NOTES AND CORRESPONDENCE

A Note on the False Short-wave Cut-off in a Simple Layer Model for Baroclinic Instability of the Zonal Wind

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1. Introduction

The instability properties of the baroclinic zonal wind with respect to quasi-geostrophic wave disturbances have been studied extensively by many authors, following the heuristic discussions by Charney (1947) and Eady (1949). As is well known, one of the notable features of Charney’s model is a long wave limit to instability due to the variation of the Coriolis parameter with latitude (so-called β-effect) for unstable waves of the first mode, and that of Eady’s model, which does not include the β-term, is a short wave limit to instability.

In attempting a numerical weather prediction with the use of a simple quasi-geostrophic layer model, Phillips (1951) and Thompson (1953) examined the possibility of representing the motions of a baroclinic atmosphere by comparing the behavior of small perturbations in a two-layer model with the results of continuous baroclinic instability theories, and they found a short wavelength cut-off to instability in the two-layer model. This short-wave cut-off has once been considered to be identical with Eady’s criterion. A more complete examination given by Green (1960) and Burger (1962) indicates, however, that there is no short wave limit to instability when the variation of the Coriolis parameter with latitude is retained in the model. The mechanism of the destabilizing effect by the β-term to shorter waves beyond Eady’s criterion is qualitatively explained by Bretherton (1966) in terms of potential vorticity. Thus the short-wave cut-off obtained in the two-layer model including the β-term is spurious.

Then a question arises as to why the false cut-off to short waves appears in a simple layer model. It seems, of course, that the cut-off comes from the truncation error due to a coarse vertical resolution of the atmosphere as pointed out by Brown (1969), but it should be required to give full details from various points of view.

In the present paper, the difference between the continuous model and the layer model is explained in terms of the phase velocity of unstable waves mathematically, rather than physically, as an algebraic restriction on the solutions of the frequency equation.

2. Frequency equation in a two-layer model

The finite difference approximations of the linearized perturbation equations of quasi-geostrophic model applied to a two-layer model yield the frequency equation

\[ c^2 - \left( \frac{2U - \beta}{k^2} \right) \left( \frac{q + 2k^2}{q + k^2} \right) + \left( \frac{U - \beta}{k^2} \right) = 0 \]  \hspace{1cm} (2-1)

where \( U \) denotes the mean zonal velocity at the mid-level, \( k \) wave number, \( \beta \) vertical wind shear, \( \beta \) variation of the Coriolis parameter with latitude, \( q = \frac{f^2}{\rho^2} \) (\( \rho \) measure of static stability, \( \rho \) pressure difference between two layers). For particulars, turn to Hirota (1968).

The eigenvalue \( c \) is given by the solution of the quadratic equation (2-1) as

\[ c = \frac{U - \beta}{2k^2(q + k^2)} \pm \sqrt{\delta} \]  \hspace{1cm} (2-2)

where

\[ \delta = \frac{\beta^2q^2}{4k^2(q + k^2)^2} - \frac{\beta q - k^2}{q + k^2} \]  \hspace{1cm} (2-3)

The eigenvalue \( c \) is complex when \( \delta \) is negative, indicating that the wave is unstable.

(1) \( \beta = 0 \)

When the β-term is excluded from the model, the condition that the discriminant \( \delta \) of the quadratic equation (2-1) shall vanish is evidently given by \( k_{CE}^2 = q \), which indicates that the critical
wave number \( k_{CB} \) depends upon only the static stability \( \sigma \) for a given value of \( f \).

The stability property in this case is then given as

\[
\begin{align*}
    k < k_{CB} : \sigma < 0 & : \text{unstable} \\
    k \geq k_{CB} : \sigma \geq 0 & : \text{stable}
\end{align*}
\]

This result means that the two-layer model without \( \beta \)-term is successful, at least qualitatively, in presenting the stability property of Eady’s model.

(II) \( \beta \neq 0 \)

On the other hand, when the \( \beta \)-term is retained the condition of \( \sigma = 0 \) gives two critical wave numbers \( k_{CB_0} \) and \( k_{CB_1} \). (We assume \( k_{CB_1} < k_{CB_0} \). One of the critical value, \( k_{CB_1} \), is considered to be an approximation to a long wave limit to instability due to the \( \beta \)-effect, which is identical with Charney’s criterion. The other critical value \( k_{CB_0} \) is, however, a false cut-off because it follows from (2-3) that \( k_{CB_0} < k_{CE} \), indicating that the \( \beta \)-term acts to stabilize the short waves.

3. An algebraic restriction on the solutions of the frequency equation

In order to have a deeper insight into the cause of the false short-wave cut-off in the simple layer model, we examine the behavior of solutions to the frequency equation. It follows from (2-1) that the two solutions of the quadratic equation have a relation

\[
c_1 + c_2 = 2U - \frac{\beta q + 2k_0^2}{k_0^2(q + k_0^2)} \quad (3-1)
\]

When either \( \beta = 0 \) or the wave number \( k \) tends to infinity, the equation (3-1) becomes

\[
c_1 + c_2 = 2U \quad (3-2)
\]

Moreover, if the eigenvalues \( c_1 \) and \( c_2 \) are complex conjugates, i.e., \( c_1, c_2 = c_r \pm \imath c_i \), then we have

\[
c_1 + c_2 = 2c_r = 2U \quad (\text{for } \beta = 0 \text{ or } k \to \infty) \quad (3-3)
\]

The algebraic restriction (3-3) means that if an unstable wave exists for sufficiently large \( k \) the phase velocity of the unstable wave must tend to the zonal velocity at the mid-level, so far as the two-layer model is concerned.

Fig. 1 shows a schematic diagram of the phase velocity given by the two-layer model with \( \beta = 0 \), where \( U_t \) and \( U_s \) denote the zonal velocity at the top and at the bottom surface respectively and the heavy broken line and heavy full line denote the phase velocity of unstable wave and that of neutral wave respectively. No discrepancy is found between the behavior of solutions in the two-layer model and that of the continuous model (Eady’s model).

Fig. 2 shows the restriction on the phase velocity of the unstable wave in the two-layer model with
approximately represented when we have a multi-layer model instead of a two-layer model. Then we examine what is the difference between the two-layer model and the multi-layer model.

The leading terms of the frequency equation corresponding to a $N$-layer model with $\beta \neq 0$ are found to be

$$c^n - (N\bar{U} - \beta^*)c^{n-1} + \ldots = 0 \quad (4-1)$$

where $\beta^*$ denotes the $\beta$-term which includes $k$ and $q$, and has a property $\beta^* \to 0$ as $k \to \infty$. Therefore the algebraic restriction on the solutions of this case is

$$\sum_{j=1}^{N} c_j = N\bar{U} \quad k \to \infty \quad (4-2)$$

Suppose that the first two of these $c_j$'s are complex conjugates, i.e., $c_1, c_2 = c_r \pm ic_i$, then (4-2) is rewritten as

$$2c_r + \sum_{j=3}^{N} c_j = N\bar{U} \quad (k \to \infty) \quad (4-3)$$

Accordingly, it is evident that $c_r$ need not tend to $\bar{U}$ as $k \to \infty$ because there are other degrees of freedom in $c_j$ ($j=3, 4, \ldots, N$). In other words, the restriction of the phase velocity of the unstable wave in a multi-layer model becomes weaker compared with that in a two-layer model.

Fig. 5 shows an example of the behavior of solutions in the multi-layer model (in practice, 6-layer model in this figure). The characteristic feature of unstable waves for the continuous model (Fig. 3) in the range of large wave numbers is approximately represented by the multi-layer model in that the phase velocity of the unstable wave tends to $U_s$ and the false cut-off shifts to sufficiently short wavelengths.

In short, we can conclude that the false short-
wave cut-off comes from the algebraic restriction on the solutions of the frequency equation for a simple layer model and so the increase of degree of freedom by the use of a multi-layer model makes the restriction weaker and makes the algebraic solutions to approach the exact solutions.

In the above discussion it is of course another problem whether the use of a quasi-geostrophic model is valid or not in the range of small wavelengths. To answer this question the examination of instability problem with the use of a non-geostrophic model should be required.

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References

傾压不安定に関する層モデルによる短な Short-wave Cut-off について

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