Effects of an Inclined Land Surface on the Land and Sea Breeze Circulation: A Numerical Experiment

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Abstract

Orographical effects of a slope behind a flat plain on the land and sea breeze circulation are investigated using a numerical model with a simplified configuration of the lower boundary. It is found that (1) when the temperature at the slope surface varies with a diurnal period similarly to that at the plain surface, both the sea breeze and the land breeze are amplified and the alternation of the sea and the land breezes occurs earlier than the land and the sea breezes over a flat plain, and (2) when the slope works merely as a barrier, both the land and the sea breezes are reduced and the circulation domain is confined in the sea and the plain regions. The solution of linear differential equations for the unsteady slope wind is examined to make more clear the physical process in the above mentioned situation. The land and sea breeze in the former case is found to be modified by the slope wind which is stronger than and is in advance of the land and sea breeze over a flat plain. The land and sea breeze for the case with a slope varying its surface temperature is highly efficient in producing the available potential energy and converting it to the kinetic energy of the breeze.

1. Introduction

A land and sea breeze, which is abbreviated as LSB later on, is one of diurnal local atmospheric circulations. Main observational facts of a LSB and its fundamental mechanism were summarized by Defant (1951). In the early theoretical studies of the LSB, effects of vertical transport of heat and momentum, Coriolis force, friction, etc. were investigated based on perturbation analysis.

Since Estoque (1961, 1962) solved the nonlinear differential equations for the LSB circulation numerically by the finite-difference method, many investigators used the same method to study a variety of aspects of LSB. Yoshikado and Asai (1972) pointed out the importance of the vertical eddy mixing in the mechanism of the LSB and investigated by numerical experiment how the pattern of the circulation depends on the eddy exchange coefficient and its vertical distribution in the transition layer. McPherson (1970) extended Estoque's model to a three-dimensional one and applied it to the area with an irregular coast line. Pielke (1974) made a three-dimensional numerical simulation of the sea breeze over South Florida.

In the review paper which summarized various aspects of the existing numerical models for the LSB, Asai and Yoshikado (1973) pointed out the necessity to study orographical effects on LSB. When a coastal plain is, as most of the plains in Japan are, narrow compared to the range over which the sea breeze is expected to be dominant (some tens of kilometers), it is important to take into account influence of the sloping terrain on the LSB.

An atmospheric flow over a mountain or a
sloping terrain has been studied in relation to slope wind or katabatic wind (e.g., Ball, 1956; Holton, 1966; Thyer, 1966; Gutman, 1969) or to formation of cumulus clouds over mountains (e.g., Braham and Draganis, 1970; Orville, 1964). Mahrer and Pielke (1975) studied, using a two-dimensional numerical model, the air flow over mountains when the mountain surface is heated or not heated. Mahrer and Pielke (1977) extended their previous study (1975) to investigate the effect of a bell-shaped mountain with the surface temperature varying as the result of surface heat budget on the LSB and showed that the combined sea breeze and mountain circulations produce a more intense circulation than when they act separately. Recently Ookouchi et al. (1978) made a numerical experiment to examine some features of the LSB in the terrain with a trapezoid-shaped mountain near the coast.

An objective of the present study is to investigate orographical effects on LSB circulation, focussing upon the following two factors: one is a thermal effect as generating slope wind and accelerating the LSB, and the other is a “barrier” effect as blocking the circulation. The atmospheric circulations obtained under thermal boundary condition at the slope given in two different ways are examined compared with the LSB circulation over a flat plain.

2. Numerical model

Numerical experiments are performed using a two-dimensional primitive equations with the boundary configuration illustrated in Fig. 1. Moisture is not taken into account.

2.1 Governing equations

The equations governing the land and sea breeze are expressed as follows.

\[
\begin{align*}
\frac{du}{dt} &= -c_p \frac{\partial \theta}{\partial x} + f v + K_H \frac{\partial^2 u}{\partial z^2} + K_v \frac{\partial^2 u}{\partial z^2}, \\
\frac{dv}{dt} &= -f u + K_H \frac{\partial^2 v}{\partial z^2} + K_v \frac{\partial^2 v}{\partial z^2},
\end{align*}
\]  

\[0 = -c_p \frac{\partial \theta}{\partial z} - g,
\]

\[\frac{\partial}{\partial z} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) = 0,
\]

\[\frac{d\theta}{dt} = K_H \frac{\partial^2 \theta}{\partial x^2} + K_v \frac{\partial^2 \theta}{\partial z^2},
\]

where

\[u, v \text{ and } w \text{ are the components of velocity in the } x, y \text{ and } z \text{ directions, respectively.} \]

\[\theta \text{ is the potential temperature defined as } \theta = T(p_0/p)^{\kappa}, \text{ where } T \text{ is the temperature, } p \text{ is the pressure, } p_0 \text{ is } 1,000 \text{ mb, the pressure at a reference level, and } \kappa = 0.2857. \pi \equiv (p/p_0)^{\kappa}, c_p \text{ is the specific heat at constant pressure, } g \text{ is the gravity acceleration and } f \text{ is the Coriolis parameter. } K_H \text{ and } K_v \text{ are the eddy diffusivities in the horizontal and vertical directions, respectively, and both } K_H \text{ and } K_v \text{ are assumed to be constant. All the variables are assumed to be uniform in the } y\text{-direction, parallel to the coast line.}

2.2 Initial conditions

The initial condition is that there is no motion and the potential temperature increases linearly with respect to height, i.e.,

\[u = v = w = 0 \]

\[\theta = \theta_s(z) = \theta_{sea} + \Gamma z \text{ at } t = 0,
\]

where \(\theta_{sea}\) is the potential temperature at the sea surface, and \(\Gamma\) is a positive constant. Suffix 0 designates the initial value.

2.3 Boundary conditions

The bottom boundary of the domain consists of four regions which are denoted by SEA, PLAIN, SLOPE, and PLATEAU in Fig. 1. The numerical experiment is carried out for three cases; Case 1, Case 2, and Case 3 which are distinguished from one another by the thermal boundary conditions at SLOPE and PLATEAU.

The boundary conditions for the potential temperature at SEA and PLAIN are

\[\theta = \theta_{sea} = \text{constant} \]

\[\text{for } -L \leq x \leq -l \text{ (SEA)}\]
Table 1. The boundary condition for the potential temperature at the lower boundary adopted for each case in the present numerical experiment. $t$ is measured in hour starting from 0800LT Day 1. $h(x)$ is the elevation of the land surface from the sea level.

<table>
<thead>
<tr>
<th>Case</th>
<th>$l$ (km)</th>
<th>SEA ((-L \leq x \leq -l))</th>
<th>PLAIN ((-l \leq x \leq 0))</th>
<th>SLOPE, PLATEAU ((0 \leq x \leq L))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>15</td>
<td>283.0</td>
<td>283.0 + 5.0 sin $2\pi \frac{t}{24}$</td>
<td>$\theta(x) + 5.0 \sin \frac{2\pi t}{24}$ ($\theta(0) = 283.0 + \Gamma z$)</td>
</tr>
<tr>
<td>Case 2</td>
<td>15</td>
<td>283.0</td>
<td>$\theta(x) + 5.0 \sin \frac{2\pi t}{24}$</td>
<td>$\theta(x) = 283.0 + \Gamma z$</td>
</tr>
<tr>
<td>Case 2'</td>
<td>30</td>
<td>283.0</td>
<td>283.0 + 5.0 sin $2\pi \frac{t}{24}$</td>
<td>$\frac{\partial \theta}{\partial x} = 0$ at $z = h(x)$</td>
</tr>
<tr>
<td>Case 3</td>
<td>15</td>
<td>283.0</td>
<td>283.0 + 5.0 sin $2\pi \frac{t}{24}$</td>
<td>$\theta(x) + 5.0 \sin \frac{2\pi t}{24}$</td>
</tr>
</tbody>
</table>

\[
\theta = \theta_\text{sea} + A \sin 2\pi \frac{t}{24}
\]

for $-l \leq x \leq 0$ (PLAIN), \hspace{1cm} (8)

where $A$ is the amplitude of the diurnal variation of the potential temperature at the PLAIN surface and $t$ (hour) is measured from 0800LT Day 1.

The boundary conditions for the potential temperature at SLOPE and PLATEAU for each case are listed in Table 1. Case 1 corresponds to the situation without SLOPE and PLATEAU, so that the bottom boundary consists of SEA \((-L \leq x \leq -l)\) and PLAIN \((-l \leq x \leq L)\). In Case 2 the potential temperature at SLOPE and PLATEAU varies diurnally around the initial potential temperature of the atmosphere at the respective level, so that

\[
\theta = \theta_\text{sloped} + A \sin 2\pi \frac{t}{24}
\]

at $z = h(x)$ and $0 \leq x \leq L$, \hspace{1cm} (9a)

where $h(x)$ denotes the elevation of the land surface from the sea level as seen in Fig. 1. In Case 3 the horizontal gradient of the potential temperature is assumed to vanish at the surfaces of SLOPE and PLATEAU, i.e.

\[
\frac{\partial \theta}{\partial x} = 0 \text{ at } z = h(x), \ 0 \leq x \leq L.
\]

(9b)

It implies that no horizontal heat flux at the slope is assumed in Case 3.

In each of the cases the velocity vanishes at the lower boundary, i.e.,

\[
u = w = 0 \text{ at } z = 0 \ (-L \leq x \leq 0)
\]

and at $z = h(x)$ \hspace{1cm} (10)

At the lateral boundaries there is no eddy flux of momentum and heat, so that

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial \theta}{\partial x} = 0 \text{ at } x = \pm L.
\]

(11)

At the upper boundary the velocity vanishes and the potential temperature and the pressure is kept constant.

\[
\left\{
\begin{array}{l}
u = w = 0 \\
\theta = \theta_\text{H} + \Gamma H \\
p = \pi H
\end{array}
\right\} \text{ at } z = H\hspace{1cm} (12)
\]

The following constants are adopted commonly to all the cases.

\[
\begin{align*}
\theta_\text{sea} &= 283.0 \text{ K}, \\
c_\rho &= 1004.0 \text{ m}^2 \text{ K}^{-1} \text{ s}^{-2}, \\
K_H &= 10^4 \text{ m}^2 \text{ s}^{-1}, \\
\theta_\text{H} &= 283.0 \text{ K}, \\
\Gamma &= 5.0 \times 10^{2} \text{ K m}^{-1}, \\
\Delta x &= 2500 \text{ m}, \\
\Delta z &= 100 \text{ m}.
\end{align*}
\]

2.4 Numerical computation

The finite differencing adopted in the present model is a staggered scheme. The grid sizes $\Delta x$ and $\Delta z$ are 2,500 m and 100 m respectively. Since the slope line passes the grid points, as

![Fig. 2](https://example.com/grid.png)

\[
\frac{\tan \alpha}{\Delta z} = \frac{\Delta x}{\Delta x} = 100 \text{ m}/2,500 \text{ m}.
\]
shown in Fig. 2, the slope angle $\alpha$ is fixed to be $\tan^{-1}(1/25) = 2.29^\circ$. Some computations were carried out for larger or smaller slope angles with various grid sizes and it turned out that the discrepancies among the results due to difference in the grid sizes are not negligible enough to allow quantitative comparison among them.

The domain, $H = 4.8 \text{ km}$ and $L = 50 \text{ km}$, is found to be large enough for all the cases, since the computations with larger domains yield practically an identical result. The height of the PLATEAU is 1,200 m, a quarter of $H$, and $l$ is 15 km except for Case 2'. The finite differencing with respect to $t$ is a centered scheme and the time increment $\Delta t$ is taken as small as 0.5 minutes to avoid computational instability.

3. Description of the Results

The time integration was carried out until the

3.1 Case 1

The land and sea breeze without SLOPE (Case 1) will be described first. The wind vector $(u, w)$ and the isopleth of the potential temperature deviation from the initial state are illustrated in Figs. 3(a) and (b) for 1500LT Day 3 and 0300LT Day 4, respectively. The strongest sea breeze is $u = 1.87 \text{ m/s}$ attained at $x = -12.5 \text{ km}$ and $z = 150 \text{ m}$ at 1500LT, an hour after the potential temperature at the land surface is highest. The sea breeze is confined in the lower layer below $z = 500 \text{ m}$, above which the offshore counterflow extends to the upper boundary, having its maxi-

![Fig. 3](image-url)
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Fig. 4 Temporal variations of \( u(A) \) for Case 1 and Case 2, \( u(C) \) for Case 2 and \( \theta_s' \).

The land breeze, which is strongest at 0300LT (\( u = -1.38 \text{ ms}^{-1} \)), is confined in the layer below \( z = 450 \text{ m} \), slightly lower than that of the sea breeze.

The time variation of \( u \) at point A (150m above the coast line), which is denoted as \( u(A) \), is shown in Fig. 4. The variation of \( u(A) \) has a delay of 1.5 hour behind the diurnal variation of the potential temperature at the lower boundary, \( \theta_s' \). The alternation of sea breeze to land breeze occurs at 2200LT on the coast line (point A) and the reverse change takes place at 0930LT.

In Fig. 5 are shown the wind holographs at \( z = 150 \text{ m} \) and at \( z = 950 \text{ m} \) above the coast line, where the LSB and their corresponding counterflows, respectively, are strongest. It is found that the land breeze is weaker than the sea breeze and that the major axis of the hodograph at \( z = 150 \text{ m} \) turns about 10° clockwise from the \( x \)-axis, while that of the counterflow at \( z = 950 \text{ m} \) is oriented about 30° and its hodograph is closer to a circle.

The features presented above coincide with the results of the numerical experiments on LSB by other authors, e.g., Estoque (1961), and Yoshikado and Asai (1972).

3.2 Case 2

The wind velocity and the potential temperature deviation in Case 2 are shown in Figs. 6(a) and (b) at 1300LT Day 3 and at 0100LT Day 4, when the maxima of the sea breeze and the land breeze are attained respectively. The maximum of the sea breeze is 3.09 \( \text{ ms}^{-1} \) obtained at \( x = 10 \text{ km} \) and \( z = 750 \text{ m} \) (point C), and the land breeze has its maximum of 2.37 \( \text{ ms}^{-1} \) at \( x = 7.5 \text{ km} \) and \( z = 650 \text{ m} \). Both the sea breeze and the land breeze in Case 2 are considerably stronger and their circulation domains are larger compared with those in Case 1.

The time variations of \( u(A) \) and \( u(C) \) are shown in Fig. 4. \( u(A) \) attains its maximum at 1400LT and minimum at 0200LT, when \( \theta_s' \) also takes its maximum and minimum, respectively. On the other hand the maximum and the minimum of \( u(C) \) are attained slightly earlier than those of \( \theta_s' \).

3.3 Case 3

In Case 3 the maximum of the sea breeze is 1.79 \( \text{ ms}^{-1} \) at \( x = -12.5 \text{ km} \) and \( z = 150 \text{ m} \) at 1500LT Day 3. The maximum of the land breeze is 1.33 \( \text{ ms}^{-1} \) at \( x = -17.5 \text{ km} \) and \( z = 150 \text{ m} \) at 0300LT Day 4. The circulation patterns at these times are shown in Figs. 7(a) and (b), respectively.
The atmospheric circulation is mostly confined in SEA and PLAIN regions, only slightly penetrating toward the lower part of SLOPE. The maximum of $u(A)$ is $1.76 \text{ms}^{-1}$ attained at 1500LT, later than in Case 2, but earlier than in Case 1. $u(C)$ attains its maximum at 1630LT, 1.5 hour later than $u(A)$.

4. Blocking effect of the slope

Comparison of the circulation patterns in Case 3 illustrated in Fig. 7 with those in Case 1 (flat plain) in Fig. 3 shows that the SLOPE in Case 3 works merely as a barrier, because in Case 3 the wind speed of the LSB circulation is evidently reduced and the inland portion of the circulation is confined within the PLAIN region. When the sea breeze driven over the coast line reaches the SLOPE, it is forced to ascend along the SLOPE against the stably stratified ambient atmosphere and to decrease its speed. Another reason for the weaker flow in Case 3 than in Case 1 is that the available potential energy produced by the heat flux at the lower boundary is smaller than in Case 1. It will be discussed in detail in Section 6.

5. Influence of the slope wind on LSB

As is shown in Fig. 4, the amplitude of $u(A)$ in Case 2 is greater than that of $u(A)$ in Case 1 and the maximum of $u(A)$ in Case 2 appears earlier than that in Case 1. It is also noticed that the maximum of $u(C)$ is greater than and in advance of that of $u(A)$ in Case 2.

Fig. 8 shows the vertical profiles of the phases of $\theta'$, $\Delta \theta'$, $-\Delta \pi'$ and $u$ above the coast line in Case 1 and above point $C$ in Case 2. The phase of any variable in Fig. 8 is defined as the difference of its phase from that of $\theta^*_s$, so that its positive value indicates that the variable is in advance of $\theta^*_s$. $\Delta \theta'$ and $\Delta \pi'$ denote the differences of $\theta'$ and $\pi'$ at two horizontally
neighbouring grid points, respectively. The diagram for Case 1 indicates that the phase of $\Delta \theta'$ coincident with that of $\theta_s'$ at the surface delays with height in the lower layer. Compared to $\Delta \theta'$, $-\Delta \pi'$ lags and $u$ is further delayed because the horizontal gradient of the pressure is the main factor accelerating $u$. While in Case 2 the phase of $\Delta \theta'$ is in advance of that of $\theta_s'$ contrary to Case 1 and the phase difference between $\Delta \theta'$ and $\theta_s'$ is as much as two hours, so that the maximum of $u$ appears in advance of the maximum of $\theta_s'$.

We derived the solution of linear differential equations describing the slope wind along the infinitely long slope with the surface temperature varying uniformly to clarify some fundamental characteristics of the slope wind and the phase relations mentioned above in Appendix. The solution is an extension of Prandtl's linear solution for steady slope wind (see Defant, 1951) to the unsteady situation. The amplitude of $u^*$ (the wind speed parallel to the slope) is mainly proportional to $A \cdot S^{-1/2}$, where $A$ is the ampli-
Fig. 9 The wind field in the limited region including PLAIN for Case 2. Both the scales of the domain and the arrow length are magnified compared to those in Fig. 6. (a) at 0700LT Day 3 and (b) at 0800LT Day 3.

Thus the greater amplitude and the precedence of $u(A)$ in Case 2 compared to $u(A)$ in Case 1 can be recognized as the influence of the slope wind.

The phase difference between $u(A)$ and $u(C)$ in Case 2 may result in separation of the circulation into two portions around the time when the sea and the land breezes alternate with each other: one over the coastal area and the other over the slope region. The flow patterns in the limited region PLAIN and SLOPE at 0700LT and 0800LT Day 3 are enlarged in Figs. 9(a) and (b), respectively. At 0800LT the upslope wind already appears along SLOPE, while the weak land breeze still remains near the coast line, thus making a calm region and descending current over PLAIN.

When the coast line is located 30 km apart from the mountain-foot (Case 2'), the separation of the two circulations is more evident. As is shown in Figs. 10(a) and (b), the upslope flow...
is recognized as early as at 0700LT.

Thus it can be concluded that the effect of the slope wind on the LSB is to intensify the LSB circulation and to make the alternation of land breeze and sea breeze earlier. This effect is smaller as the slope is farther from the coast line.

6. Efficiency of LSB circulation

Deriving the equations for kinetic energy and available potential energy from Eqs. (1)–(5) and averaging them throughout the whole computation domain with the aid of the boundary conditions (7)–(12), we obtain

$$\frac{\partial}{\partial t} \langle KE \rangle = \langle KE, AP \rangle + \langle DKE \rangle + \langle AKE \rangle + \langle PKE \rangle$$

$$\frac{\partial}{\partial t} \langle AP \rangle = -\langle KE, AP \rangle + \langle DAP \rangle + \langle AAP \rangle + \langle PAP \rangle$$

where $\langle A \rangle \equiv \frac{1}{S} \int \int A \; dz \; dx$, denotes the area average of $A$ throughout the whole computation domain $S$, $KE \equiv \frac{\rho g}{2} (u^2 + v^2)$ the kinetic energy,

$$AP \equiv \frac{\rho g}{2} \frac{\theta_0}{\partial \theta_0/\partial z}$$

the available potential energy. $\langle KE, AP \rangle$ denotes the conversion of available potential energy to kinetic energy through upward heat transport, $\langle DKE \rangle$ the viscous dissipation of kinetic energy, $\langle PAP \rangle$ the production of available potential energy caused by heat flux at the lower boundary, $\langle DAP \rangle$ the diffusional dissipation of available potential energy. $\langle AKE \rangle$ and $\langle AAP \rangle$ are the advections of KE and AP, respectively, which are negligibly small compared with the other terms. $\langle PKE \rangle$ is the work done by the pressure, which is not significant in the present study.

Integration of (13) and (14) with respect to time yields

$$\langle KE \rangle = \langle KE, AP \rangle + \langle DKE \rangle$$

$$\langle AP \rangle = -\langle KE, AP \rangle + \langle DAP \rangle + \langle PAP \rangle$$
Fig. 11 Schematic representation of the energy flow. The value of each term in $J m^{-3}$ and its percentage relative to $<PAP>$ in the round bracket are shown for each case.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Case} & <PAP> & <KE, AP> & <KE> & <KE> \\
\hline
1 & 31.1 (100) & 1.6 (5) & 0.07 (0.21) & \\
2 & 61.2 (100) & 13.7 (22) & 0.57 (0.93) & \\
3 & 13.3 (100) & 1.3 (10) & 0.02 (0.18) & \\
\hline
\end{array}
\]

where $\tilde{A}\equiv\int_{0}^{t} A \, dt$ denotes the integrated value of $A$ with respect to time from 0 to $t$.

The values of $<PAP>$, $<KE, AP>$ and $<KE>$ averaged over the second period (from 0800LT Day 2 to 0800LT Day 3) are represented in units of $J m^{-3}$ for Case 1, Case 2 and Case 3, respectively, in Fig. 11. The production of the available potential energy is largest in Case 2, and is smallest in Case 3. The large value of $<PAP>$ in Case 2 is mainly caused by the contribution of the heat flux at the SLOPE surface, while the small value of $<PAP>$ in Case 3 is because of the very little heat flux at SLOPE and PLATEAU where the temperature perturbations are smaller than that at PLAIN. The ratio of the kinetic energy to the accumulated production of the available potential energy, $<KE>/<PAP>$, can be taken as a measure of the efficiency of LSB, a kind of energy input-output system which transforms heat energy supplied at the ground into atmospheric motion. The ratio in Case 2 is about five times greater than those in Case 1 and Case 3.

Thus the energy conversion in Case 2 is highly efficient compared with the other cases.

7. Conclusion

Effects of the slope on the land and sea breeze circulation were examined using numerical models for a slope without heat exchange at the surface (Case 3) and a slope with diurnally varying surface temperature (Case 2). One is the barrier effect as reducing the LSB, while the other is the thermal effect as enhancing the LSB. One of the outstanding effects of the slope in Case 2 is that the slope wind generated along the slope advances the phase of the LSB, so that alternation of the sea breeze and the land breeze takes place earlier than that over the flat plain. The discrepancy in the phase between the LSB and the slope wind may result in separation of each circulation and induce a downward current over the plain region at the outbreak of the upslope wind.

The consideration on the energy budget leads to the conclusion that (1) the slope in Case 2 supplies the much more available potential energy to atmosphere than the flat plain, while the slope in Case 3 supplies much less of it, and (2) the slope in Case 2 converts the supplied energy into kinetic energy more effectively than the flat plain.

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Appendix

Analytical Solution for the Atmospheric Flow over an Infinitely Long Slope at which Temperature Varies Periodically with respect to Time.

The perturbation equations of momentum, heat and mass are written as follows in the orthogonal coordinates $x^*$ and $z^*$, in which the $x^*$ axis is along the slope with an inclination angle of $\alpha$, as shown in Fig. A1.

\[
\begin{align*}
\frac{\partial u^*}{\partial t^*} &= -\frac{\partial p^*}{\partial x^*} + \frac{g}{\Theta} \theta^* \sin \alpha + K \frac{\partial^2 u^*}{\partial z^* \partial x^*} \\
\frac{\partial \theta^*}{\partial z^*} &= \frac{g}{\Theta} \theta^* \cos \alpha \\
\frac{\partial \theta^*}{\partial t^*} &= -Su^* \sin \alpha - Sw^* \cos \alpha + K \frac{\partial^2 \theta^*}{\partial z^* \partial x^*} \\
\frac{\partial u^*}{\partial x^*} + \frac{\partial w^*}{\partial z^*} &= 0
\end{align*}
\]

Fig. A1 The orthogonal $x^*-z^*$ coordinate inclined by the angle of $\alpha$ from the horizontal direction.
where \( u^* \) and \( w^* \) are the velocity components in the \( x^* \) and \( z^* \) directions respectively, \( \theta^* \) is the deviation of the potential temperature from that of the basic state \( \Theta(z) \) and

\[
\pi^* \equiv c_p \Theta \left( \frac{p^*}{p_0} \right)^{\frac{1}{2}}, \quad S = \frac{\partial \Theta}{\cos \alpha \cdot \partial z^*},
\]

\( K \) is the eddy diffusion coefficient.

The boundary condition is

\[
\theta^* = A \cos \omega^* t^* \quad \text{and} \quad u^* = 0 \quad \text{at} \quad z^* = 0.
\]

Since the slope is assumed to be infinitely long and uniform in the \( x^* \) direction, all the variables are independent of \( x^* \). Thus (A1) and (A3) are reduced to

\[
\frac{\partial u^*}{\partial t^*} - \frac{\partial}{\partial z^*} \left( \frac{\partial \pi^*}{\partial z^*} - \Theta^* \Theta \sin \alpha + K \frac{\partial u^*}{\partial z^*} \right) = 0, \tag{A5}
\]

\[
\frac{\partial \theta^*}{\partial t^*} = -S u^* \sin \alpha + K \frac{\partial \theta^*}{\partial z^*} \tag{A6}
\]

Here we introduce nondimensional variables \( \theta, u, z \) and \( t \) defined as follows.

\[
\theta = \theta^*/A, \quad u = (S/AN) u^*, \quad z = (N/K)^{1/2} z^*, \quad t = N t^*.
\]

where \( N = g S/\Theta^{1/2} \) is the Brunt-Väisälä frequency. Eliminating \( u \) from (A5) and (A6), we obtain

\[
\frac{d \theta}{dz^2} - 2 \frac{d^2 \theta}{dz^2 \partial t} + \frac{d^2 \theta}{dz^2} + \sin^2 \alpha = 0, \tag{A7}
\]

Putting \( \theta(z, t) \equiv \theta(z) e^{i\omega t} \), where \( \omega = \omega^*/N \), (A7) reduces to an ordinary differential equation for \( \theta(z) \).

\[
\frac{d^2 \theta}{dz^2} - 2i \omega \frac{d \theta}{dz^2} + \sin^2 \alpha - \omega^2 \frac{d \theta}{dz^2} = 0. \tag{A8}
\]

The solution of \( \theta \) has quite different characteristics according as \( \omega < \sin \alpha \), \( \omega = \sin \alpha \), or \( \omega > \sin \alpha \). The respective solutions for \( \theta \) and \( u \) are as follows.

(i) \( \omega < \sin \alpha \)

\[
\begin{aligned}
\theta &= (1/2) \left( R_1 z + I_1 \right) \cos (\omega t - \delta_1) \\
u &= (1/2) \left( R_1 z + I_1 \right) \cos (\omega t - \delta_1'), \\
R_1 &= e^{x \beta} \cos \beta z + e^{-x \beta} \cos \gamma z \\
I_1 &= e^{x \beta} \sin \beta z - e^{-x \beta} \sin \gamma z \\
R_1' &= e^{x \beta} \sin \beta z + e^{-x \beta} \sin \gamma z \\
I_1' &= e^{x \beta} \cos \beta z + e^{-x \beta} \cos \gamma z \\
\delta_1 &= \tan^{-1} (I_1/R_1), \quad \delta_1' = \tan^{-1} (I_1'/R_1'),
\end{aligned}
\]

(ii) \( \omega = \sin \alpha \)

\[
\begin{aligned}
\theta &= \frac{1}{2} \{ \cos \omega t + \exp (-\sqrt{\sin \alpha} z) \} \\
&\times \cos (\omega t - \sqrt{\sin \alpha} z) \\
u &= \frac{1}{2} \{ \sin \omega t - \exp (-\sqrt{\sin \alpha} z) \} \\
&\times \sin (\omega t - \sqrt{\sin \alpha} z),
\end{aligned}
\]

(iii) \( \omega > \sin \alpha \)

\[
\begin{aligned}
\theta &= (1/2) \left( R_1 z + I_1 \right) \cos (\omega t - \delta_1) \\
u &= (1/2) \left( R_1 z + I_1 \right) \cos (\omega t - \delta_1') \\
R_1 &= e^{x \beta} \cos \beta z + e^{-x \beta} \cos \gamma z \\
I_1 &= e^{x \beta} \sin \beta z + e^{-x \beta} \sin \gamma z \\
R_1' &= e^{x \beta} \sin \beta z - e^{-x \beta} \sin \gamma z \\
I_1' &= e^{x \beta} \cos \beta z - e^{-x \beta} \cos \gamma z \\
\delta_1 &= \tan^{-1} (I_1/R_1), \quad \delta_1' = \tan^{-1} (I_1'/R_1'),
\end{aligned}
\]

The parameters adopted in the present numerical model are as follows; \( S = 5.0 \times 10^{-3} \text{Km}^{-1} \), \( \theta = 300 \text{K} \), \( K = 10 \text{m}^2 \text{s}^{-1} \), \( \sin \alpha = \tan \alpha = 0.04 \), and \( \omega^* = 7.272 \times 10^{-5} \text{s}^{-1} \), which yield \( N \sin \alpha = 5.12 \times 10^{-4} \text{s}^{-1} \). Thus the situation concerned here corresponds to case (ii).

In the present case \( \omega^* \) is so much smaller than \( N \sin \alpha \), i.e., \( \omega \ll \sin \alpha \), that the solutions of \( \theta(z) \) and \( u(z) \) are found to be quite similar to those for the steady state \( (\omega = 0) \) obtained by Prandtl (see Defant, 1951).

Vertical profile of the phases of \( \theta', \partial \theta'/\partial x, -\partial \pi'/\partial x \) and \( u \) are shown in Fig. 8. Their phases in the limit of \( z \to 0 \) are as follows;

\[
\begin{aligned}
limit_{z \to 0} (\text{phase of } \theta') &= 0, \\
limit_{z \to 0} (\text{phase of } -\partial \pi'/\partial x) &= \tan^{-1} (\gamma - \beta / \beta + \gamma), \\
&= -0.072(\text{rad}),
\end{aligned}
\]

which corresponds to -16 minutes for the diurnal period.

\[
\begin{aligned}
limit_{z \to 0} (\text{phase of } -\partial \pi'/\partial x) &= 0, \\
limit_{z \to 0} (\text{phase of } u) &= \tan^{-1} (\beta - \gamma / \beta + \gamma), \\
&= 0.072(\text{rad}),
\end{aligned}
\]

which corresponds to 16 minutes for the diurnal period.

As the slope angle decreases, the behavior of the solutions deviates from that for the steady state, and at the limit of \( \sin \alpha \to 0 \) (case (ii)) the solutions of \( \theta(z) \) and \( u(z) \) have, as expressed in (A10), parts not reducing with increasing \( z \). This
is understood as a "resonance" between the buoyancy oscillation of the stably stratified atmosphere along the slope and the oscillation of the surface temperature of the inclined ground. When the slope angle decreases further (case (iii)), the behavior of the solution is quite different from case (i).

In the situation of diurnal oscillation of the surface temperature, in which \( \omega^* \) is \( 2\pi/86400 \) \( \text{s}^{-1} \), the 'critical' slope angle is about 0.33°.

References


