Numerical Experiments of the Airflow over Mountains

1. Uniform current with constant static stability

By Takehiko Furukawa

Meteorological Research Institute, Tokyo
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Abstract

In order to investigate the characteristics of the lee waves, numerical experiments are performed by integrating the equations of the so-called Boussinesq system as an initial value problem for a given topography with rigid upper boundary. Some considerations on the linear theory are presented for a better understanding of the results. For almost all internal Froude number $F_i$, a quasi-steady pattern is obtained. Calculated patterns are discussed in connection with the linear theory, and good agreements are obtained. For relatively large values of the $F_i$, a laminar sinusoidal flow is obtained. On the other hand, for small values of the $F_i$, a S-shaped or Rotor-like flow pattern is obtained, in which a statically unstable region is found to survive. Such a flow may be considered as a result of a nonlinear interference of the waves. Occurrence of such an overturning flow also depends on the height of the mountain. The criterion for the overturning flow seems to be linear dependency in terms of the nondimensional mountain height $D$ and $F_i$, which appears to be consistent with Long's analytical study (1955).

Strong lee-side surface winds are discussed in relation to the simulated lee waves. The possibility of a strong downslope wind or horizontal wind at the lee side of the mountain is stressed.

1. Introduction

How does a fluid behave when it crosses over an obstacle? It has been long considered as one of the most interesting problems in fluid dynamics. Challenges to this subject have been attempted by many investigators and it needs much space simply to list them all.

From a meteorological point of view the so-called “Airflow over a mountain” seems to be one of the most important subjects. Almost all studies have been devoted to the so-called two-dimensional “mountain wave equation (or lee wave equation)”, which is largely characterized by the assumption of non-hydrostatic and infinitely small perturbations in steady state. Attention was focused on the effect of a mountain chain on the strong surface wind in the lee and on influences on flight operations. Recently the energy propagation of disturbances generated by mountains into the upper troposphere or stratosphere has been noticed. The interaction due to wave drag between small-scale disturbances caused by mountains and synoptic-scale disturbances is also an interesting problem.

We shall briefly review the basic linear model and some of the important studies which have contributed greatly to our understanding of the airflow over mountains. Work on the airflow over mountains can date back to Lyra (1943) and Queney (1947, 1948). They considered a simple atmosphere of constant wind and constant static stability (i.e. Lyra-Queney model). Their results threw a fresh light on the problem of the airflow over mountains in spite of the adoption of a simple one-layer model with semi-infinite space, in which the lee wave motion is substantially due to the interference of the continuous harmonics of disturbances.

Scorer (1949) developed Queney’s approach toward a two-layer model. He indicated that the lee wave can be interpreted as a resonance wave which corresponds to one of the eigenvalues of a boundary value problem. A single wave train, which is predominant only in the middle or lower troposphere and never damps out downstream, was simulated. In the immediate vicinity of a finite mountain, however, Scorer’s solution was modified semi-experimentally so as to fit a bottom topography. Wurtele (1953) dealt
with a case of linear shear and showed that much more resonance waves are possible in weak shear, while only a few resonance waves are possible in strong shear. That is, it has been recognized that the lee wave mode depends on the vertical distribution of the general current and static stability.

On this basis Sawyer (1960) solved the lee wave equation semi-numerically for an arbitrary distribution of wind and static stability. Onishi (1965) proposed a numerical scheme to solve the equation by a finite difference method. A natural extension to a three-dimensional lee wave was discussed by many authors.

All the studies mentioned above aimed to solve the linear lee wave equation so as to satisfy the adequate linearized boundary conditions. However, there exists an inevitable inconsistency in a physical sense that the effect of the mountain is generally introduced as if source and sink were located at the horizontal surface beneath the "finite" mountain. The intensity of the source and sink is a priori specified, depending on both the slopes of the mountain and the basic current of the lowest level. Therefore, the behavior of the flow in the vicinity of mountains can not be recognized through the linear theory. A more reasonable explanation of the problem is possible only through the nonlinear approach.

In order to take account of the effect of the bottom topography realistically, Krishnamurti (1964) expressed the steady state lee wave equation system including a nonlinear term by adopting an isentropic coordinate and obtained reasonable nonlinear solutions by a finite difference marching scheme. He applied the Kelvin monotony condition that there is no wave far upstream from the mountain for uniqueness of the solution. This condition seems to be practically useful when it is applied so far upstream, but some doubt is cast on the uniqueness of the solution of the exponential type (Vergeiner, 1971). Vergeiner (1971) proposed a revised operational linear lee wave model by adopting a nonlinear consideration at the bottom topography to avoid an exaggerated lower boundary condition.

A time-dependent approach for the lee wave problem was successfully made by Hirota (1965). He estimated a wave drag due to the steady state lee wave disturbances in nonlinear form. Foldvik and Wurtele (1967) discussed a transient lee wave by integrating the inviscid Boussinesq equation system. Cyclic continuity on the lateral boundary condition was assumed there, so that the time integration was stopped at the time when downstream disturbances were beginning to introduce perturbations through the upstream boundary. Their results show a remarkably strong updraft just above the lee slope, and a total horizontal velocity is negative for some grid points, which is referred to as a lee jump. Further computation could not be performed in their model after the set up of such a jump. It seems that it was not their main intention to fully discuss the property of the steady state resonance wave itself, though the upper rigid boundary was introduced.

It is interesting to note what conditions control the occurrence of the jump and whether the jump will come to develop a violent turbulence. Resonance property and nonlinearity due to the introduction of an increased finite height mountain in the channel with rigid upper boundary have not been fully explained. In these respects, Long's (1953, 1954, 1955) analytical studies and laboratory experiments on the channel flow need to be re-examined and compared with an initial value problem. He suggested some interesting criteria on the property of finite amplitude lee waves in the so-called Long's model. Some considerations on this model will be presented in the next section.

Magata (1969) attempted to simulate some kinds of cloud which are observed above Mt. Fuji by numerical integration of a two-layer model of the Boussinesq system. He stressed that effects of the cooling and heating from the surface play a major role in the deformation of the airflow over mountains. However, his model seems to be not enough to discuss the thermal effect from the ground because of the crude vertical resolutions.

Finally some comments must be given on the adoption of a hydrostatic assumption. Under this approximation extensive work has been performed by many authors as an initial value problem: Hovermale (1965) by adopting $\alpha$-coordinate, Rousseau (1969) with a multi-layer model. Larsen (1966), Houghton and Kasahara (1968), Arakawa and Oobayashi (1968), Arakawa (1968) and Oobayashi (1970) applied a shallow fluid equation to the airflow over mountains. Some
of their interest is directed to nonlinearity. The hydrostatic assumption, however, has some limitation for the validity of the scale or property of the phenomena. This approximation is not adequate when the atmosphere has a strong resonance tendency due to buoyancy or when a topography has a curvature effect, as was indicated by Hovermale (1965) and Gutman (1969)*.

In the light of the lee wave theory mentioned above, the main purposes of this paper are summarized as follows. 1) How are lee waves simulated in a non-hydrostatic and nonlinear equation system, when it is treated as an initial value problem? 2) How does a finite height mountain modify an airflow in the vicinity of a mountain? 3) How about information on a strong lee-side wind, foehn and the so-called Rotor phenomenon?

2. Basic equations and some considerations on the lee wave equation

Physical processes which are essential for the setup of a mountain lee wave may be analyzed intuitively as follows: 1) blocking effect of a mountain against the incident airstream, and a subsequent forced displacement, 2) exertion of a restoring force due to a static stability against the vertical displacement, 3) development of wave motion by the general current and the external and stationary forcing due to the topography. It is advantageous, therefore, to adopt the so-called Boussinesq approximation system in order to simulate such lee wave phenomena. This equation system may be written in the vertical two-dimensional plane as follows:

Equations of motion

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -C_p \frac{\partial \pi}{\partial x} + K_m p^2 u \tag{2.1},
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -C_p \frac{\partial \pi}{\partial z} + \frac{\theta}{\Theta} g + K_m p^2 w \tag{2.2}.
\]

Equation of continuity

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.3}.
\]

Thermodynamic equation

\[
\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + w \frac{\partial \theta}{\partial z} = K_s p^2 \theta \tag{2.4},
\]

where \( u, w \), are the \( x- \) and \( z- \) components of velocity, \( \Theta \) is the homogeneous potential temperature, \( \theta \) is the deviation from the temperature of the adiabatic atmosphere with potential temperature \( \pi = (p/P)^{\kappa} \), Exner function, \( \kappa \) is \( R/C_p \), \( R \) is the gas constant for dry air and \( C_p \) is the specific heat for dry air at constant pressure. \( K_m \) and \( K_s \) are the eddy coefficient for momentum and heat, respectively.

In order to reach a better understanding of the results of the numerical experiments, we shall first investigate the characteristics of the linear lee wave equation which is derived from the Boussinesq system mentioned above. Suppose the total field is composed of a basic state which is a function of \( z \) only and has infinitely small amplitude perturbations superposed upon it, then the dependent variables may be written as follows:

\[
\begin{pmatrix}
    u \\
    w \\
    \pi \\
    \theta
\end{pmatrix}
\equiv
\begin{pmatrix}
    \bar{u}(z) \\
    0 \\
    \bar{\pi}(z) \\
    \bar{\theta}(z)
\end{pmatrix} +
\begin{pmatrix}
    u' \\
    w' \\
    \pi' \\
    \theta'
\end{pmatrix} \tag{2.5}.
\]

If we assume a harmonic solution with wave number \( k \) and with phase velocity \( c \) in the horizontal direction, then infinitely small amplitude perturbations may be expressed as follows:

\[
\begin{pmatrix}
    u' \\
    w' \\
    \pi' \\
    \theta'
\end{pmatrix}
\equiv
\begin{pmatrix}
    U(z) \\
    W(z) \\
    \Pi(z) \\
    T(z)
\end{pmatrix} e^{ik(x-ct)} \tag{2.6}.
\]

In the classical linear lee wave theory the viscous and diabatic process have generally been neglected because of their minor effects. In this case, substituting (2.5) and (2.6) into (2.1)–(2.4) and linearizing them, we obtain an equation for \( W \):

\[
\frac{d^2 W}{dz^2} + \left[ \frac{g \beta}{(\bar{u}-c)^2} - \frac{\bar{u}''}{\bar{u}-c} - \delta k^2 \right] W = 0 \tag{2.7},
\]
where

\[ \beta = \frac{1}{\Theta} \frac{d\Theta}{dz} \]

\[ \tilde{u}'' = \frac{d^2\tilde{u}}{dz^2} \]

\[ \delta = \begin{cases} 1 & \text{non-hydrostatic treatment} \\ 0 & \text{hydrostatic assumption} \end{cases} \]

In (2.7) \( \delta \) is the tracer to check the effect of vertical acceleration due to the buoyancy. As easily seen from (2.7), when the lee wave is treated under the hydrostatic assumption the so-called resonance lee wave can not be included.

The simplified equation (2.7) with \( c=0 \) is considered as a valid approximation for the discussion of the lee wave, as compared with the primitive linear lee wave equation (See Queney et al. (1960)).

First we shall discuss in brief what kind of free waves exist in the absence of mountain. As a first step of lee wave simulation, we shall hereafter treat the case of \( \tilde{u} \) and \( \tilde{u}'' \) with rigid boundary, which will enable us to make easy comparison with the linear theory or Long’s (1953, 1955) finite amplitude analysis as is shown below. We shall assume that the vertical motion vanishes at the upper and lower boundaries, that is,

\[ W = 0 \text{ at } z = 0 \text{ and } z = H \] (2.11),

where \( H \) is the height of the upper rigid boundary. In this case the eigenvalues of (2.7) for the non-hydrostatic treatment are as follows:

\[ C_n = \tilde{u} \pm \sqrt{\frac{g\beta H^2}{n^2\pi^2 + k^2H^2}} \quad (n = 1, 2, \ldots) \] (2.12).

We shall restrict ourselves to a steady state solution. In this case the vertical mode has an upper limit. The horizontal wave number of a wave corresponding to the \( n \)-th vertical mode can be obtained from (2.12) as follows:

\[ k_n = \sqrt{\frac{g\beta}{\tilde{u}^2} - \frac{n^2\pi^2}{H^2}} \quad (n = 1, 2, \ldots, n_1) \] (2.13),

where \( \tilde{u} = g\beta/u^2 \) is the Scorer parameter with no wind shear and \( n_1 \) is the largest positive integer for which \( k_n \) is real. Next we shall define the internal Froude number, which is expressed by

\[ F_i = \frac{\tilde{u}}{\sqrt{g\beta H}} = \frac{1}{lH} \] (2.14).

Long’s (1953) original definition was \( F_i = \tilde{u}/\sqrt{g\beta H} \), where \( \tilde{\beta} = d(d\rho)/dz \). From (2.13) the horizontal wavelength of free waves may be written in terms of \( F_i \) as

\[ \lambda_n = \frac{2\pi H}{\sqrt{F_i^2 - n^2\pi^2}} \quad (n = 1, 2, \ldots, n_1) \] (2.15).

As easily seen from (2.15), when \( F_i \) is larger than \( F_i^* = 1/\pi \), there is no free wave, where \( F_i^* \) is called the critical internal Froude number. When \( F_i \) is smaller than \( F_i^* \), there exist \( n \) free waves for \( F_i^*/(n+1) < F_i < F_i^*/n \). The cases of \( F_i = 1/\pi \) are singular with an infinitely long wavelength. The horizontal wavelengths of free waves \( \lambda_n \) are shown as functions of \( F_i \) in Fig. 1, where we take \( H = 8.7 \text{ km} \).

It is important to see that when the internal Froude number is given as well as the depth of the fluid, the number of free waves and their horizontal wavelengths can be determined. It should be remarked that only discrete free wave modes are possible when we impose an upper rigid boundary at a finite altitude. This is an essential difference between the present model.
and semi-infinite space model with constant Scorcer parameter.

The validity of an application of the present model with a rigid upper boundary for the atmosphere is considered as follows. It is generally expected in a two-layer model that the lee wave amplitude is largely confined in the lower layer, when Scorcer parameter shows a marked decrease through the layers (Scorer, 1949; Corby and Sawyer, 1958). This is the case even with a model showing a continuous and rapid decrease of the Scorcer parameter (Onishi, 1960; Doos, 1961). A marked increase of the basic current or decrease of static stability with height means a large decrease of the Scorcer parameter above $H$. In such cases almost all disturbances are expected to be confined below $H$. The present model, therefore, may simulate the characteristics of the lee wave disturbances in the depth of a constant Scorcer parameter with a marked decrease above $H$.

The lee wave train has been considered to correspond to one of the eigenvalues which are obtained under a specified distribution of the Scorcer parameter. In our one-layer atmospheric model, however, the resonance property itself is entirely due to the existence of an upper rigid boundary. The present model, then, is considered to have some physical analogy to the real atmosphere in the sense that our model has a possibility of occurrence of the lee wave.

Our next task is to investigate 1) which free mode can be excited as lee waves by the introduction of the mountain, 2) how the lee waves are modified by nonlinearity, and ultimately what flow pattern can be formed.

Long (1953, 1955) studied the characteristics of the lee wave in steady state by using a nonlinear vorticity equation, and indicated that the lee wave equation is exactly linear only when the so-called Long's model is applied, assuming that linear density gradient and $u^2=\text{const.}$

Some of Long's results, which are interesting for our numerical experiments, are reproduced in Fig. 3, which shows the criterion for an overturning instability (hereafter to be called the OTI) in terms of $h/H$ and $b$, where $h$ is the height of the mountain and $b$ the horizontal extent of the mountain. The instability is defined as the increase of density with height or $u<0$ in a certain region of the channel. According to Long's (1955) analysis, it is expected that the OTI takes place in the region above the solid lines for $F_i<F_i^*$. It should be noted that the condition for the OTI has a special behavior in the vicinity of $F_i=F_i^*/n$ ($n=1, 2, \cdots n$). That is, there is a strong tendency of the OTI for the range of $F_i$ which is slightly smaller than $F_i^*/n$. As is seen from Fig. 1, this range of $F_i$ corresponds to the situation in which there exists a free wave with a very long horizontal wavelength. Long showed that except such special values of the $F_i$, the curves indicate roughly $D=F_i$ when the OTI begins. Assuming a steady state, it is impossible by this criterion to see how the flow will evolve after the beginning of the instability. The dependency of $F_i$ and $D$ on the flow pattern and on the OTI in Long's model should be compared with the present initial value problem, since the present model is physically similar to Long's except that the effect of internal friction is introduced and a uniform basic current is assumed in our model.

3. Numerical calculation

As has been discussed in the previous section, the Boussinesq system is adopted for the initial value problem. The equations are already written as (2.1)-(2.4) in section 2. The stream function $\varphi$ may be defined by

$$u = -\frac{\partial \varphi}{\partial z}$$

$$w = \frac{\partial \varphi}{\partial x}$$

Equations (2.1) and (2.2) may be reduced to a vorticity equation, which may be written in the form,

$$\frac{\partial}{\partial t} (p^2 \varphi) = J(p^2 \varphi, \varphi) + \frac{g}{\Theta} \frac{\partial \theta}{\partial x} + K_m p^2 \eta$$

and the thermodynamic equation (2.4) may be rewritten as follows:

$$\frac{\partial \theta}{\partial t} = J(\theta, \varphi) = K_m p^2 \theta$$

where $p^2 \varphi = \eta$.
Boundary and initial conditions

Vanishing of the stream function,

\[ \varphi = 0 \]  \hspace{1cm} (3.6),

along the lower boundary surface was assumed and that surface was also assumed to be isentropic in order to obtain the steady state,

\[ \theta = \text{const}(=\bar{\theta}(0)) \]  \hspace{1cm} (3.7)

and a stress-free condition was assumed along the surface,

\[ \eta = 0 \]  \hspace{1cm} (3.8).

In the viscous fluid, it is natural to take account of the effect of the surface stress at the lower surface. However, it is not our intention in this paper to consider the effect of the surface boundary layer or to simulate a resolution from the boundary. Therefore, the above slipping condition was adopted. This assumption seems to have some limitation in its application to a steeper mountain.

The upper boundary condition is similar to the lower one,

\[ \varphi = \varphi(H) \]  \hspace{1cm} (3.9),

\[ \theta = \bar{\theta}(H) \]  \hspace{1cm} (3.10),

\[ \eta = 0 \]  \hspace{1cm} (3.11).

As the inflow boundary condition, a constant inflow was assumed but potential temperature was set equal to its value at one grid point into the grid. Therefore, the static stability near the inflow boundary was kept almost constant. The inflow boundary was located at about 6 km upstream from the mountain ridge.

At the outflow boundary, vanishing of \( x \)-derivative, \( \frac{\partial}{\partial x} = 0 \), was assumed for all dependent variables. In practice it was assumed that the values of all dependent variables are set equal to their values at one grid point upstream. This Neuman condition was also used in solving the diagnostic equation (3.4). A cyclic boundary condition is not adequate in our problem because the possibility of resonance wave itself is our main interest. At the initial step of the time integration (say, \( t = 0 \)), the mountain was abruptly introduced into the horizontally uniform current.
The external parameter points which are selected for the numerical experiments are shown, in terms of $F_i$ and $D$, by the black circle or the symbol, *.

These points may be characterized by the number of the normal mode ($N_x$) and the height of the montain ($M_y$). Both are shown in Tables 1 and 2. Solid curved lines correspond to the Long's (1955) criterion of the overturning instability. At the parameter points, which are located above the straight line, $D=1.3F_i$, the overturning flow was obtained. $H$ is the total depth of the channel and $h$ the height of the mountain. $l$ is the square root of the Scorer parameter.

(c) External parameters

The parameters of intensity of the basic current and static stability for numerical experiments are plotted in the form of $F_i$ in Fig. 3. Some of the parameters are OTI.

(d) Computational stability

Time increment was set to satisfy a linear computational stability, $\Delta t < \Delta S |V_{max}|$. It was usually several seconds or so. Most of the integrations were carried out up to 180 time steps, which corresponds to 30-60 minutes. Some of the runs were executed up to 240 or 360 time steps.

First, several test runs were performed for the neutral atmosphere (i.e. $\beta=0$ or $F_i=\infty$). These cases are considered the ultimate limit of the supercritical flow region ($F_i>F_i^*$), as was discussed in section 2. Since there is no vorticity generation due to buoyancy, the steady state which is shown in Fig. 4 was finally established after dissipation of the initial vorticity, which was nearly advected downstream. The flow pattern is similar to the mountain shape. Its amplitude monotonously damps out upward and only one crestline is vertical with a nearly symmetric pattern. Following the evolution of this case, a steady symmetric flow is formed.
around 90 time steps (16 minutes after the initial state). The flow field in Fig. 4 is fairly consistent with that obtained under the assumption of an irrotational flow. A slight asymmetry in Fig. 4 seems to depend on the fact that the mountain ridge is placed upstream from the center of the computational domain. For the other different sizes of mountains and basic currents, the results were quite similar. This means that our computational method used in the integration is satisfactory. When the effect of the static stability is introduced, wave-like motion is expected to occur, as will be shown below.

4. Development of lee waves

It is interesting to investigate how a lee wave...
develops from the initial state. Our initial condition, which includes a shock disturbance, is in principle the same as in the method adopted by Houghton and Kasahara (1968), which is different from the start of an irrotational flow (Hirotta, 1965; Foldvik and Wurtele, 1967).

We shall show, among many runs, one of the interesting evolutions of the flow pattern at the parameter point $G$ in Fig. 3, which we call case N5M2, in which N5 denotes the numbers of the normal modes shown in Table 2, and M2 the type of the mountain shown in Table 1. A closed circulation pattern was obtained only in this case, although such patterns were often obtained in the two-layer model with different internal Froude numbers. The evolution into the lee wave with the closed circulation is interesting for a discussion of large amplitude lee wave.

Fig. 5(a) shows the flow pattern of cases N5M2 after 30 time steps representing about

![Fig. 5(c)](image)

Fig. 5(c). Similar to Fig. 5(a), but after 120 time steps representing 28 minutes.

![Fig. 5(d)](image)

Fig. 5(d). Similar to Fig. 5(a), but after 240 time steps.
7 minutes. A first ridge and through with windward tilt are already formed. A thin back current seems to set up along an upslope of the mountain, which may be due to abrupt blocking against an incident current. An S-shaped meandering flow is formed in the vicinity of the lee side of the mountain.

Figs. 5(b) and 6(b) show the flow field and potential temperature field after 60 time steps. A lee wave train is successively generated. The tilt of the first through becomes large and a subsequent second ridge with windward tilt is almost set up. Compared with these figures, the first three axes of the streamlines are completely in phase with those of isentropic lines around these time steps. The S-shaped flow is developed into a closed circulation, which hereafter we call the “Rotor”. In Fig. 6(b) isentropic lines drop

![Fig. 6(b)](image)

**Fig. 6(b).** Calculated potential temperature field after 60 time steps. Solid lines denote isentropic lines in which unit is °K. Dashed lines denote an axis of a trough or ridge.

![Fig. 6(c)](image)

**Fig. 6(c).** Similar to Fig. 6(b), but after 120 time steps representing 28 minutes.
like a tongue along the first lee side trough and spread toward the surface beneath the Rotor. The static stability is, then, considerably reduced in the vicinity of the Rotor and becomes locally unstable.

Figs. 5(c) and 6(c) show the establishment of a major pattern of the lee wave train including the Rotor after 120 time steps representing about 28 minutes. A closed isentrope, which corresponds to the Rotor, is also obtained in Fig. 6(c). Almost all of the streamline axes are found to coincide with those of the isentropic surfaces except far downstream. Furthermore a streamline nearly overlaps the isentropic line. The first lee side trough, however, shows a gradual deepening.

After 150 time steps no significant change in the flow pattern is seen except a slight windward shift of the axes. A gradual weakening of the Rotor circulation is recognized. The horizontal wind speed decreases remarkably in the wake of the Rotor. Some adjustments still take place after 180 time steps, especially far downstream. The flow, however, keeps sufficiently the same pattern as that at an earlier time. It should be noticed that the Rotor can still survive in spite of the existence of a statically unstable region.

Figs. 5(d) and 6(d) show the flow pattern and the potential temperature field after 240 time steps. In Fig. 5(d) the Rotor still exists and continues to do so until about 40 minutes after the initial formation, though the closed isentropic is no longer found in Fig. 6(d). The vertical velocity field is shown in Fig. 7. Corresponding to the lee wave train, a systematic up- and downdraft is obtained. A remarkably strong downslope wind representing more than 10 m/sec is seen along the lee of the mountain, and an updraft of 3 m/sec or more exists in front of the Rotor. The horizontal wavelength is longer at the higher level than at the lower level. The tilt of the axis is large especially near the mountain.

Time integration was stopped after 360 time steps representing 74 minutes. The distinctive features of the flow are still maintained. A periodic updraft and downdraft pattern of the order of 1 m/sec is still excited.

Even if the parameter point is selected in the region of the OTI in Fig. 3, as is seen above, the lee wave pattern is considered to show the property of a quasi-steady state. For the other parameter points except point Q, a change after the establishment of the lee wave pattern was so small in the whole computed domain. This does not necessarily mean the establishment of an exact steady state but that of a relatively long-lived flow.

The change was so small especially for the parameter point below $D=1.3F_1$ that the flow patterns were considered with sufficient certainty in the quasi-steady state.

We shall next show some figures which represent time sequences at specified grid points.
Fig. 7. Calculated vertical velocity field after 240 time steps. Stippled areas denote downdraft in which unit is m/sec.

Fig. 8. Time sequences of stream function are shown for four cases, (a) case N5M2, (b) case N2M2, (c) case N2M3 and (d) the neutral case.
As the grid points those near the mountain crest and the first lee side trough near the surface were selected.

Fig. 8(a) shows the time change of the stream function for case N5M2 mentioned above. A major adjustment from the initial impulsive flow occurs by 60 time steps. After 180 time steps the changes are small. Figs. 8(b) and 8(c) show the change for cases N2M2 and N2M3, respectively. After 120 time steps the change becomes small for case N2M2, but for the higher mountain M3 changes are still recognized after 120 time steps due to a secondary development of the S-shaped flow. Fig. 8(d) corresponds to the neutral case. Steady state was obtained after 90 time steps.

5. Resonance lee waves and occurrences of the S-shaped flow and Rotor-like circulation

Among the great number of runs made, some are plotted in terms of $F_i$ and $D$ in Fig. 3. Some comparison between the result and the linear theory is made, especially in connection with the wavelength of the lee waves in Table 2.

Table 2. Summary of the computational results. $D$ is the nondimensional mountain height, $F_i$ is the internal Froude number, $U$ is the basic current (m/sec), $d\theta/dz$ is estimated by the potential temperature increase for 1 km depth, $L$ is the wavelength (km) calculated from (2.15). Concerning the overturning instability, OTI denotes whether the parameter point belongs to the region of the instability or not according to Long (1955).

<table>
<thead>
<tr>
<th>Case</th>
<th>External parameters</th>
<th>Normal mode</th>
<th>OTI</th>
<th>Calculated</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$D=h/H$ $F_i$ $U$ $d\theta/dz$</td>
<td>$n$ $L$ (km)</td>
<td></td>
<td>pattern $L$ (km) $A_e$ $A_t$</td>
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<tr>
<td>N5M1(A)</td>
<td>0.07 0.048 5 4.5</td>
<td>0 2.6</td>
<td>Yes</td>
<td>S-shaped 3 2.7 3.0</td>
</tr>
<tr>
<td>N5M2(G)</td>
<td>0.14 0.048 5 4.5</td>
<td>[1 2.8]</td>
<td>Yes</td>
<td>Rotor like 3 2.7 4.8</td>
</tr>
<tr>
<td>N3M1(B)</td>
<td>0.07 0.096 10 4.5</td>
<td>0 5.2</td>
<td>Yes</td>
<td>Laminar 6-7 1.9 1.5</td>
</tr>
<tr>
<td>N3M2(H)</td>
<td>0.14 0.096 10 4.5</td>
<td>[1 6.6]</td>
<td>Yes</td>
<td>S-shaped 6-7 3.3 2.7</td>
</tr>
<tr>
<td>N3M3(M)</td>
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<td>[3 12.0]</td>
<td>Yes</td>
<td>S-shaped 6-7 3.7 3.4</td>
</tr>
<tr>
<td>N2M1(C)</td>
<td>0.07 0.117 10 3.0</td>
<td>0 6.4</td>
<td>No</td>
<td>Laminar 8 1.9 1.3</td>
</tr>
<tr>
<td>N2M2(I)</td>
<td>0.14 0.117 10 3.0</td>
<td>[1 6.9]</td>
<td>No</td>
<td>Laminar 8 2.6 1.9</td>
</tr>
<tr>
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<td>[2 9.5]</td>
<td>Yes</td>
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</tr>
<tr>
<td>N2M5(Q)</td>
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<td></td>
<td>Yes</td>
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<tr>
<td>N2*M1(D)</td>
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<td>0 7.9</td>
<td>No</td>
<td>Laminar 10 1.8 1.2</td>
</tr>
<tr>
<td>N2*M2(J)</td>
<td>0.14 0.144 15 4.5</td>
<td>[1 8.9]</td>
<td>Yes</td>
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<tr>
<td>N2*M3(O)</td>
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<td>[2 18.3]</td>
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<tr>
<td>N2*M4(P)</td>
<td>0.21 0.144 15 4.5</td>
<td></td>
<td>Yes</td>
<td>S-shaped 10 3.7 2.3</td>
</tr>
<tr>
<td>N1M1(E)</td>
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<td>0 11.1</td>
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<td>Laminar 14 1.8 1.1</td>
</tr>
<tr>
<td>N1M2(K)</td>
<td>0.14 0.203 10 1.0</td>
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<td>No</td>
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<tr>
<td>N1*M1(F)</td>
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<td>Yes</td>
<td>Laminar 18? 1.8 1.1</td>
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<tr>
<td>N1*M2(L)</td>
<td>0.14 0.305 15 1.0</td>
<td></td>
<td>Yes</td>
<td>Laminar 17? 2.3 1.2</td>
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</table>
Free modes were calculated from (2.15). Since more than two waves with different wavelengths usually exist, an adequate mean of the distances from ridge to ridge or trough to trough in the middle layer was considered as the estimated wavelength. Occurrences of the OTI are also presented in connection with Long's diagram. In the present numerical experiment, the S-shaped or Rotor-like flow is conveniently called the overturning flow, though the meaning of the overturning instability was originally defined by Long (1955) as is shown in section 2. In order to obtain information on a strong lee-side wind, the amplification rate is presented in the last column of Table 2. As the rate, we define the ratio of the calculated surface wind speed which is averaged for about 300 m depth, to that of the approaching flow. In general a wavy motion was predominantly formed and the estimated wavelength was quite consistent with that obtained from the linear theory, as is presented in Table 2. First we will discuss the estimated wavelength due to the linear theory.

In cases N1M1 and N1M2, the flow patterns resemble the potential flow in the vicinity of the mountain, but a weak lee wave with 17 km in wavelength appeared with nearly vertical axes. This wavelength is very close to the critical wavelength of \( n=0 \) mode, though the wavelength which is expected from (2.15) is 58 km. Our computed domain is, however, obviously too small to simulate such a long wave resonance.

Fig. 9 shows the flow field for case N1M2. Excitation of only the wave of one wavelength which corresponds to \( n=1 \) mode in the linear theory took place in case N1-series. A single sinusoidal wave with wavelength of about 14 km was obtained. The tilt of the first lee-side trough is small. The lee wave amplitude of case N1M2 was larger than that of case N1M1.

The streamlines for case N2*M2 are shown in Fig. 10. A wavelength of 10 km, which is near \( n=1 \), was obtained. The tilt of the first trough is larger than that of case N1M2 (Fig. 9). According to the above figures, the lee wave amplitudes show the intensification with decreasing in \( F_\lambda \). We shall discuss the dependency of lee wave properties on the height of the mountain. The amplitudes of the wave disturbances for case N2*-series at 6 km level are presented in Fig. 11 with different mountain height. Troughs which are located at about 6 km from the inflow boundary in Fig. 11 correspond to the first lee-side trough, respectively. It is worthy to note that troughs and ridges are located at the same position. That is, for a given \( F_\lambda \), the horizontal wavelengths of lee waves are almost the same and are not affected by difference of mountain height. The double amplitudes of the lee waves for different mountains are 2.3, 5.0, 8.3 and 12.4 for the mountains M1, M2, M3 and M4, respectively. The amplitude at the ridge of the streamline is considerably large in case N2M4, in which an S-shaped flow pattern was obtained. It should be noticed that the amplitude is not always proportional to the height of the mountain. Nonlinearity seems to increase with the height of the mountain.

Lee waves which correspond to \( n=2 \) mode were predominant for case N2-series, one of which, \( i.e. \) for case N2M2, is shown in Fig. 12. Comparing Figs. 9, 10 and 12, in which \( F_\lambda \) is 0.203, 0.144 and 0.117, respectively, we see that the tilt of the streamline axis becomes large with decreasing in \( F_\lambda \). For the decrease in \( F_\lambda \), it is clearly seen that the meandering of the streamlines becomes large. The meandering is especially prominent between the first lee-side trough and the subsequent ridge.

Fig. 13 shows the flow pattern for case N3M2, in which the estimated wavelength is 6–7 km. The S-shaped flow is obtained in the vicinity of the mountain.

The flow pattern for case N5M1 is shown in
Fig. 11. Amplitudes of the lee waves at 6 km height for different height of the mountain but for the same internal Froude number.

Fig. 12. Similar to Fig. 9, but for case N2M2.

Fig. 13. Similar to Fig. 9, but for case N3M2.

Fig. 14. Similar to Fig. 9, but for case N5M1.
Fig. 15. Theoretical flow pattern over an obstacle. $F_i=0.091$, $D=0.090$, $b=0.4$, $a=0.15$. After Long (1955).

Fig. 16. Stratified shear flow over a semicircular obstacle for $k=1.5$. After Miles and Huppert (1968).

Fig. 14. Horizontal wavelength is rather long at the higher levels, especially over the mountain. The tilt of the streamline axes far downstream is not so large as in the neighborhood of the mountain. The wavy motion obtained is not a simple sinusoidal wave. The wavelength varies locally from about 2.5 to 4.0 km. Therefore, one can not find one-to-one correspondence between the linear and the nonlinear case for such a region of small $F_i$. Such incoherent waves, after all, may be regarded as a result of interference between the different vertical modes (i.e., the different horizontal wavelengths). In connection with interference of waves, the interesting point is the development of an S-shaped or Rotor-like flow which was obtained in case N5M1 (Fig. 14), case N5M2 (Fig. 5(d)) and case N3M2 (Fig. 13). In these cases the tilt is large especially in the vicinity of the mountain. This means a large vertical phase shift. These large meandering flow patterns are still quasi-steady, as was shown in section 4. It should be noticed that there is no tendency for such an overturning flow to develop into a turbulent one, as is suggested by the instability of the static stability or by a shearing instability.

At the parameter points A, G, H, M, N, Q and P, which are all marked with the symbol $\triangle$, the overturning flow was obtained. All the parameters above are located in the domain of the OTI in Fig. 3. It may be noticed that it took a relatively long time before an S-shaped flow set up at the point H. The flow pattern at this point is shown in Fig. 13. This flow field is similar to Fig. 15, which was obtained by Long (1955). However, at the parameter points of J and O in Fig. 3 neither an S-shaped flow nor a Rotor-like flow appeared, though time integration was further performed up to 270 time steps as compared with the integration of 180 time steps at the parameter points I and K, which are not so far from the points J and O. These flows were still quasi-steady in spite of their positions in the OTI region. The limit above which an overturning flow appears seems to lie near $D=1.3F_i$, according to our numerical experiment, in which mountains with the same shape were used. Miles and Huppert (1968) obtained a similar limit, $k=k_c=1.27$, in the model of a semi-circular obstacle in a half space, which is considered to correspond to $D=1.27F_i$ in our definition. It is interesting to note that the S-shaped flow patterns, which are obtained above the limit, are similar to that obtained by Miles and Huppert for $k>k_c$, whose result is reproduced in part in Fig. 16.

6. Strong surface wind at the lee-side of a mountain

When the air stream crosses over a mountain ridge, a strong surface wind is observed at the lee side of the mountain, and sometimes the wind is accompanied by foehn or roll cloud. Up to the present the hydraulic jump or the strong mountain lee wave has been presented to explain the mechanism of the strong wind (Long, 1953; Hovermale, 1965; Houghton and Kasahara, 1968; Arakawa, 1968; Foldvik and Wurtele, 1967). In the linear theory for the isolated mountain, a strong wind is expected to occur in the vicinity of the first lee-side trough in accordance with the upstream tilt of the axes. The extent and intensity of the strong surface wind may depend on the shape and scale of the mountain and on the properties of the basic current (i.e. the Scorer parameter or the internal Froude number). We shall see how this strong wind depends on these parameters. At first we shall see the cases of the lower mountain, in which the nonlinearity is considered to be small as is discussed in the previous section.
Distributions of the surface wind in cases N5M1, N3M1 and N2*M1 are shown in Fig. 17(a). The intensities of the basic currents used in these cases are 5, 10 and 15 m/sec, respectively. The surface wind is almost constant at the place about several kilometers apart from the inflow boundary in each case. As it approaches the mountain, the wind is reduced in intensity. The upslope wind which is weaker than the basic current is obtained with a small dip at the adjacent upstream side of the mountain. The maximum wind occurs at the mountain crest except in case N5M1. It must be noticed that the distribution of the wind along the surface is underestimated at the adjacent upstream side and downstream side of the mountain and is overestimated at the crest by a corner effect. It is seen that the behaviors of the air stream are similar in all cases at the upstream side of the mountain, though these are different in the lee as will be shown below. Near the mountain crest the amplification rate in case N5M1 in Table 2, which is 2.7, is rather different from those in the other cases: 1.9 and 1.8 in cases N3M1 and N2*M1, respectively. This difference becomes larger near the first lee-side trough. However, the behaviors of the surface wind have a similar sense except for the region near the lee-side maximum. The surface wind is generally strong at the lee-side of the mountain and has a maximum, which is prominent in case N5M1. It should be noted that the wind speed which is associated with the first lee-side trough is stronger in case N5M1 than in case N3M1 in spite of the half speed in the approaching current. As is shown in Table 2, the amplification rate at the trough is 3.0 in case N5M1, where the S-shaped flow pattern is obtained. This rate is considerably larger than those in the others: The rates are 1.5 and 1.2 in cases N3M1 and N2*M1, respectively. In these cases the flow shows a quasi-sinusoidal wave.

Clear difference, which may be caused by the nonlinearity, is seen in Fig. 17(b), where the mountain height is twice as large as that in Fig. 17(a). Three cases correspond to the parameter points G, H and J in Fig. 3, respectively. The lee-side maxima of the surface wind in the three cases are all associated with the first lee-side trough and the magnitudes of the amplitude are comparable. The strong downslope wind which appeared in case N5M1 shows a remarkable development in case N5M2. This intense downslope wind is clearly related to the Rotor-like flow which is formed by the strong nonlinear effect. The flow pattern was shown in section 4. The Rotor itself seems to play the role of "Fluid-Barrier" against the air stream and to intensify the surface wind.

It should be noticed that the lee-side trough which corresponds to the downstream maximum of the surface wind is found to be located at a relatively similar position from the mountain for a given \( F_i \), not depending on the obstacle height. This maximum position shifts depending on the magnitude of \( F_i \). It is especially sensitive to the intensity of the basic current. This is consistent with the fact that the surface wind maximum
The distribution of the surface wind may be classified into two types. The first type is the case where the strong wind is associated with the usual first lee-side trough and the nonlinearity is less effective. In the second type, the S-shaped or Rotor-like flow is so predominant that the surface wind is remarkably intensified both at the downslope of the mountain and the lee-side trough. Besides, the tilt of the axis is generally stronger than in the first case. It is worthy to note that the surface wind, in this Rotor-type flow, can become stronger at the downslope or at the trough than at the mountain crest.

7. Summary and conclusions

Lee wave properties were investigated by integrating numerically the Boussinesq equation system with the upper rigid boundary surface. The system may excite the resonance wave. The results obtained may be summarized as follows:
1) In almost all the cases calculated, only one lee wave component was obtained as the nearly quasi-steady state solution, which is considered to correspond to one of the several free waves shown by the linear theory. 2) Relatively speaking, for large values of the internal Froude number and for low mountains, the calculated patterns were an almost laminar and sinusoidal wave, which were in good agreement with the linear theory. 3) For small values of the internal Froude number and for high mountains, roughly in the domain $D > 1.3 F_i$, an overturning flow (i.e., S-shaped or Rotor-like flow) was obtained near the mountain. However, except for the region of the overturning, it was still a sinusoidal laminar flow. These flows were found to survive for a long time in spite of the existence of a statically unstable region. The large meandering flow may be regarded as a result of the interferences of free waves due to the introduction of a mountain with a finite height. Calculated flow patterns are supported qualitatively by Long's (1955) analysis. At a few parameter points near $F_i = 1$ which belong to Long's overturning instability region, our calculated flows show no overturning. The reason for this is not clear, though the roles of internal friction and nonlinearity are inferred to be important. 4) The results may offer some information on the flight operation for the clear air turbulence caused by topography. 5) The strong surface wind at the lee side of the mountain was simulated in the simplest model, which was found to be remarkably intensified for small values of the internal Froude number and for high mountains.

Our first intention was to simulate the lee wave pattern in a constant flow with constant static stability. Therefore, the effects of the variations of the basic flow upon lee waves were left to be solved in future, as well as the problem of the surface boundary layer.

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山越気流の数値実験
1. 安定度一定のユニホームな流れ

古川武彦
気象研究所

山越気流（特に山岳波）の問題は、今まで多く線型定常解のままで論じられている。その取扱いは山脈が十二分に低く、且つおだやかな傾斜をしている場合には有効であるが、いわゆる“有限高の山”については未だ良好に理解されていない。山越の風下近傍で観測される強風などを論じる際は、やはり有限振幅の扱いをする必要がある。

この論文は、約9 km 上空に水平剛体壁を仮定して、Boussinesq 近似の非線型方程式を時間積分することにより、山脈気流の性質を調べようとしたものである。Boussinesq 系をこのような山越気流の問題に適用する場合の妥当性及び剛体壁を仮定したことによるモデルの特性などが先ず線型の範囲で考察された。

次に時間積分によって得られた lee wave のパターンが線型理論と比較され、良い対応が得られている。相対的に内部フールド数が大きくなる山が低い場合、流れは正弦波的であるが、山が高くなると内部フールド数が小さくなるにつれて、流れは破壊を増し、山の近傍で S 字型や Rotor 流に似た流れ（オーバー ターニング流）が形成される。この時、山の背後面や、風下側で風は著しく強化される。

オーバー ターニング流れには、局所的に静的不安定の領域が形成されるが、それには持続性が見られる。このような流れは、波の非線型的な干渉の結果と考えられる。オーバー ターニングはまた、山が高くなるにつれて起こりやすくなる。オーバー ターニングに対する臨界は \( D > 1.3 \) \( F_i \) （\( D \): 無次元化された山の高さ, \( F_i \): 内部フールド数）である。

流れのタイプの、山の高さおよび内部フールド数に対する依存性は Long (1955) の予測にほぼ対応している。