q_{Bi}^{1-x_i} G_B(\mathbf{X}) \quad (2.7)$$

where $G_B(\mathbf{X})$ has a similar form to $G_A(\mathbf{X})$ obtained by substituting subscript B for subscript A in (2.6), and $\rho_{ij}(A)$ is the correlation coefficient between X_i and X_j in the case $Y \in A$, and its estimator is evidently equivalent to the coefficient of association.

In order to induce the discriminator for Y , the next condition must be considered fundamentally:

$$\frac{P_A(\mathbf{X})}{P_B(\mathbf{X})} \geq 1 \text{ or } \log \frac{P_A(\mathbf{X})}{P_B(\mathbf{X})} \geq 0,$$

that is to say, if $I = \log\{P_A(\mathbf{X})/P_B(\mathbf{X})\}$ is positive for the actual data set $\mathbf{X} = (X_1, X_2, \dots, X_s)$, Y can be regarded as belonging to A and if I is negative, the reverse action will be taken up on the contrary.

The explicit form of discriminator I can be written by

$$\begin{aligned} I &= \sum_{i=1}^s \log(q_{Ai}/q_{Bi}) + \sum_{i=1}^s x_i \log(p_{Ai}q_{Bi}/p_{Bi}q_{Ai}) \\ &+ \log\{G_A(\mathbf{X})/G_B(\mathbf{X})\} \end{aligned} \quad (2.8)$$

and we can discriminate for category A or B by use of the sample estimate of I .

However, the actual computation of I is rather difficult since the 3rd term of the right-hand side of (2.8) can hardly be computed without the help of the electronic computer's routine work except for the case of small s , even if all the parameters were readily estimated by a given sample set. Also, the estimation procedures of sample error variances of parameters were never obtained easily.

For the case when s is equal to 2 and 3 only,

the actual examples of computation of (2.8) will be shown later.

To serve for the operational routine of prediction, we may consider the best linear form of discriminator I as follows:

If we put

$$\begin{aligned} f(x_1, x_2, \dots, x_s) &\equiv \log \{G_A(\mathbf{X})/G_B(\mathbf{X})\} \\ &= \log \left[1 + \sum_{i < j}^s \rho_{ij} (A) \frac{(x_i - p_{Ai})(x_j - p_{Aj})}{\sqrt{p_{Ai} q_{Ai} p_{Aj} q_{Aj}}} + \dots \right. \\ &\quad + E \left\{ \prod_{i=1}^s (x_i - p_{Ai}) \right\} \frac{\prod_{i=1}^s (x_i - p_{Ai})}{p_{Ai} q_{Ai}} \Big] - \log \left[1 + \right. \\ &\quad \left. \sum_{i < j}^s \rho_{ij} (B) \frac{(x_i - p_{Bi})(x_j - p_{Bj})}{\sqrt{p_{Bi} q_{Bi} p_{Bj} q_{Bj}}} + \dots \right. \\ &\quad \left. + E \left\{ \prod_{i=1}^s (x_i - p_{Bi}) \right\} \frac{\prod_{i=1}^s (x_i - p_{Bi})}{p_{Bi} q_{Bi}} \right] \end{aligned} \quad (2.9)$$

then the linear approximation of $f(\dots)$ can be written by

$$f(x_1, x_2, \dots, x_s) = \sum_{i=1}^s a_i x_i + \varepsilon(x_1, x_2, \dots, x_s) \quad (2.10)$$

and the next two methods will be considered:

Least square method:

$$\sum_{x_1=0}^1 \sum_{x_2=0}^1 \dots \sum_{x_s=0}^1 \varepsilon(x_1, x_2, \dots, x_s)^2 \rightarrow \text{Min.}$$

Method of mini-max approximation:

$$\text{Max. } |\varepsilon(x_1, x_2, \dots, x_s)| \rightarrow \text{Min.}$$

Since the former method is used in general, almost equal results are expected from these two methods except for cases where extremes values are taken by parameters p_{Ai} , q_{Ai} , p_{Bi} , q_{Bi} and ρ_{ij} etc. (The telescoping method due to C. Lanczos is quite suggestive for the latter method but its computational procedure is rather difficult in this case).

Then, we have the following theorem:

[Theorem]

Let the best linear form of I be

$I = \text{const.} + \sum_{i=1}^s w_i X_i$ in the sense of the least square error, then we have

$$\begin{aligned} \text{const.} &= \sum_{i=1}^s \log(q_{Ai}/q_{Bi}) \quad w_i = \log(p_{Ai} q_{Bi}/p_{Bi} q_{Ai}) \\ &+ 2^{2-s} \left\{ g_i - \sum_{i=1}^s g_i / (s+1) \right\} \end{aligned} \quad (2.11)$$

and the minimum sum of square error for this linear approximation can be shown by

$$\begin{aligned} \text{Min. } S &= \sum_{x_1=0}^1 \sum_{x_2=0}^1 \dots \sum_{x_s=0}^1 [\log \{G_A(\mathbf{X})/G_B(\mathbf{X})\}]^2 \\ &+ 2^{2-s} \left\{ \left(\sum_{i=1}^s g_i \right)^2 / (s+1) - \sum_{i=1}^s g_i^2 \right\} \end{aligned} \quad (2.12)$$

where

$$g_i = \sum_{x_k} \sum_{(\neq x_i)} \dots \sum_{x_{i-1}} [\log \{G_A(\mathbf{X})/G_B(\mathbf{X})\}]$$

Proof of this theorem is shown in the Appendix. And so the mean standard error of linearization can be tentatively written by

$$\sqrt{\text{Min. } S/2^s}$$

To have a clarified example, we may consider two special cases $s=2$ and $s=3$, in connection with the actual computations to be shown later.

Case. 1. $s=2$.

$$\begin{aligned} f(x_1, x_2) &= \log \left\{ 1 + \rho_{12(A)} \frac{(x_1 - p_{A1})(x_2 - p_{A2})}{\sqrt{p_{A1} q_{A1} p_{A2} q_{A2}}} \right\} \\ &- \log \left\{ 1 + \rho_{12(B)} \frac{(x_1 - p_{B1})(x_2 - p_{B2})}{\sqrt{p_{B1} q_{B1} p_{B2} q_{B2}}} \right\} \end{aligned} \quad (2.13)$$

$$\begin{aligned} S(a_1, a_2) &= f(0, 0)^2 + \{f(1, 0) - a_1\}^2 \\ &+ \{f(0, 1) - a_2\}^2 + \{f(1, 1) - a_1 - a_2\}^2 \end{aligned} \quad (2.14)$$

$$\partial S / \partial a_1 = 0 \rightarrow 2a_1 + a_2 = f(1, 0) + f(1, 1)$$

$$\partial S / \partial a_2 = 0 \rightarrow a_1 + 2a_2 = f(0, 1) + f(1, 1)$$

$$\left. \begin{aligned} a_1 &= \frac{2f(1, 0) - f(0, 1) + f(1, 1)}{3} \\ a_2 &= \frac{2f(0, 1) - f(1, 0) + f(1, 1)}{3} \end{aligned} \right\} \quad (2.15)$$

$$\begin{aligned} \text{Min. } S &= f(0, 0)^2 + \{f(1, 0) + f(0, 1) \\ &- f(1, 1)\}^2 / 3 \end{aligned} \quad (2.16)$$

Case 2. $s=3$.

$$\begin{aligned} f(x_1, x_2, x_3) &= \log \left\{ 1 + \rho_{12(A)} \frac{(x_1 - p_{A1})(x_2 - p_{A2})}{\sqrt{p_{A1} q_{A1} p_{A2} q_{A2}}} + \rho_{13(A)} \frac{(x_1 - p_{A1})(x_3 - p_{A3})}{\sqrt{p_{A1} q_{A1} p_{A3} q_{A3}}} \right. \\ &\quad \left. + \rho_{23(A)} \frac{(x_2 - p_{A2})(x_3 - p_{A3})}{\sqrt{p_{A2} q_{A2} p_{A3} q_{A3}}} + \rho_{123(A)} \frac{(x_1 - p_{A1})(x_2 - p_{A2})(x_3 - p_{A3})}{\sqrt{p_{A1} q_{A1} p_{A2} q_{A2} p_{A3} q_{A3}}} \right\} \\ &- \log \left\{ 1 + \rho_{12(B)} \frac{(x_1 - p_{B1})(x_2 - p_{B2})}{\sqrt{p_{B1} q_{B1} p_{B2} q_{B2}}} + \rho_{13(B)} \frac{(x_1 - p_{B1})(x_3 - p_{B3})}{\sqrt{p_{B1} q_{B1} p_{B3} q_{B3}}} \right. \\ &\quad \left. + \rho_{23(B)} \frac{(x_2 - p_{B2})(x_3 - p_{B3})}{\sqrt{p_{B2} q_{B2} p_{B3} q_{B3}}} + \rho_{123(B)} \frac{(x_1 - p_{B1})(x_2 - p_{B2})(x_3 - p_{B3})}{\sqrt{p_{B1} q_{B1} p_{B2} q_{B2} p_{B3} q_{B3}}} \right\} \end{aligned}$$

$$+ \rho_{23(B)} \frac{(x_2 - \bar{p}_{B2})(x_3 - \bar{p}_{B3})}{\sqrt{\bar{p}_{B2}\bar{q}_{B2}\bar{p}_{B3}\bar{q}_{B3}}} + \rho_{123(B)} \frac{(x_1 - \bar{p}_{B1})(x_2 - \bar{p}_{B2})(x_3 - \bar{p}_{B3})}{\sqrt{\bar{p}_{B1}\bar{q}_{B1}\bar{p}_{B2}\bar{q}_{B2}\bar{p}_{B3}\bar{q}_{B3}}} \} \quad (2.17)$$

$$\begin{aligned} S(a_1, a_2, a_3) = & f(0, 0, 0)^2 + \{f(1, 0, 0) - a_1\}^2 + \{f(0, 1, 0) - a_2\}^2 + \{f(0, 0, 1) - a_3\}^2 \\ & + \{f(1, 1, 0) - a_1 - a_2\}^2 + \{f(1, 0, 1) - a_1 - a_3\}^2 + \{f(0, 1, 1) - a_2 - a_3\}^2 \\ & + f\{(1, 1, 1) - a_1 - a_2 - a_3\}^2 \end{aligned} \quad (2.18)$$

$$\left. \begin{aligned} 4a_1 + 2(a_2 + a_3) &= f(1, 0, 0) + f(1, 1, 0) + f(1, 0, 1) + f(1, 1, 1) \\ 4a_2 + 2(a_1 + a_3) &= f(0, 1, 0) + f(0, 1, 1) + f(1, 1, 0) + f(1, 1, 1) \\ 4a_3 + 2(a_1 + a_2) &= f(0, 0, 1) + f(1, 0, 1) + f(0, 1, 1) + f(1, 1, 1) \end{aligned} \right\}$$

$$\left. \begin{aligned} a_1 &= [3f(1, 0, 0) - f(0, 1, 0) - f(0, 0, 1) + 2\{f(1, 1, 0) + f(1, 0, 1) - f(0, 1, 1)\} + f(1, 1, 1)]/8 \\ a_2 &= [3f(0, 1, 0) - f(1, 0, 0) - f(0, 0, 1) + 2\{f(1, 1, 0) + f(0, 1, 1) - f(1, 0, 1)\} + f(1, 1, 1)]/8 \\ a_3 &= [3f(0, 0, 1) - f(1, 0, 0) - f(0, 1, 0) + 2\{f(1, 0, 1) + f(0, 1, 1) - f(1, 1, 0)\} + f(1, 1, 1)]/8 \end{aligned} \right\} \quad (2.19)$$

$$\therefore \text{Min. } S = f(0, 0, 0)^2 + \{f(1, 0, 0) + f(0, 1, 0) + f(0, 0, 1) - f(1, 1, 1)\}^2/4 \quad (2.20)$$

3. Quantification for qualitative predictors and linear discriminator.

The ordinary linear discrimination function has been computed by using continuous variables only, and not yet given in the case when qualitative or discrete variables are to be taken into account.

However, in such a case, discriminant analysis will be carried out through a certain suitable quantification for these variables as will be stated below, since the linear discriminant function can be constituted by considering these quantified variables as continuous variables tentatively.

Now, let us take up the case where all predictors $X_1, X_2, \dots, X_s$ are categorical variables only and consider an optimum quantification for these variables.

The actual procedure is schematically shown in Table 2, to explain the quantification process.

In Table 2, let the category size of each variable be $k_1, k_2, \dots, k_s$ and $(x_{11}, \dots, x_{1k_1}), (x_{21}, \dots, x_{2k_2}), \dots, (x_{s1}, \dots, x_{sks})$ be the sets of quantification result for category sets $(C_{11}, \dots, C_{1k_1}), (C_{21}, \dots, C_{2k_2}), \dots, (C_{s1}, \dots, C_{sks})$ respectively, then the total sample size n is divided into n_A (sample size in the case $Y \in A$) and n_B (sample size in the case $Y \in B$) and check mark $\vee$ can be assigned to any one category of each predictor by the dependent data.

If quantification is carried out by an appropriate method, their totals will be attainable as shown in Table 2.

Therefore, we put each mean value and variance as

$$\begin{aligned} \bar{X}_{Ai} &= \sum_j X_{ijA}/n_A, \quad \bar{X}_{Bi} = \sum_j X_{ijB}/n_B \\ s^2_{Ai} &= \sum_j (X_{ijA} - \bar{X}_{Ai})^2/n_A, \\ s^2_{Bj} &= \sum_i (X_{ijB} - \bar{X}_{Bi})^2/n_B \quad (i=1, 2, \dots, s) \end{aligned} \quad (3.1)$$

then we have squared mean distance relative to variance within two cases as shown by the next formula.

$$\eta^2_{AB}(i) = n_A n_B (\bar{X}_{Ai} - \bar{X}_{Bi})^2 / (n_A s^2_{Ai} + n_B s^2_{Bi}) \quad (3.2)$$

and $\eta^2_{AB}(i) (i=1, 2, \dots, s)$ is essentially proportional to the correlation ratio between the two cases A and B for a certain predictor X_i .

To attain the optimum quantification in a statistical sense, two approaches at least can be considered as follows:

(i) If we put $\mathbf{W}_s, \mathbf{B}_s$ as $s \times s$ variance-covariance matrix of predictors $X_1, X_2, \dots, X_s$ within two cases $Y \in A$ and $Y \in B$, and between these two cases respectively, as has often been used in the field of Statistical Analysis of Variance, then $\text{tr}(\mathbf{W}_s^{-1} \mathbf{B}_s)$ is a fundamentally proportional quantity to the generalized relative squared distance defined by P. C. Mahalanobis already, so the first approach is to maximize $\text{tr}(\mathbf{W}_s^{-1} \mathbf{B}_s)$ in regard to x_{ij} .

(ii) The second approach is to maximize $\eta^2_{AB}(i)$ in regard to each $X_{ij} (i=1, 2, \dots, s)$. Statistical dependency or relation between X_i and X_j is taken into consideration in the first approach but not in the second one; so, if possible, (i) is more preferable than (ii).

Table 2. Schematic illustration of actual summarizing process of dependent samples.

Predictor		X_1	X_2		X_s	Sum of assigned values
Category		$C_{11} \cdots \cdots C_{1k_1}$	$C_{21} \cdots \cdots C_{2k_2}$		$C_{s1} \cdots \cdots C_{sks}$	
assigned value		$x_{11} \cdots \cdots x_{1k_1}$	$x_{21} \cdots \cdots x_{2k_2}$		$x_{s1} \cdots \cdots x_{sks}$	
Sample No. (A)	1	✓	✓	⋮	✓	$\sum_i x_{i1A}$
	2	✓	✓	⋮	✓	$\sum_i x_{i2A}$
	3	✓	✓	⋮	✓	$\sum_i x_{i3A}$
	⋮	⋮	⋮	⋮	⋮	⋮
	n_A	✓	✓	⋮	✓	$\sum_i x_{in_A A}$
Total (sum)		$m_{11} \cdots \cdots m_{1k_1}$	$m_{21} \cdots \cdots m_{2k_2}$		$m_{s1} \cdots \cdots m_{sks}$	$(\sum_i \sum_j x_{ijA})$
Sample No. (B)	1	✓	✓	⋮	✓	$\sum_i x_{i1B}$
	2	✓	✓	⋮	✓	$\sum_i x_{i2B}$
	3	✓	✓	⋮	✓	$\sum_i x_{i3B}$
	⋮	⋮	⋮	⋮	⋮	⋮
	n_B	✓	✓	⋮	✓	$\sum_j x_{in_B B}$
Total (sum)		$n_{11} \cdots \cdots n_{1k_1}$	$n_{21} \cdots \cdots n_{2k_2}$		$n_{s1} \cdots \cdots n_{sks}$	$(\sum_i \sum_j x_{ijB})$

Remark: $\sum_i x_{ijA} (j=1, \dots, n_A)$ and $\sum_i x_{ijB} (j=1, \dots, n_B)$ show the summations of assigned values in the columns for the cases of A and B respectively.

However, an analytical procedure to maximize $\text{tr}(\mathbf{W}^{-1}\mathbf{B})$ in respect to all variables $x_{ij} (j=1, 2, \dots, k_i, i=1, 2, \dots, s)$ is very difficult to attain and since results essentially equivalent to those by the first approach are expected at least from constituting an ordinary linear discriminant function after quantification for each predictor X_i by use of the second approach we will make use of this latter.

The following results have already been reported by C. Hayashi (1952) and S. Chino (1963), to maximize $\eta^2_{AB}(i)$ in respect to x_{ij} :

$$\frac{\partial \eta^2_{AB}(i)}{\partial x_{ij}} = 0 \quad (j=1, 2, \dots, k_i) \quad (3.3)$$

$$\text{condition 1. } \bar{X}_i = (n_A \bar{X}_{Ai} + n_B \bar{X}_{Bi}) / (n_A + n_B) = 0$$

$$\text{condition 2. } \sum_{j=1}^{k_i} m_{ij} = n_A, \quad \sum_{j=1}^{k_i} n_{ij} = n_B$$

solution:

$$x_{ij} = C \left(\frac{m_{ij}}{n_A} - \frac{n_{ij}}{n_B} \right) / (m_{ij} + n_{ij}) \quad (3.4)$$

$$(j=1, 2, \dots, k_i, i=1, 2, \dots, s)$$

where C = arbitrary const. and, m_{ij} n_{ij} are such

numbers as given in Table 2.

After quantification for each category by using the above solution, we can compute sample variance $\text{Var}(X_i)$ and sample covariance $\text{Cov.}(X_i, X_j)$ as follows:

$$\begin{aligned} \text{Var}(X_i) &= \left\{ \sum_j (X_{ijA} - \bar{X}_{iA})^2 \right. \\ &\quad \left. + \sum_j (X_{ijB} - \bar{X}_{iB})^2 \right\} / (n_A + n_B) \equiv V_{ii} \\ \text{Cov.}(X_i, X_j) &= \left\{ \sum_k (X_{ikA} - \bar{X}_{iA})(X_{jkA} - \bar{X}_{jA}) \right. \\ &\quad \left. + \sum_k (X_{ikB} - \bar{X}_{iB})(X_{jkB} - \bar{X}_{jB}) \right\} / \\ &\quad (n_A + n_B) \equiv V_{ij} \end{aligned} \quad (i, j=1, 2, \dots, s) \quad (3.5)$$

And if we let the linear discriminant function be

$$I = w_1 X_1 + w_2 X_2 + \dots + w_s X_s \quad (3.6)$$

as in the case containing continuous variables only, then the coefficient vector $(w_1, w_2, \dots, w_s)$ can be obtained by solving the next equation:

$$\begin{pmatrix} V_{11} & V_{12} & \cdots & V_{1s} \\ V_{21} & V_{22} & \cdots & V_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ V_{s1} & V_{s2} & \cdots & V_{ss} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_s \end{pmatrix} = \lambda \begin{pmatrix} \bar{X}_{A1} - \bar{X}_{B1} \\ \bar{X}_{A2} - \bar{X}_{B2} \\ \vdots \\ \bar{X}_{As} - \bar{X}_{Bs} \end{pmatrix} \quad (3.7)$$

where λ is an arbitrary constant.

To select effective predictors in discrimination, the following procedure has been proposed by R. G. Miller (1962),

The first predictor $X_{(1)}$ must satisfy the condition $\text{Mix. } \gamma^2_{AB}(i)$, and the successive effective predictors must satisfy the condition $\text{Max tr}(\mathbf{W}^{-1} \mathbf{B})$ sequentially under the assumption that both $\mathbf{W}$ and $\mathbf{B}$ are computed only by taking up the predictors adopted in each step of this sequential process.

The above-mentioned procedures can also be applied to the case of quantified predictor variables as will be mentioned below.

4. Application Examples of the theoretical results obtained above

(i) Discrimination between heavy rainfall and not heavy rainfall (including no rainfall) based on the state of the low level Jet.

The June-August data, which are necessary for predicting the rainfall state at the Hokuriku district, have been obtained for the period 1956–1959 by S. Miyazawa and I. Maeda, and the next notations and assumptions are adopted here:
Event $Y \in A$ = Daily maximum rainfall is equal to or greater than 100mm at any station in the Hokuriku district.

Event $Y \in B$ = Any possible cases other than $Y \in A$.

X_1 = Variable is equal to 1 or 0 corresponding to the existence or non-existence of the low-level Jet at Wajima respectively.

(This existence can be recognized by

a conspicuous maximum wind velocity in the lower wind profile)

X_2 = Similar variable to above at Yonago.

X_3 = Variable is equal to 1 or 0 corresponding to the case that the mean Showalter's stability index is less than or equal to 2°C in Akita, Yonago and Wajima. Variable showing the mean values of Showalter's Stability index may be obtained by a similar process even when we make use of any other three points.

Summarized contingency tables are shown in Table 3, to show the various corresponding relations among Y , X_1 , X_2 and X_3 , and sample estimates of parameters are computed by using Table 3 and shown in Table 4.

Computation results are obtained for example as follows:

$$\hat{\pi}_1 = \frac{n_A}{n} = 0.417 \quad \hat{\pi}_2 = 1 - \hat{\pi}_1 = 0.583$$

$$\hat{p}_{A1} = 32/50 = 0.640 \quad \hat{q}_{A1} = 1 - \hat{p}_{A1} = 0.360$$

$$\hat{p}_{B1} = 29/70 = 0.414 \quad \hat{q}_{B1} = 1 - \hat{p}_{B1} = 0.586$$

$$\tilde{\rho}(Y, X_1) = \frac{18 \times 29 - 41 \times 32}{\sqrt{59 \times 61 \times 50 \times 70}} = -0.223$$

$$\tilde{\rho}(X_1, X_2) = \frac{34 \times 39 - 25 \times 22}{\sqrt{59 \times 61 \times 56 \times 64}} = 0.216$$

$$\tilde{\rho}_{12(A)} = \frac{14 \times 21 - 7 \times 8}{\sqrt{21 \times 29 \times 22 \times 28}} = 0.389$$

$$\tilde{\rho}_{12(B)} = \frac{23 \times 21 - 11 \times 15}{\sqrt{38 \times 32 \times 34 \times 36}} = 0.261$$

(a) Case 1. $s=2$

To predict discriminately the case A or the case B of Y by use of two predictors X_1 and X_2 , it is necessary to obtain the discriminator I by the following computations:

$$\text{const.} = \log(\hat{q}_{A1}/\hat{q}_{B1}) + \log(\hat{q}_{A2}/\hat{q}_{B2}) = -1.0666$$

Table 3. Contingency tables showing corresponding relations among Y , X_1 , X_2 and X_3 .

$Y \backslash X_1$	0	1	Total
A	18	32	50
B	41	29	70
Total	59	61	120

$Y \backslash X_2$	0	1	Total
A	16	34	50
B	40	30	70
Total	56	64	120

$Y \backslash X_3$	0	1	Total
A	21	29	50
B	50	20	70
Total	71	49	120

$X_2 \backslash X_1$	0	1	Total
0	34	22	56
1	25	39	64
Total	59	61	120

$X_3 \backslash X_1$	0	1	Total
0	40	31	71
1	19	30	49
Total	59	61	120

$X_3 \backslash X_2$	0	1	Total
0	38	33	71
1	18	31	49
Total	56	64	120

Case $Y \in A$

$X_2 \backslash X_1$	0	1	Total
0	14	8	22
1	7	21	28
Total	21	29	50

$X_3 \backslash X_1$	0	1	Total
0	17	21	38
1	4	8	12
Total	21	29	50

$X_3 \backslash X_2$	0	1	Total
0	16	22	38
1	6	6	12
Total	22	28	50

Case $Y \in B$

$X_2 \backslash X_1$	0	1	Total
0	23	11	34
1	15	21	36
Total	38	32	70

$X_3 \backslash X_1$	0	1	Total
0	21	12	33
1	17	20	37
Total	38	32	70

$X_3 \backslash X_2$	0	1	Total
0	19	14	33
1	15	22	37
Total	34	36	70

Table 4. Statistical quantities and estimates of parameters

n	n_A	n_B	$\hat{\pi}_1$	$\hat{\pi}_2$
120	50	70	0.417	0.583

i	$\hat{p}_{Ai}$	$\hat{q}_{Ai}$	$\hat{p}_{Bi}$	$\hat{q}_{Bi}$	$\tilde{\rho}(Y, X_i)$
1	0.640	0.360	0.414	0.586	-0.2226
2	0.680	0.320	0.429	0.571	-0.2485
3	0.580	0.420	0.286	0.714	-0.2952

		$\tilde{\rho}(X_i, X_j)$			$\tilde{\rho}_{ij(A)}$			$\tilde{\rho}_{ij(B)}$		
$i \backslash j$		1	2	3	1	2	3	1	2	3
1	1	1.0000	0.2161	0.1727	1.0000	0.3886	0.0987	1.0000	0.2607	0.1773
2	2		1.0000	0.1654		1.0000	0.0038		1.0000	0.1701
3	3			1.0000			1.0000			1.0000

Remark:

 n : total sample size $\hat{p}_{Ai}$, $\hat{q}_{Ai}$, etc. : sample estimates of probability $\tilde{\rho}(\quad)$ etc. : sample correlation coefficients between two variables shown in brackets

$$f(1, 1) = \log \left\{ 1 + \tilde{\rho}_{12(A)} \sqrt{\frac{\hat{q}_{A1} \hat{q}_{A2}}{\hat{p}_{A1} \hat{p}_{A2}}} \right\} - \log \left\{ 1 + \tilde{\rho}_{12(B)} \sqrt{\frac{\hat{q}_{B1} \hat{q}_{B2}}{\hat{p}_{B1} \hat{p}_{B2}}} \right\} = -0.1237$$

$$f(1, 0) = \log \left\{ 1 - \tilde{\rho}_{12(A)} \sqrt{\frac{\hat{q}_{A1} \hat{p}_{A2}}{\hat{p}_{A1} \hat{q}_{A2}}} \right\}$$

$$-\log \left\{ 1 - \tilde{\rho}_{12(B)} \sqrt{\frac{\hat{q}_{B1} \hat{p}_{B2}}{\hat{p}_{B1} \hat{q}_{B2}}} \right\} = -0.2405$$

$$f(0, 1) = \log \left\{ 1 - \tilde{\rho}_{12(A)} \sqrt{\frac{\hat{p}_{A1} \hat{q}_{A2}}{\hat{q}_{A1} \hat{p}_{A2}}} \right\}$$

$$-\log \left\{ 1 - \tilde{\rho}_{12(B)} \sqrt{\frac{\hat{p}_{B1} \hat{q}_{B2}}{\hat{q}_{B1} \hat{p}_{B2}}} \right\} = -0.1473$$

$$f(0, 0) = \log \left\{ 1 + \tilde{\rho}_{12(A)} \sqrt{\frac{\hat{p}_{A1} \hat{p}_{A2}}{\hat{q}_{A1} \hat{q}_{A2}}} \right\}$$

$$-\log \left\{ 1 + \tilde{\rho}_{12(B)} \sqrt{\frac{\hat{p}_{B1} \hat{p}_{B2}}{\hat{q}_{B1} \hat{q}_{B2}}} \right\} = 0.3888$$

$$a_1 = \frac{2f(1, 0) - f(0, 1) + f(1, 1)}{3} = -0.1525$$

$$a_2 = \frac{2f(0, 1) - f(1, 0) + f(1, 1)}{3} = -0.0592$$

$$w_1 = \log(\hat{p}_{A1} \hat{q}_{B1} / \hat{p}_{B1} \hat{q}_{A1}) + a_1 = 0.7691$$

$$w_2 = \log(\hat{p}_{A2} \hat{q}_{B2} / \hat{p}_{B2} \hat{q}_{A2}) + a_2 = 0.9822$$

$$\text{Min. } S = f(0, 0)^2 + \frac{\{f(1, 0) + f(0, 1) - f(1, 1)\}^2}{3} = 0.033$$

To verify the prediction accuracy of the exact discriminator I defined by (2.8) and of the approximate linear discriminator for the dependent data, the correspondence between discriminated results and actual cases is given by Table 5.

Sample means $\bar{I}_A$ and $\bar{I}_B$ are obtained by

$$\bar{I}_A = \text{Const.} + w_1 \times \hat{p}_{A1} + w_2 \times \hat{p}_{A2} = 0.0936$$

$$\bar{I}_B = \text{Const.} + w_1 \times \hat{p}_{B1} + w_2 \times \hat{p}_{B2} = -0.3269$$

$$\bar{I} = (n_A \bar{I}_A + n_B \bar{I}_B) / (n_A + n_B) = -0.1517$$

Theoretically, the value of $\bar{I}$ is expected to be equal to 0 by (2.8), but the actual value of $\bar{I}$ is slightly different from 0 by computational error.

So we have the next discrimination rule by neglecting the approximation error Min. S

Actual value of $I > -0.1517$

.....case $Y \in A$ is predicted.

Actual value of $I < -0.1517$

.....case $Y \in B$ is predicted.

In Table 5, the percentage of success and skill score are computed in the usual way as follows:

For example, in the case of an approximate linear discriminator,

percentage of success

$$= \frac{n_{11} + n_{22}}{n} = \frac{33 + 48}{120} = 0.68$$

$$\text{skill score} = \frac{n(n_{11} + n_{22}) - (n_A^2 + n_B^2)}{n^2 - (n_A^2 + n_B^2)} = 0.31$$

If the approximation error Min.S is to be taken into account, we have $\sqrt{\text{Min. S}/2} \doteq 0.091$ as the mean standard error, so the range $(-0.1517 + 0.091, -0.1517 - 0.091)$ may be regarded as a kind of doubtful region.

Table 5. Contingency tables of verification in the prediction of rainfall at the Hokuriku district ($s=2$).

(a) Exact discriminator

Pred \ Actual	A	B	Total
A	34	16	50
B	21	49	70
Total	55	65	120

Percentage of success = 0.70

Skill score = 0.34

(b) Approximate discriminator

Pred \ Actual	A	B	Total
A	33	17	50
B	22	48	70
Total	55	65	120

Percentage of success = 0.68

Skill score = 0.31

(b) Case 2. $s=3$

To predict Y by use of the three predictors X_1 , X_2 and X_3 similarly to Case 1, the next computations are performed:

$$\text{const.} = \sum_{i=1}^3 (\hat{q}_{Ai} / \hat{q}_{Bi}) = -1.5976$$

$$\tilde{\rho}_{123(A)} = \frac{\tilde{E}_A(X_1 - \hat{p}_{A1})(X_2 - \hat{p}_{A2})(X_3 - \hat{p}_{A3})}{\sqrt{\hat{p}_{A1} \hat{q}_{A1} \hat{p}_{A2} \hat{q}_{A2} \hat{p}_{A3} \hat{q}_{A3}}} \doteq 0.0016$$

$$\tilde{\rho}_{123(B)} = \frac{\tilde{E}_B(X_1 - \hat{p}_{B1})(X_2 - \hat{p}_{B2})(X_3 - \hat{p}_{B3})}{\sqrt{\hat{p}_{B1} \hat{q}_{B1} \hat{p}_{B2} \hat{q}_{B2} \hat{p}_{B3} \hat{q}_{B3}}} \doteq 0.0012$$

$$f(1, 1, 1) = \log \{ 1 + \tilde{\rho}_{12(A)} \sqrt{\frac{\hat{q}_{A1} \hat{q}_{A2}}{\hat{p}_{A1} \hat{p}_{A2}}} \}$$

$$\begin{aligned}
& + \tilde{\rho}_{13(A)} \sqrt{\hat{q}_{A1}\hat{q}_{A3}/\hat{p}_{A1}\hat{p}_{A3}} + \tilde{\rho}_{23(A)} \sqrt{\hat{q}_{A2}\hat{q}_{A3}/\hat{p}_{A2}\hat{p}_{A3}} \\
& + \tilde{\rho}_{123(A)} \sqrt{\hat{q}_{A1}\hat{q}_{A2}\hat{q}_{A3}/\hat{p}_{A1}\hat{p}_{A2}\hat{p}_{A3}} \Big\} \\
& - \log \left\{ 1 + \tilde{\rho}_{12(B)} \sqrt{\hat{q}_{B1}\hat{q}_{B2}/\hat{p}_{B1}\hat{p}_{B2}} \right. \\
& + \tilde{\rho}_{13(B)} \sqrt{\hat{q}_{B1}\hat{q}_{B3}/\hat{p}_{B1}\hat{p}_{B3}} + \tilde{\rho}_{23(B)} \sqrt{\hat{q}_{B2}\hat{q}_{B3}/\hat{p}_{B2}\hat{p}_{B3}} \\
& \left. + \tilde{\rho}_{123(B)} \sqrt{\hat{q}_{B1}\hat{q}_{B2}\hat{q}_{B3}/\hat{p}_{B1}\hat{p}_{B2}\hat{p}_{B3}} \right\} \doteq -0.4656
\end{aligned}$$

similarly

$$f(1, 1, 0) = 0.0094, \quad f(1, 0, 1) = -0.2724$$

$$f(0, 1, 1) = -0.4303, \quad f(1, 0, 0) = -0.3038$$

$$f(0, 1, 0) = 0.1149, \quad f(0, 0, 1) = 0.8180$$

$$f(0, 0, 0) = 0.3165$$

and so we have

$$a_1 = -0.2483, \quad a_2 = -0.1159, \quad a_3 = 0.0947$$

$$w_1 = 0.6730, \quad w_2 = 0.9255, \quad w_3 = 1.3338$$

$$\text{Min. } S = 0.1048, \quad \bar{I}_A = 0.2361, \quad \bar{I}_B = -0.5410$$

$$\bar{I} = -0.2172$$

The next discrimination rule can be approximately induced:

Actual $I > -0.2172$case $Y \in A$ is predicted.

Actual $I < -0.2172$case $Y \in B$ is predicted.

As in Case 1 ($s=2$), the correspondence between discriminated result and actual case is given by Table 6, and the percentages of success and skill score are obtained in the same way.

Table 6. Contingency tables of verification in the prediction of rainfall at the Hokuriku district ($s=3$).

(a) Exact discriminator

Pred \ Actual	A	B	Total
A	37	13	50
B	19	51	70
Total	56	64	120

Percentage of success = 0.73

Skill score = 0.42

(b) Approximate discriminator

Pred \ Actual	A	B	Total
A	36	14	50
B	19	51	70
Total	55	65	120

Percentage of success = 0.72

Skill score = 0.40

Certainly, the success rate of prediction is larger in Case 2 than in Case 1 for the dependent data, as is expected from a consideration of predictors, and there is almost no difference between exact discriminator and approximate linear discriminator so long as the success rate of prediction is concerned. However, it is not always sufficient in the present forecast routine, and numerous effective predictors must be taken up in this way to attain to practical accuracy.

(ii) Discrimination of two modes of snowslide by means of various vegetational and geomorphological conditions.

Snowslides can be conventionally classified into avalanche (Nadare) and snow-crack (Yukiwareme).

Relevant data have been obtained by graphical representation of the aerial photographic observation projected in 1968, by the Meteorological (Counter measure) Committee of Express Highway Research Foundation of Japan and they are used in this paper with the permission of the chief of the above Committee.

To have a discriminator for the snowslide, the following notations and assumptions are conventionally adopted:

To make it available for actual use, the decision rule concerning the occurrence or non-occurrence of snowslide at any given slope must of course be established so as to make possible a preventive measure, but such data have not been obtained so far.

Event $Y \in A$ = avalanche occurred on a mountain slope.

Event $Y \in B$ = snow crack occurred in part of a mountain slope.

The total number of factors is equal to 63 and the sample size $N=516$, which were measured by T. Ogawa and the computation was carried out by M. Hirose using these possible factors.

However, in this paper, ten factors are selected from among them as empirically effective ones suggested by several experts, to show only a computational example of theoretical result mentioned in Section 3.

Explanation of the ten selected factors (predictors) and their category definitions are given in Table 7.

Sample sizes are as follows:

Total sample size $n = n_A + n_B = 516$

Sample size of case $Y \in A$: $n_A = 123$

Sample size of case $Y \in B$: $n_B = 393$

Table 7. Explanation of the selected geomorphological and vegetational factors and each category.

Factor No. (Predictor)	Explanation of Factor	Category	
		size (ki)	Categorical classification
X_1	Outline of slope	6	$C_{11} = \nabla$ $C_{12} = \square$ $C_{13} = \triangle$ $C_{14} = \nabla$ $C_{15} = \nabla$ $C_{16} = \nabla$
X_2	Longitudinal profile of slope	4	$C_{21} = \text{flat}$, $C_{22} = \text{concave}$ $C_{23} = \text{convex}$, $C_{24} = \text{mixed type uneven}$
X_3	Traverse profile of slope	4	$C_{31} = \text{flat}$, $C_{32} = \text{concave}$ $C_{33} = \text{convex}$, $C_{34} = \text{mixed type uneven}$
X_4	Dimensions of slope	3	$C_{41} = \text{narrow}$, $C_{42} = \text{moderate}$ $C_{43} = \text{wide}$
X_5	Snow-cover	3	$C_{51} = \text{light}$, $C_{52} = \text{moderate}$ $C_{53} = \text{heavy}$
X_6	Wind direction inferred from Cornice	4	$C_{61} = \text{unknown}$, $C_{62} = \text{upward}$ $C_{63} = \text{downward}$, $C_{64} = \text{sideward}$
X_7	Kind of forest trees (Upper part of slope)	3	$C_{71} = \text{broad leaf}$, $C_{72} = \text{needle leaf}$ $C_{73} = \text{mixed}$
X_8	Height of forest trees	2	$C_{81} = \text{lower}$, $C_{82} = \text{higher}$
X_9	Density of forest trees	4	$C_{91} = \text{sparse}$, $C_{92}, C_{93} = \text{intermediate}$ $C_{94} = \text{dense}$
X_{10}	Land-surface condition or vegetation	5	$C_{101} = \text{naked rocks}$, $C_{102} = \text{grassland}$ $C_{103} = \text{intermediate}$, $C_{104} = \text{shrub}$ $C_{105} = \text{forest (high) trees}$

Table 8. Quantification of each category for $X_1, X_2, \dots, X_{10}$.

Factor	Category	Quatification	Factor	Category	Quatification
X_1	C_{11}	0.00	X_3	C_{33}	2.54
	C_{12}	1.38		C_{34}	-2.38
	C_{13}	-0.01	X_4	C_{41}	0.17
	C_{14}	-0.22		C_{42}	-0.33
	C_{15}	-0.51		C_{43}	0.30
	C_{16}	-0.01	X_5	C_{51}	0.00
X_2	C_{21}	0.00		C_{52}	-5.46
	C_{22}	0.96		C_{53}	1.19
	C_{23}	-0.49	X_6	C_{61}	-1.45
	C_{24}	-0.20		C_{62}	0.78
X_3	C_{31}	0.05		C_{63}	0.48
	C_{32}	0.03			

Factor	Category	Quantification	Factor	Category	Quantification
X_6	C_{64}	1.89	X_9	C_{93}	-0.34
X_7	C_{71}	0.73		C_{94}	0.56
	C_{72}	-0.81	X_{10}	C_{101}	0.60
	C_{73}	-0.12		C_{102}	2.54
X_8	C_{81}	0.73		C_{103}	-0.74
	C_{82}	-0.89		C_{104}	-1.61
X_9	C_{91}	0.73		C_{105}	0.16
	C_{12}	-0.64			

Table 9. Statistical computations necessary to constitute discriminator I for snowslide.

(a) Sample mean value

i	1	2	3	4	5	6	7	8	9	10
$\bar{X}_{Ai}$	5.33^{-2}	4.15^{-4}	2.25^{-2}	9.37^{-3}	7.96^{-1}	2.32^{-1}	9.38^{-2}	8.05^{-2}	1.08^{-1}	1.38^{-1}
$\bar{X}_{Bi}$	-1.70^{-1}	-1.33^{-3}	-7.18^{-2}	-2.99^{-2}	-2.54^{-1}	-7.40^{-1}	-3.00^{-1}	-2.57^{-1}	-3.46^{-1}	-4.42^{-1}

(b) Sample variance and covariance matrix

$j \backslash i$	1	2	3	4	5	6	7	8	9	10
1	6.50^4	6.47^5	5.23^5	4.97^4	4.26^5	2.42^5	4.91^4	-7.88^2	-6.10^2	4.24^1
2		6.29^6	6.30^6	6.02^5	5.26^6	2.46^6	5.24^5	-4.36^4	-1.16^4	7.74^2
3			6.39^6	5.03^5	4.85^6	2.20^6	5.07^5	-5.19^4	-1.44^4	2.18^3
4				1.32^5	4.65^5	2.18^5	4.55^4	-4.15^3	-4.80^2	-3.27^2
5					5.30^6	1.17^6	4.18^5	-2.06^4	-6.31^3	2.20^3
6						2.44^6	2.26^5	-2.60^4	-1.10^4	-1.19^3
7							4.92^4	-3.41^3	-9.55^2	1.95^2
8								5.00^3	1.62^2	-3.08^1
9									5.91^2	1.29^2
10										7.15^2

(c) Discriminator coefficient

i	1	2	3	4	5	6	7	8	9	10
w_i	8.45^{-3}	2.78^{-4}	-7.89^{-4}	3.82^{-4}	-3.13^{-4}	-4.26^{-3}	2.04^{-2}	-1.38^{-2}	-9.14^{-2}	-4.75^{-3}

Remark: For example, 5.33^{-2} , -1.70^{-1} , 6.50^4 and 8.45^{-3} are 5.33×10^{-2} , -1.70×10^{-1} , 6.50×10^4 and 8.45×10^{-3} respectively.

Table 10. Contingency tables of verification (Snowside).

(a) $s=10, X$

Pred \ Actual	A	B	Total
A	300	93	393
B	10	113	123
Total	310	206	516

Percentage of success = 81%
Skill score = 0.34

(b) $s=52, X$

Pred \ Actual	A	B	Total
A	341	52	393
B	6	117	123
Total	347	169	516

Percentage of success = 90%
Skill score = 0.58

Quantified results of categorical variables are given in Table 8, and the necessary statistical values are summarized in Table 9. As is shown in Table 9, X_7 is the most effective factor for our discrimination, and discriminator I can be expressed as

$$I = 0.00845X_1 + 0.00028X_2 - 0.00079X_3 \\ + 0.00038X_4 - 0.00031X_5 - 0.00426X_6 \\ + 0.0204X_7 - 0.0138X_8 - 0.0914X_9 \\ - 0.00475X_{10}$$

by use of Table 9. (c).

In this case, the discrimination rule is determined as in the following:

Actual values of $I > 0$ case $Y \in A$ is predicted.

Actual values of $I < 0$ case $Y \in B$ is predicted.

Verification of the results of snowslide discrimination can be performed based on the above discrimination rule for dependent data, and a part of the summarized results is shown in Table 10.

As is seen in Table 10, the result of snowslide discrimination is not always satisfactory for practical use, since the sufficiently effective predictors may not be always selected from the possible factors of size 63 in this example.

However, judging from the result shown in Table 10 (b), it will be possible to obtain a practically useful discrimination procedure through a screening method similar to the one that R.G.

Miller proposed for the construction of quantitative linear discriminant function.

5. Concluding remark

Two statistical procedures of discrimination have been given in this paper by use of several qualitative or categorical variables (predictors) with their applications to meteorological problems.

Speaking generally of the essential points, in order to predict either of two mutually exclusive events, each variable taken up as an effective predictor has to be specified as a simple qualitative variable having a tendency to be equal to 1 when it is more suitable for one case (Event $Y \in A$) and to (equal to) 0 for the other case (Event $Y \in B$). So, there is no need to have three or more values or categories for each variable, so long as the two events only are considered.

Consequently, discriminator I based on the likelihood ratio of two multi-variate binomial distributions has a sufficient generality for discrimination of the two exclusive cases of predictand, and we may consider the sequential approach to constitute such discriminator in a stepwise manner for discrimination of the three or more exclusive cases in view of fundamental standpoint as follows:

For example, let the possible cases of Y be $A_1, A_2, \dots, A_k$ (they are assumed generally to be mutually exclusive and exhaustive), then we can have the following stepwise discriminators:

The 1st discriminator for the one Event $Y \in A$, and the other Event $Y \in \sum_{i=2}^k A_i$

The 2nd discriminator for the one Event $Y \in A_2$ and the other Event $Y \in \sum_{i=3}^k A_i$

The $k-1$ -th discriminator for the one Event $Y \in A_{k-1}$ and the other last Event $Y \in A_k$

And for each stepwise discriminator, we can devise so as to take predictor variables having tendencies to equal to 1 when they are suitable for the one Event and to 0 for the other exclusive Event.

The procedure combining the categorical variables of the simultaneous equation (3.7) to have the discrimination coefficients whose methodological originality is to be ascribed to T. Hayashi and S. Chino, was first applied in this paper to meteorological problems in a slightly different way from the usual method.

Acknowledgement

The author wishes to express his sincere thanks to the staff members of the Forecast Research Laboratory in M.R.I. for their valuable discussions and especially to Mr. M. Hirose for his programming and computation throughout this. He is also indebted to Miss J. Hashiba for her unstinted assistance. The author would like to dedicate this paper to Prof. E. Fukui in commemoration of his retirement from Tokyo Educational University.

Appendix

[Proof of Theorem referred to in P. 3]

For this linear approximation, the residual sum of square error can be formally written as

$$S = \sum_{x_1=0}^1 \sum_{x_2=0}^1 \cdots \sum_{x_s=0}^1 \left\{ f(x_1, x_2, \cdots, x_s) - \sum_{i=1}^s a_i x_i \right\}^2$$

and S is a function of $(a_1, a_2, \cdots, a_s)$

In order to minimize S in respect to $a_1, a_2, \cdots, a_s$, the necessary conditions

$$\partial S / \partial a_i = 0 \quad (i=1, 2, \cdots, s)$$

are explicitly shown by the next simultaneous linear equation:

$$\begin{aligned} & 2^{s-1}a_1 + 2^{s-2}(a_2 + a_3 + \cdots + a_s) \\ &= \sum_{x_2=0}^1 \sum_{x_3=0}^1 \cdots \sum_{x_s=0}^1 f(1, x_2, x_3, \cdots, x_s) \\ & 2^{s-1}a_2 + 2^{s-2}(a_1 + a_3 + \cdots + a_s) \\ &= \sum_{x_1=0}^1 \sum_{x_3=0}^1 \cdots \sum_{x_s=0}^1 f(x_1, 1, x_3, \cdots, x_s) \\ & \cdots \cdots \cdots \\ & 2^{s-1}a_s + 2^{s-2}(a_1 + a_2 + \cdots + a_{s-1}) \\ &= \sum_{x_1=0}^1 \sum_{x_2=0}^1 \cdots \sum_{x_{s-1}=0}^1 f(x_1, x_2, \cdots, x_{s-1}, 1) \end{aligned}$$

Let the notations $g_1, g_2, \cdots, g_s$ be put as the first, second, and the s -th column in the right multiple summations respectively, then the solution of this simultaneous equation can be easily written as

$$\begin{aligned} a_1 &= 2^{2-s} \left\{ g_1 - \sum_{i=1}^s g_i / (s+1) \right\}, \\ a_2 &= 2^{2-s} \left\{ g_2 - \sum_{i=1}^s g_i / (s+1) \right\}, \\ & \cdots \cdots \cdots \\ a_s &= 2^{2-s} \left\{ g_s - \sum_{i=1}^s g_i / (s+1) \right\} \end{aligned}$$

Consequently, the weight coefficients w_i ($i=1, \cdots, s$) will be shown from (2.8) as follows:

$$\begin{aligned} w_i &= \log(p_{Ai}q_{Bi}/p_{Bi}q_{Ai}) + a_i \\ &= \log(p_{Ai}q_{Bi}/p_{Bi}q_{Ai}) + 2^{2-s} \left\{ g_i - \sum_{i=1}^s g_i / (s+1) \right\} \end{aligned}$$

In the next place, let Min. S be the minimum value of S obtained by substituting these results of $a_1, a_2, \cdots, a_s$, then we have

$$\begin{aligned} \text{Min. } S &= \sum_{x_1=0}^1 \sum_{x_2=0}^1 \cdots \sum_{x_s=0}^1 f(x_1, x_2, \cdots, x_s)^2 - 2 \sum_{i=1}^s a_i g_i \\ &+ 2^{s-1} \left(\sum_{i=0}^s a_i^2 + \sum_{i=0}^{s-1} a_i a_{i+1} \right) \\ &= \sum_{x_1=0}^1 \sum_{x_2=0}^1 \cdots \sum_{x_s=0}^1 f(x_1, x_2, \cdots, x_s)^2 \\ &- 2^{3-s} \left\{ \sum_{i=1}^s g_i^2 - \sum_{i=1}^s g_i^2 / (s+1) \right\} \\ &+ 2^{2-s} \left\{ \left(\sum_{i=1}^s g_i \right)^2 / (s+1)^2 \right. \\ &+ \sum_{i=1}^s g_i^2 - 2 \left(\sum_{i=1}^s g_i \right)^2 / (s+1) \\ &+ s \left(\sum_{i=1}^s g_i \right)^2 / (s+1)^2 \left. \right\} \\ &= \sum_{x_1=0}^1 \sum_{x_2=0}^1 \cdots \sum_{x_s=0}^1 f(x_1, x_2, \cdots, x_s)^2 \\ &+ 2^{2-s} \left\{ \left(\sum_{i=1}^s g_i \right)^2 / (s+1) - \sum_{i=1}^s g_i^2 \right\} \end{aligned}$$

References

- Chino, S., 1963: The relation between correlation ratio and success rate in the classification by the quantification method. *Proc. Inst. Stat. Math.*, 11, 7-24, (in Japanese)
- Dwyer, P.S., 1951: Linear computations. John Wiley and Sons.
- Hayashi, C., 1952: Fundamental concept of the theory of quantification and prediction. *Proc. Inst. Stat. Math.* 7, 43-64. (in Japanese)
- Householder, A.S., 1953: *Principles of numerical analysis*. McGraw-Hill, New York.
- Krishnamoorthy, A.S., 1951: Multi-variate binomial and Poisson distribution. *Sankhya*, 11, 117-124.
- Maruyasu, T. and his collaborators, 1964: A study on landslide at the neighbour region of Yoshino river by aerial photographs. *Sokuryo* 6, 8-13 (in Japanese)
- Miller, R.G., 1962: Statistical prediction by discriminant analysis. *Met. Monogr.*, 4, 25. Amer.

Met. Soc.

Suzuki, E., 1964: Categorical prediction schemes of rainfall types by discriminant analysis. *Papers in Met. and Geophys.* **15**, 119-160.

Suzuki, E., 1966: Correlation analysis containing discrete variables used in certain meteorological problems. *Papers in Met. and Geophys.* **17**, 10-30.

定性的な変量による統計的判別理論とその気象への応用

鈴木 栄 一

気象研究所

これまでの判別関数による気象予測は R. G. Miller らと筆者により研究されてきたが、それは数値として得られる連続変量の総合化したものであった。ここでは大雨や大雪、なだれなどの発生有無を事前に予測する場合、その手がかりとなる因子が定性的な変量（たとえば下層ジェットや前線の有無、地形、植生形態を示すもので数値として得られていない変量）である場合の判別方式を開発した。

まず筆者らによって定義された多変量2項分布の一般式から出発し、尤度比の対数を線形近似によって現場でも容易に利用できる形になおし、次に統計的数量化理論を気象予測の目的に合うよう定式化して判別予報式を導びき、こうした2つの方式を気象の具体例に適用してその判別精度を検討した結果、かなり良い成績が得られた。