Lagrangian Mean Motion Induced by a Growing Baroclinic Wave

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Abstract

Lagrangian mean motion induced by a growing baroclinic wave is discussed, based on the solution of Eady type problem of baroclinic instability including non-geostrophic effect.

It is shown that to the leading order of Rossby number, the Lagrangian mean meridional motion is convergent toward the center of the channel. This means that air particles are mixed horizontally as a consequence of the instability. It is also shown that to the second order, air particles move downward near the northern wall and upward near the southern wall, while in the central region they move southward in the upper layer and northward in the lower, except for weak reverse flows near the top and the bottom. This Lagrangian mean picture is completely different from the usual Eulerian mean picture, and agrees qualitatively well with the result of Kida's (1977) numerical experiment as far as the behaviors of tropospheric particles are concerned, and with the result by Riehl and Fultz (1957) obtained in a rotating annulus experiment as far as the distribution of Lagrangian mean vertical flow is concerned.

The result that the Lagrangian mean velocity field is convergent (divergent) even under Boussinesq assumption (cf. Andrews and McIntyre, 1978) is attributed mainly to horizontal mixing term and partly to a term of transverse-gradient transport. Eliminating the horizontal mixing term from the latitudinal component of Stokes drift and also a term of transverse-gradient transport from the vertical component, we can obtain the solenoidal part of Lagrangian mean meridional velocity field. This residual circulation is somewhat similar to the Eulerian mean meridional circulation, and it may be equivalent to the Lagrangian mean meridional circulation induced by a dissipating planetary wave (cf. Matsuno and Nakamura, 1978).

It is shown that only a part of the second order field mentioned above can be responsible to the change in Lagrangian mean zonal flow. As a result, the direction of the mean zonal flow acceleration is reverse to that in the Eulerian mean problem.

Finally, we estimate the so-called eddy diffusivity, to obtain that $K_H = 9.6 \times 10^6 \text{ cm}^2/\text{sec}$ and $K_V = 8.1 \times 10^5 \text{ cm}^2/\text{sec}$ under the assumed condition of baroclinic wave which is chosen as a typical cyclone. It is further pointed out that latitudinal buoyancy (heat) flux consists of down-gradient transport (or particle mixing) term and transverse-gradient transport term, and that the latter is about 20% of the former in magnitude.

1. Introduction

In this paper, we shall discuss the Lagrangian mean meridional motions induced by a baroclinically unstable wave, in order to see how air particles move and circulate in the mid-latitudinal region as a consequence of the development of cyclones, and compare the results with the well-known Eulerian mean picture of the tropospheric general circulation.

It is of common knowledge that in the earth’s troposphere there exists the so-called 3-cell circulation consisting of two direct circulations in the tropical and in the polar regions, and of an indirect circulation in the mid-latitude atmosphere. This indirect circulation is interpreted as a consequence of heat transport due to cyclone waves. The tropical direct circulation is the
Hadley cell induced by strong convective motions in ITCZ, while the polar one is also Hadley-type circulation associated with the subsidence of cold air mass, although the secondary effects of cyclone waves may be superposed on these circulations. Anyhow, this is the Eulerian mean picture of the tropospheric general circulation.

However, these flow patterns, especially, the mid-latitudinal indirect circulation does not show the actual trajectories of ensemble of air particles in a mean meridional plane. It is then interesting to ask how air particles move as baroclinic waves grow. This problem may be also connected with large-scale circulation and diffusion of atmospheric constituents, although the difference of motion of each particle from that of ensemble-mean should be emphasized. So far, except for a few numerical experiments (e.g., Kida, 1977), the Lagrangian mean picture of atmospheric general circulation seems not to have been drawn.

On the other hand, since QBO\(^{(1)}\)-model by Lindzen and Holton (1968, 1972) and SSW-model by Matsuno (1971), the theory of planetary wave-zonal flow interaction has been developed energetically, and Lagrangian mean motion of air particles caused by waves has been discussed, in order to study how waves accelerate mean zonal flow or equivalently how waves transfer momentum in a material medium (Bretherton, 1969, 1979; Uryu, 1974 a, b; Andrews and McIntyre, 1976, 1978 etc.). Concerning the Lagrangian mean meridional circulation, Uryu (1974b) has shown that a vertically propagating planetary wave packet cannot induce such a circulation because of cancellation between the Eulerian mean vertical flow and the Stokes drift up to the first order of small parameter expressing the slowness of amplitude change. This result, however, does not hold in general, especially in case of internal gravity wave in a horizontal wave guide (McIntyre, 1973) and in case of wave with time-dependent amplitude (Andrews and McIntyre, 1978). It should be noted here that as pointed out by Andrews and McIntyre (1978) in their general theory on Lagrangian mean motion, wave-induced Lagrangian mean flow field is divergent in general even in case of Boussinesq fluid. Based on these works, the present work is motivated to examine concretely the Lagrangian mean motions in case of growing baroclinic wave.

Recently, Matsuno and Nakamura (1978) have proposed the Lagrangian mean picture on SSW, by assuming that a basic zonal flow varies from westerly to easterly wind at a certain altitude (critical level) and that a stationary planetary wave is incident on the critical level. According to their picture, a strong Lagrangian mean northward flow is induced along the critical level and it is associated with converging vertical flow toward the level near the southern wall and diverging vertical flow near the northern wall. Thus, the Lagrangian mean meridional circulations are reversed below and above the critical level. It should be noticed that the lower one is completely reverse to the Eulerian mean circulation with upward flow to the north and downward flow to the south, which has been shown by Matsuno (1971). It is due to the existence of critical level that the Lagrangian mean meridional circulation described above is induced in spite of stationary wave. That is, as mentioned by the authors based on the Lagrangian mean dynamics, the strong northward mass transport is attributed to the discontinuous change of radiation stress at the critical level. It is noted, however, that as is recognized by frequent use of word, ‘circulation’, each Lagrangian mean velocity field appearing above and below the critical level is solenoidal because of stationary wave.

In the present paper, as will be shown later (Section 4), the Lagrangian mean meridional velocity field induced by a growing baroclinic wave is divergent, showing that air particles in the southern half region move northward and upward along almost horizontal trajectories, while in the northern half region they move southward and downward. This result means that baroclinic instability causes an almost horizontal overturning. In Lagrangian mean dynamical sense, such large latitudinal motions of air particles are attributed to Coriolis force balancing with, to the first order in Rossby number, a zonally directed force due to radiation stress caused by systematic correlation between pressure disturbance and latitudinal displacement of isentropic (equivalently, isopycnic, in Boussinesq fluid) surface. It is noted that a part of the departure flow from such a geostrophic-like balance can contribute to the change in Lagrangian mean zonal flow. Other various properties of Lagrangian mean flow field obtained will be mentioned in what follows.

\(^{(1)}\) QBO and SSW are abbreviations of Quasi-Biennial Oscillation and Stratospheric Sudden Warming, respectively.
Although the baroclinic wave treated here is assumed to be of small amplitude, our results may be useful to understand one of elementary processes operating in numerical experiments such as performed by Kida (1977), and also may be a concrete example of general theory by Andrews and McIntyre (1978).

2. A brief review of the theory on Eady type baroclinic instability

In order to determine the functional form of a growing baroclinic wave, we shall briefly review the theory of baroclinic instability. For simplicity, we are concerned here with Eady type problem.

We consider an inviscid Boussinesq fluid channel bounded by two horizontal rigid walls at the bottom and the top, and by two vertical rigid walls at two different latitudes. The basic zonal flow is assumed to be of constant vertical shear.

Under quasi-geostrophic, quasi-hydrostatic and constant Coriolis parameter approximations, the linearized Eulerian equations describing perturbations superposed on the basic zonal flow can be written as follows, using conventional local Cartesian coordinates;

\[
\left( \frac{\partial}{\partial t} + U(z) \frac{\partial}{\partial x} \right) \nabla^2 \rho_0 - f^2 \rho_0 \frac{\partial v_0}{\partial z} = 0 ,
\]

\[
\left( \frac{\partial}{\partial t} + U(z) \frac{\partial}{\partial x} \right) \frac{\partial \rho_0}{\partial z} - \nabla^2 \rho_0 = 0 ,
\]

(2-1)

(2-2)

where \( \rho_0 \) is pressure disturbance and \( v_0 \) is vertical component of velocity perturbation. Suffix 0 indicates 0-order in Rossby number. \( U(z) \) and \( A \) are the basic zonal flow and its constant vertical shear respectively. \( f \) is constant Coriolis parameter at a reference latitude. \( \rho_{00} \) and \( N \) are mean density of the fluid and Brunt-Väisälä frequency which is assumed to be constant, respectively. \( \nabla^2 \) stands for horizontal Laplacian operator.

Eliminating \( v_0 \) from eqs. (2-1) and (2-2), we have the following equation of potential vorticity conservation.

\[
\left( \frac{\partial}{\partial t} + U(z) \frac{\partial}{\partial x} \right) \left( \nabla^2 \rho_0 + N^2 f^2 \nabla^2 \rho_0 \right) = 0 ,
\]

(2-3)

According to the model situation mentioned above, we can write the boundary conditions as follows;

\[
v_0 = \frac{1}{f \rho_{00}} \frac{\partial \rho_0}{\partial x} = 0 \quad \text{at} \quad y = 0, D
\]

\[
v_0 = \frac{1}{N^2 \rho_{00}} \left[ \frac{A}{\nabla^2} \frac{\partial \rho_0}{\partial x} - \left( \frac{\partial}{\partial t} + U(z) \frac{\partial}{\partial x} \right) \frac{\partial \rho_0}{\partial z} \right] = 0 \quad \text{at} \quad z = 0, H
\]

(2-4)

where \( D \) and \( H \) are the width and the depth of the channel respectively.

Considering the boundary condition at \( y = 0 \) and \( D \), we assume that

\[
p_0 = p_0(z) \sin ly \cdot e^{ik(z-c_0 t)} ,
\]

(2-5)

where \( k \) is wave number, \( l = \pi / D \) and \( c_0 \) is complex phase velocity; \( c_0 = c_{0r} + i c_{0i} \).

Substituting (2-5) into eq. (2-3) and considering the boundary conditions at \( z = 0 \) and \( H \), we obtain

\[
p_0(z) = A e^{a z} + B e^{-a z} ,
\]

(2-6)

where

\[
\mu = \frac{N}{f} \left( k^2 + l^2 \right)^{1/2} .
\]

A and B are determined as eigensolutions, if \( c_0 \) is an eigenvalue given as

\[
c_{0r} = \frac{1}{2} \left( U_1 + U_2 \right) ,
\]

\[
c_{0i} = \frac{\Delta H}{2} \left[ \frac{1}{\mu H} \coth \mu H \right] ,
\]

(2-7)

where \( U_1 \) and \( U_2 \) are the speeds of basic zonal flow at the bottom and the top respectively.

The well-known Eady's (1949) criterion for baroclinic instability is

\[
\frac{1}{2 \mu H} < 1.997 ,
\]

(2-8)

The buoyancy (or heat) flux due to unstable wave can be written as

\[
\frac{g}{\rho_{00}} \frac{v_0 p_0}{v_0 p_0} = - \frac{1}{f \rho_{00}} \frac{\partial \rho_0}{\partial x} \frac{\partial \rho_0}{\partial z} = \frac{k}{2 f \rho_{00}} \sin^2 l y \cdot I_m \left( p_0 \frac{\partial \rho_0}{\partial z} \right) e^{2 k c_{0i} t} = \frac{\mu k}{f \rho_{00}} \sin^2 l y \cdot I_m (A^* B) e^{2 k c_{0i} t} = 2 \Delta H \mu^2 k c_{0i} \left[ \mu (U_1 - c_{0r}) + A \right]^2 + \mu^2 c_{0i}^2 ,
\]

(2-9)
where $p_0^*$ and $A^*$ are complex conjugates of $p_0$ and $A$ respectively, and $\text{Im}(\ )$ means the imaginary part of quantity in the bracket.

As is seen from (2-9), the buoyancy (heat) flux is constant in the vertical and negative (positive heat flux). This quantity plays a crucially important role in the second order Eulerian mean problem, as will be shown in the next section. That is, since the functional form of buoyancy flux divergence is $\sin^2\eta y$, a dipole-like buoyancy source and sink of the second order in wave amplitude is set up along the bottom and the top, to induce the second order mean meridional circulation.

The solutions obtained above are of zero-order in Rossby number. As will be seen later (Section 4), the first order correction is required when we calculate the Lagrangian mean meridional velocity component. Thus, we shall briefly review here the first order solutions of Eady type problem, based on the results by Derome and Dolph (1970). A little more detailed discussion is mentioned in Appendix.

The first order equations can be written as follows.

$$\frac{\partial^2 p_1}{\partial z^2} + N^2 \left( \frac{\partial^2 p_1}{\partial y^2} - k^2 p_1 \right) = - \frac{2Al}{f} \cos ly \cdot \frac{dp_0}{dz},$$  

(2-10)

with boundary conditions

$$p_1 = \frac{(U-c_0)}{f} l \cos ly \cdot p_0 \quad \text{at} \quad y=0, D,$$

$$(U-c_0) \frac{\partial p_1}{\partial z} - Ap_1 = C_1 \cos ly \cdot \frac{dp_0}{dz}$$

$$= \frac{(U-c_0) L}{f} \cos ly \cdot p_0 \quad \text{at} \quad z=0, H,$$

(2-11)

where $p_1$ is the amplitude function of the first order pressure disturbance\(^{(1)}\) and $C_1$ is the first order correction to eigenvalue.

Eq. (2-10) with boundary condition (2-11) forms the first order eigenvalue problem. We note here that $C_1$ vanishes if instability condition (2-8) is satisfied (Derome and Dolph, 1970). The eigensolution can be found by expanding $p_1$ in Fourier series with the boundary condition at $y=0$ and $D$ in mind as follows;

$$p_1 = \frac{(U-c_0)}{f} l \cos ly \cdot p_0(z) + \sum_{m: \text{even}} p_m(z) \sin mly,$$  

(2-12)

as well as

$$\cos ly = \sum_{m: \text{even}} \frac{4m}{\pi (m^2-1)} \sin mly,$$  

(2-13)

Substituting (2-12) and (2-13) into eq. (2-10), we have

$$\frac{d^2 p_m}{dz^2} - \mu_m^2 p_m = - \frac{4Al}{f} \cdot \frac{4m}{\pi (m^2-1)} \cdot \frac{dp_0}{dz},$$  

(2-14)

where

$$\mu_m = \frac{N}{f} (k^2 + m^2 f^2)^{1/2}.$$

This can easily be solved under the boundary condition at $z=0$ and $H$. Making use of equation

$$v_1 = \frac{ik}{f} \left[ (U-c_0) u_0 + \frac{p_1}{\rho_0} \right] e^{ik(x-c_0t)},$$  

(2-15)

we obtain

$$v_m(z) = C_m e^{\mu m z} + D_m e^{-\mu m z}$$

$$- \frac{4Al}{f^2 \rho_0} \cdot \frac{4m}{\pi (m^2-1)} \cdot \frac{ik}{\mu^2 - \mu_m^2} \times (A e^{\mu m z} - B e^{-\mu m z}),$$

where the detailed forms of $C_m$ and $D_m$ are given in Appendix and $A$ and $B$ are the constants appearing in (2-6).

3. Second order Eulerian mean flow

In this section, we shall examine the Eulerian mean flow induced by the second order effect of a growing baroclinic wave which has been obtained in the last section.

By averaging zonally the basic equations under quasi-geostrophic, quasi-hydrostatic and Boussinesq approximations, the following set of equations for zonal mean flows in Eulerian framework, to the second order in wave amplitude, is obtained;

$$\frac{\partial \bar{U}}{\partial t} - f \bar{V} = - \frac{\partial \bar{u}_0 \bar{v}_0}{\partial y}$$  

(3-1)

$$j \bar{\Omega} = - \frac{1}{\rho_0} \frac{\partial \bar{P}}{\partial y},$$  

(3-2)

$$0 = - \frac{1}{\rho_0} \frac{\partial \bar{P}}{\partial z} - \frac{g}{\rho_0} \bar{p},$$  

(3-3)

\(^{(1)}\) In what follows, suffix 1 indicates 1-st order term.
It should be noted that in (3-1) the Reynolds stress term \( u_0v_0 \) identically vanishes because the phase difference between \( u_0 \) and \( v_0 \) is \( \pi/2 \) in the present problem. Thus, we omit this term in the following discussion. Further, the contribution of non-geostrophic terms such as \( u_1v_0 \) and \( u_0v_1 \) can also be neglected in this problem.

The above set of equations is the same as that used by Uryu (1974b), in order to study the second order mean flows induced by a vertically propagating Rossby wave packet. In that case, the right-hand side of eq. (3-4) has been calculated from a wave packet solution, while in the present it is given by (2-9). It is noticed that in both cases, a wave is associated with negative buoyancy (positive heat) flux.

Making use of (2-9), we can deduce the zonal mean potential vorticity equation from the above set of equations as follows;

\[
\frac{\partial}{\partial t} \left( \frac{g}{\rho_{00}} \tilde{\rho} \right) - N^2 \tilde{W} = - \frac{\partial}{\partial y} \left( \frac{g}{\rho_{00}} \rho_0 v_0 \right),
\tag{3-4}
\]

\[
\frac{\partial \tilde{V}}{\partial y} + \frac{\partial \tilde{W}}{\partial z} = 0,
\tag{3-5}
\]

Further, the contribution of non-geostrophic terms such as \( u_1v_0 \) and \( u_0v_1 \) can also be neglected in this problem.

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\[
\frac{\partial}{\partial t} \left( \frac{g}{\rho_{00}} \tilde{\rho} \right) = N^2 \tilde{W} = - \frac{\partial}{\partial y} \left( \frac{g}{\rho_{00}} \rho_0 v_0 \right),
\tag{3-4}
\]

\[
\frac{\partial \tilde{V}}{\partial y} + \frac{\partial \tilde{W}}{\partial z} = 0,
\tag{3-5}
\]

As the corresponding boundary conditions, we impose that \( \tilde{V} \) and \( \tilde{W} \) vanish at the vertical walls and the horizontal walls respectively. Thus, we obtain that

\[
\frac{\partial}{\partial y} (\frac{\partial \tilde{P}}{\partial y}) = 0 \quad \text{at} \quad y = 0, D
\tag{3-7}
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial \tilde{P}}{\partial z} \right) = - \frac{1}{f \rho_{00}} \frac{\partial}{\partial y} \left( \frac{\partial \rho_0}{\partial y} \frac{\partial \rho_0}{\partial z} \right)
= \frac{kl}{2f \rho_{00}} \sin 2ly \cdot \frac{d}{dz} \left[ \rho_0 \left( \rho_0 \frac{d \rho_0^*}{dz} \right) \right]
\times e^{k z \cos \psi} = 0,
\tag{3-6}
\]

As the corresponding boundary conditions, we impose that \( \tilde{V} \) and \( \tilde{W} \) vanish at the vertical walls and the horizontal walls respectively. Thus, we obtain that

\[
\frac{\partial}{\partial y} \left( \frac{\partial \tilde{P}}{\partial y} \right) = 0 \quad \text{at} \quad y = 0, D
\tag{3-7}
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial \tilde{P}}{\partial z} \right) = - \frac{1}{f \rho_{00}} \frac{\partial}{\partial y} \left( \frac{\partial \rho_0}{\partial y} \frac{\partial \rho_0}{\partial z} \right)
= \frac{kl}{2f \rho_{00}} \sin 2ly \cdot \frac{d}{dz} \left[ \rho_0 \left( \rho_0 \frac{d \rho_0^*}{dz} \right) \right]
\times e^{k z \cos \psi} = 0,
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\[
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\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial \tilde{P}}{\partial z} \right) = - \frac{1}{f \rho_{00}} \frac{\partial}{\partial y} \left( \frac{\partial \rho_0}{\partial y} \frac{\partial \rho_0}{\partial z} \right)
= \frac{kl}{2f \rho_{00}} \sin 2ly \cdot \frac{d}{dz} \left[ \rho_0 \left( \rho_0 \frac{d \rho_0^*}{dz} \right) \right]
\times e^{k z \cos \psi} = 0,
\tag{3-8}
\]

Combining eq. (3-6) with (3-8), we see that the Eulerian zonal mean field can be changed by the distribution of buoyancy flux divergence along the top and the bottom. In other words, the buoyancy flux divergence in the present case plays a role like the so-called Ekman pumping (cf. Holton, 1965). In this point, the present problem is definitely different from the case of vertical propagation of Rossby wave packet discussed by Uryu (1974b), in which the forcing term on the right-hand side of eq. (3-6) is distributed vertically as well as horizontally in the fluid layer with the extent of the wave packet. Further, if we consider an accelerating bottom corrugation, the right-hand side of (3-8) can be replaced by a pressure drag force (Uryu, 1974a).

In order to solve eq. (3-6) under the boundary conditions (3-7) and (3-8), we expand \( \tilde{P} \) in Fourier cosine series as follows,

\[
\frac{\partial \tilde{P}}{\partial t} = \sum_{m: \text{odd}} Q_m(z, t) \cos ml y,
\tag{3-9}
\]

as well as

\[
\sin 2ly = - \frac{8}{\pi} \sum_{m: \text{odd}} \frac{1}{m^2 - 4} \cos ml y,
\tag{3-10}
\]

(Uryu, 1974a), so that the boundary condition (3-7) is automatically satisfied.

Then, eq. (3-6) and boundary condition (3-8) can be reduced to

\[
\frac{\partial^2 Q_m}{\partial z^2} - \lambda_m^2 Q_m = 0,
\tag{3-11}
\]

where \( \lambda_m = m N l / f \) and

\[
\frac{\partial Q_m}{\partial z} = - \frac{8}{\pi} \cdot \frac{q_0}{m^2 - 4} \quad \text{at} \quad z = 0, H,
\tag{3-12}
\]

where

\[
q_0 = kl \cdot I_m \left( \rho_0 \frac{d \rho_0^*}{dz} \right) \cdot e^{k z \cos \psi}/2f \rho_{00}.
\]

Eq. (3-11) is easily solved, to obtain that

\[
Q_m = - \frac{f}{N l} \cdot \frac{8}{\pi} \cdot \frac{q_0}{m^2 - 4} \cdot \frac{1}{\sinh \lambda_m H}
\times [\cosh \lambda_m z - \cosh \lambda_m(z - H)],
\tag{3-13}
\]

Then, by substituting \( \partial \tilde{P}/\partial t \) thus obtained into eqs. (3-1) to (3-5), we obtain \( \tilde{V} \) and \( \tilde{W} \) as follows;

\[
\tilde{V} = - \frac{1}{f \rho_{00}} \cdot \sum_{m: \text{odd}} \frac{8}{\pi} \cdot \frac{q_0}{m^2 - 4} \cdot \frac{1}{\sinh \lambda_m H}
\times [\cos \lambda_m H - \cosh \lambda_m(z - H)],
\tag{3-14}
\]

From these solutions, we can obtain Eulerian mean meridional stream function \( \tilde{\xi}_E \) which is
defined as
\[ \bar{V} = -\frac{\partial \bar{x}}{\partial x}, \quad \bar{W} = -\frac{\partial \bar{x}}{\partial y}, \] as follows.
\[ \bar{x} = \frac{1}{N^2 \rho_0} \sum_{m=1}^{8} \frac{q_0}{m(m^2-4)} \frac{\sin ml \gamma}{\sinh \lambda m H} \times \left[ \sinh \lambda m z - \sinh \lambda m (z-H) - \sinh \lambda m H \right], \] (3-16)

It should be noted here that \( \bar{X}_E \) satisfies the following equation.
\[ f^2 \frac{\partial^2 \bar{x}}{\partial z^2} + N^2 \frac{\partial^2 \bar{x}}{\partial y^2} = \frac{1}{f \rho_0} \frac{\partial^2}{\partial y^2} \left( \frac{\partial \rho_0}{\partial z} \right) \]
\[ = -\frac{kl^2}{f \rho_0} I_m \left( \rho_0 \frac{\partial \bar{x}}{\partial z} \right) e^{2k_c n^i} \cos 2l y \]
\[ = -\frac{2l q_0}{\rho_0} \cos 2l y \] (3-17)

If we solve eq. (3-17) with boundary condition that \( \bar{x}_E = 0 \) along the boundary walls, we can reproduce the solution (3-16).

In what follows, in order to present the results numerically, we use the following values of parameters: \( f = 10^{-4} \text{sec}^{-1}, N = 10^{-2} \text{sec}^{-1}, \rho_0 = 1.3 \times 10^3 \text{gcm}^{-3}, D = 5 \times 10^8 \text{cm}, H = 10^6 \text{cm}, U_1 = 0, U_2 = 3 \times 10^3 \text{cmsec}^{-1}, A = 3 \times 10^{-3} \text{sec}^{-1} \) (= 30 msec^{-1}/10 km), \( k = 1.26 \times 10^{-8} \text{cm}^{-1} \) (corresponding to wave length \( 5 \times 10^3 \text{km} \), \( l = 0.63 \times 10^{-8} \text{cm}^{-1} \), \( C_{0r} = 1.5 \times 10^3 \text{cmsec}^{-1}, C_{0i} = 6.46 \times 10^2 \text{cmsec}^{-1} \) (corresponding to growth rate \( k C_{0i} = 0.7 \text{day}^{-1} \), \( \mu = 1.41 \times 10^{-6} \text{cm}^{-1} \) and \( |\rho_0(0)| = 11.3 \text{mb} \).

Further, in the presentation of the results, we omit time factor \( e^{2k_c n^i} \) because we are concerned only with the flow structure, and fortunately time-integration is not required for our purpose.

In Fig. 1, Eulerian mean meridional stream function is shown. This is essentially a continuous version of the meridional circulation obtained by Phillips (1954) using a two-layer model, though he considered the \( \beta \)-effect. The flow pattern indicates the so-called 3-cell circulation. It should be, however, noted that two direct circulations near the vertical walls are induced as a result that buoyancy flux diminishes towards the walls as \( \sin^2 ly \), while the change in mean buoyancy is zero along the walls due to \( \bar{V} = 0 \) there. In this sense, these direct circulations—especially, the southern one is essentially different from the Hadley circulation in the actual tropical atmosphere.

The indirect circulation in the central region of Fig. 1 is caused by the convergence (or divergence) of buoyancy flux along the bottom and the top, which sets up the second order mean buoyancy excess in the northern part and deficit in the southern part. As is seen from (3-14) or eq. (3·4), mean vertical motion \( \bar{W} \) consists of the buoyancy flux part which is independent of height and the mean buoyancy (temperature) tendency part which becomes largest at the horizontal walls. These two parts counteract with each other, but the former is larger, apart from the bottom and the top. Thus, upward motion is induced to the north and downward motion to the south. Further, we note that it is confirmed by the so-called rotating annulus experiment that a baroclinic wave induces such an indirect circulation (Riehl and Fultz, 1957; Matsuwo et al., 1977).

It should be remarked, however, that the Eulerian mean meridional circulations obtained above do not show mean motion of air particles projected onto a mean meridional plane.

4. Lagrangian mean motion

In this section, we shall discuss mean motion of air particles, i.e., Lagrangian mean motion, resulting from baroclinic instability.

(a) Concept of Lagrangian mean

In the basic state, the trajectory of each particle is a straight line at a constant height, parallel to a latitudinal line so far as local Cartesian geometry is applied. When baroclinic instability

Fig. 1 Eulerian mean meridional stream function.
occurs, individual air particles move in all directions, though horizontal movement is dominant, and each trajectory is deviated from a straight line by a growing wave. The position of each particle at any time or the trajectory could be described in principle in Lagrangian framework.

However, if a wave can be assumed to be of small amplitude, the following consideration leads to Lagrangian mean dynamics without handling complicated Lagrangian equations (cf. Bretherton, 1971; Andrews and McIntyre, 1978; Matsuno and Nakamura, 1978). A straight line on which each particle moves in the basic state can be regarded as a material line (or tube), and motion of one member particle due to wave cannot be distinguished from that of other member particle if each particle is not ‘labeled’ by its initial position on the line, i.e., initial phases. Thus, the mean motion of member particle A over a wave-length is same as that of member particle B, and also same as that obtained by averaging with respect to initial phases, i.e., labels, or equivalently by averaging over many member particles.

According to this consideration, the system of Lagrangian mean equations can be constructed, as done by Andrews and McIntyre (1978), although their theory can describe more general situation than that mentioned above. However, in the present work, we shall proceed our discussions, based on the solutions obtained in Eulerian framework in the previous sections, as done by Uryu (1974b) and Matsuno and Nakamura (1978).

The Lagrangian displacement of fluid particle due to wave motion can be connected to Eulerian perturbations, to the first order in wave amplitude, through the following kinematic relationships.

\[
\begin{align*}
\frac{\partial}{\partial t} + U(z) \frac{\partial}{\partial x} \xi &= u + \Lambda \zeta, \\
\frac{\partial}{\partial t} + U(z) \frac{\partial}{\partial x} \eta &= v, \\
\frac{\partial}{\partial t} + U(z) \frac{\partial}{\partial x} \zeta &= w,
\end{align*}
\]

where \( \xi, \eta \) and \( \zeta \) are \( x, y \) and \( z \) components of displacement respectively, which are functions of space and time in Eulerian sense, and \( u, v \) and \( w \) are \( x, y \) and \( z \) components of Eulerian perturbation velocity respectively. It is noted that the Lagrangian displacement introduced above is solenoidal to the first order in wave amplitude, corresponding to the non-divergent property of Eulerian velocity perturbation field, and that \( \zeta \) in (4-1) is one order smaller than \( u \) under quasi-geostrophic assumption.

Since we have already known the functional forms of \( u, v \) and \( w \) to the first order in Rossby number in the previous sections, we can calculate \( \xi, \eta \) and \( \zeta \) by the above relationships; to the leading order in Rossby number in each,

\[
\begin{align*}
\xi_0 &= \frac{\cos \theta}{f\rho_0 k (U - c_0)} \hat{p}_0 e^{ik(z-c_0 t)}, \\
\eta_0 &= \frac{\sin \theta}{f\rho_0 (U - c_0)} \hat{p}_0 e^{ik(z-c_0 t)}, \\
\zeta_0 &= \frac{\sin \theta}{N^2 \rho_0 (U - c_0)} \left[ \Lambda \hat{p}_0 - (U - c_0) \frac{dp_0}{dz} \right] e^{ik(z-c_0 t)},
\end{align*}
\]
the following relationship;
\[ \zeta \omega_v + \eta_0 \omega_w = - \frac{\partial}{\partial t} \gamma_0 \zeta_0, \]  
(4-7)
where we have made use of the kinematic relationships connecting \( v_0 \) with \( \eta_0 \) and \( w_0 \) with \( \zeta_0 \). Thus, in order to obtain the Lagrangian mean meridional motion correctly to the first order in Rossby number, \( \eta_0 \omega_w \) should be corrected by non-geostrophic components described in Section 2. Further, in (4-6), \( \zeta \omega \) is one order smaller than the other two terms, to be neglected in the following discussions.

Then, \( \mathcal{V}_L \) and \( \mathcal{W}_L \) are written as follows.
\[ \mathcal{V}_L = \mathcal{V} + \frac{\partial}{\partial y} \left[ \eta_0 v_0 + \gamma_0 v_0 + \gamma_0 v_0 \right] + \frac{\partial}{\partial z} \zeta_0 v_0, \]
(4-5')
\[ \mathcal{W}_L = \mathcal{W} + \frac{\partial}{\partial y} \eta_0 \omega_0 \]
where \( \gamma_1 \) can be obtained from the following relationship.
\[ \gamma_1 = - \frac{i v_1}{k (U - c_0)}. \]  
(4-8)

We note here that \( \overline{\eta v} \) in \( \overline{\mathcal{V}_L} \) implies particle dispersion or mixing and it accompanies latitudinal buoyancy (heat) flux as is seen from the relationship
\[ \frac{g}{\rho_0} \rho_0 \omega_0 = - f \zeta_0 v_0 + N^2 \zeta_0 \omega_0, \]
(4-9)

where use is made of Eulerian perturbation equation of adiabatic motion (2-2) in which \( 1/\rho_0 (\partial \rho_0/\partial z) \) and \( f/\rho_0 (\partial \rho_0/\partial z) \) are replaced by \( -g/\rho_0^2 \) and \( v_0 \) respectively. In other words, meridional buoyancy flux consists of down-gradient transport \( f \zeta_0 \omega_0 \) and transverse-gradient transport \( N^2 \zeta_0 \omega_0 \)

The functional forms of \( \mathcal{V}_L \) and \( \mathcal{W}_L \) are written as follows;
\[ \mathcal{V}_L = \mathcal{V}_L^{(1)} + \mathcal{V}_L^{(1)}, \]
\[ \mathcal{V}_L^{(1)} = \frac{\partial}{\partial y} \left( \frac{C_0 k l}{2 f^2 \rho_0^2} \sin 2 l y \right) \frac{|\rho_0|^2}{|U - c_0|^2} \mathcal{V} \]
\[ \times e^{2 z c_0 l t}, \]  
(4-10-1)
\[ \mathcal{V}_L^{(1)} = \mathcal{V} + \frac{\partial}{\partial y} \left( \frac{\gamma_0 v_0 + \gamma_1 v_0}{U - c_0} \right) + \frac{\partial}{\partial z} \zeta_0 v_0 \]
\[ \mathcal{W}_L^{(1)} = \mathcal{W} + \frac{\partial}{\partial y} \left( \frac{\eta_0 v_0 + \gamma_0 v_0}{U - c_0} \right). \]  
(4-11)

In Figs. 2(a, b, c), the distribution of Lagrangian mean velocity components \( \mathcal{V}_L \) and \( \mathcal{W}_L \) are presented. Arrows in Fig. 2(c) indicate the direction and magnitude of mean velocity, though the vertical component is multiplied by 500. We see in these figures that baroclinic instability causes almost horizontal motions of air particles converging toward the central region of the channel, with slow downward motion in the northern half region and upward motion in the southern half region. It is noted that the usual intuitive explanation that baroclinic instability occurs when the tangent of particle trajectory projected onto a meridional plane is smaller than that of the basic isentropic surface (e.g., Green, 1960) is verified by Fig. 2(c). The result mentioned above can be also confirmed from the functional form of \( \frac{\partial}{\partial y} \frac{\eta_0 v_0}{U - c_0} \) which shows that to the leading order in Rossby number, air particles move southward (northward) in the northern (southern) half region. The converging meridional flow shows horizontal mixing of particles, and Coriolis force acting on this flow is balanced by a force due to radiation stress \( \frac{\partial}{\partial y} \left( \frac{\rho_0}{\rho_0} (\frac{\partial \gamma_0}{\partial z}) \right) \) (see subsection (e)).

We note that Fig. 2(c) is qualitatively similar to the result obtained by Kida's numerical experiment (1977, Fig. 11) as far as the behaviors of tropospheric air particles are concerned, except a small difference that in the present case air particles move northward and upward in the
Fig. 2(a) Distribution of Lagrangian mean meridional velocity, $\overline{V}_L$.

Fig. 2(b) Distribution of Lagrangian mean vertical velocity, $\overline{W}_L$.

Fig. 2(c) Distribution of Lagrangian mean meridional velocity vectors. Arrows are drawn by $(\overline{V}_L, \overline{W}_L \times 500)$.

Further, according to Riehl and Fultz (1957) who analysed the data obtained in rotating fluid annulus experiments, the distribution of mean vertical motion obtained by averaging along the 'meandering' surface jet shows a similar pattern to Fig. 2(b). Since the surface jet stream line can be regarded as an undulating material line in the steady wave regime, their averaging procedure is almost equivalent to the present Lagrangian mean, and hence the qualitative agreement between two results seem to be reasonable. It should be emphasized, however, that their result has been obtained in a steady state, while ours for a growing wave. Thus, the similarity of $\overline{W}_L$-distribution may be attributed to that of releasing process of available potential energy. The distribution of meridional velocity component $\overline{V}_L$ in their result is considered not to be convergent as in Fig. 2(b), though Riehl and Fultz have not shown it. As will be discussed in detail below, the convergence (divergence) is caused by unsteadiness of wave. The result of Riehl and Fultz should be explained, based on the solution of steady finite amplitude problem.
dependence of wave amplitude (cf. Andrews and McIntyre, 1978). It is further noted that the major part of the horizontal motion \( (\partial / \partial z) \overline{\zeta_0 v_0} \) seen in Fig. 3 is not responsible to the acceleration of Lagrangian mean zonal flow, because as will be discussed in subsection (e) the horizontal motion due to \( (\partial / \partial z) \overline{\zeta_0 v_0} \) is a consequence of balance between Coriolis force and a force caused by radiation stress \( \frac{\partial}{\partial z} \left( \frac{p_0}{\rho_0} \frac{\partial \zeta_0}{\partial x} \right) \).

(c) Divergent property of Lagrangian mean velocity field

As is seen from Figs. 2(e) and 3, in spite of Boussinesq assumption, the Lagrangian mean meridional velocity field is not solenoidal not only to the leading order but also to the second order in Rossby number. Andrews and McIntyre (1978) have shown that the Lagrangian mean velocity field is generally divergent (or convergent). In the present case, the latitudinal component of Stokes drifts includes \( \overline{V_L}^{(1)} \) \( (= \overline{\gamma_0 v_0} / \partial y) \) which is the largest among various components, and the convergence in the leading order field is due to this term, i.e.,

\[
\partial \overline{V_L}^{(1)} / \partial y = \frac{\partial^2}{\partial y^2} \overline{\gamma_0 v_0} = \frac{1}{2} \left( \frac{\partial^2}{\partial y^2} \gamma_0 \right)
\]

\[
= \frac{c_0 k f^2}{f^2 \rho_0^2} \cos 2ly \cdot \left| \hat{p}_i \right|^2 \cdot e^{i k c_0 t},
\]

(4-12)

Further, even in the second order field, the divergence appears as shown in Fig. 3(a). This can be confirmed again by the following equation.

\[
\overline{V_L}^{(2)} + \partial \overline{V_L} / \partial z = \frac{\partial^2}{\partial y^2} [\overline{\gamma_0 v_1 + \gamma_1 v_0}]
\]

\[
+ \frac{\partial^2}{\partial y \partial z} \left[ \zeta_0 v_0 + \zeta_1 v_1 \right],
\]

(4-13)

where we have made use of the kinematic relations (4-2) and (4-3) and the solenoidality of Eulerian mean meridional velocity field (3-5).

Dotted lines in Fig. 3(a) present the distribution of the right-hand side of eq. (4-13). It is found that in the main body of the fluid layer a quadrupole-like sink-source appears, while weak dipoles near the vertical walls. As is seen from eqs. (4-12) and (4-13), the divergent behavior of Lagrangian mean meridional velocity field is due to time-dependence of wave amplitude. In case of stationary wave, the right-hand sides of these equations vanish, i.e., the velocity field is

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\]

\[
= \frac{c_0 k f^2}{f^2 \rho_0^2} \cos 2ly \cdot \left| \hat{p}_i \right|^2 \cdot e^{i k c_0 t},
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\overline{V_L}^{(2)} + \partial \overline{V_L} / \partial z = \frac{\partial^2}{\partial y^2} [\overline{\gamma_0 v_1 + \gamma_1 v_0}]
\]

\[
+ \frac{\partial^2}{\partial y \partial z} \left[ \zeta_0 v_0 + \zeta_1 v_1 \right],
\]

(4-13)

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Comparing with Eulerian mean picture, we consider here the reason why the divergence of Lagrangian mean velocity field appears.

Boussinesq or incompressibility assumption requires that the density of a fluid particle must conserve during its motion. As is well known, this condition is given by

\[ F \cdot V = 0 \]  
\tag{4-14} 

Eulerian mean value of any quantity is specified by an average of the quantity along a latitudinal line (at a constant height) which was identical with a material line in the basic state but now merely 'geometrical' one. For any particle or at any position on this line, eq. (4-14) must hold if the fluid is incompressible. Therefore, the averaged velocity along this line, i.e., Eulerian mean velocity must also be solenoidal.

On the other hand, in Lagrangian mean specification, we concern an average with respect to particles on a material line which is now undulated by wave. Thus, as already mentioned, any physical quantity carried by a particle must be evaluated not at its original position \((x, y, z)\) but at its displaced position \((X, Y, Z) = (x + \xi, y + \eta, z + \zeta)\). Therefore, incompressibility condition must be expressed in terms of velocity at \((X, Y, Z)\):

\[ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} = 0, \]  
\tag{4-15} 

where \(U, V\) and \(W\) are velocity components at \((X, Y, Z)\).

Making use of the following relationship

\[ J(X, Y, Z) \left\{ \frac{\partial U}{\partial X}, \frac{\partial V}{\partial Y}, \frac{\partial W}{\partial Z} \right\} = \{J(\bar{U}, Y, Z), J(\bar{V}, Z, X), J(\bar{W}, X, Y)\}, \]  
\tag{4-16} 

(cf., Tomotika, 1940), we can rewrite eq. (4-15) in a form of referred to the original position \((x, y, z)\):

\[ J(\bar{U}, Y, Z) + J(\bar{V}, Z, X) + J(\bar{W}, X, Y) = 0, \]  
\tag{4-15'} 

In the above expressions, \(J(P, Q, R) = \frac{\partial(P, Q, R)}{\partial(x, y, z)}\) and especially \(J(X, Y, Z)\) is the Jacobian determinant of the mapping \((x, y, z) \rightarrow (X, Y, Z)\).

Eq. (4-15') (or equivalently (4-15)) means that Lagrangian time derivative of \(J(X, Y, Z)\) is zero.\(^{(1)}\) Then, considering \((X, Y, Z) = (x + \xi, y + \eta, z + \zeta)\) and making use of the property of Jacobian determinant, eq. (4-15') can be rewritten as follows:

\[ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = \frac{\partial \eta}{\partial (x, y)} + \frac{\partial \zeta}{\partial (x, y)} + \frac{\partial (\xi, \eta)}{\partial (y, z)} + \frac{\partial (\xi, \zeta)}{\partial (z, x)} + \frac{\partial (\xi, \zeta)}{\partial (z, x)} + J(\bar{U}, \eta, \zeta) + J(\bar{V}, \xi, \zeta) + J(\bar{W}, \xi, \zeta), \]  
\tag{4-15''} 

Thus, averaging (4-15'') in terms of particle label \(x\) and taking account of \(\bar{U} = \bar{U}_L, \bar{V} = \bar{V}_L\) and \(\bar{W} = \bar{W}_L\), we see that Lagrangian mean velocity field is generally divergent.\(^{(2)}\)

In practice, when wave is of small amplitude, we can expand \(\bar{U}, \bar{V}\) and \(\bar{W}\) near \((x, y, z)\) as follows:

\[ \bar{U} = U + U + u + (\xi \cdot \Gamma) U + (\xi \cdot \Gamma) u \]
\[ + \frac{1}{2} \xi \cdot (\xi \cdot \Gamma) U + \text{higher orders}, \]
\[ \bar{V} = V + v + (\xi \cdot \Gamma) v + \text{higher orders}, \]
\[ \bar{W} = W + w + (\xi \cdot \Gamma) w + \text{higher orders}, \]  
\tag{4-17} 

where the assumption that the basic zonal flow is of linear profile in the vertical and homogeneous in \(y\)-direction is relaxed.

Substituting (4-17) and \((X, Y, Z) = (x + \xi, y + \eta, z + \zeta)\) into (4-15''), averaging the result with respect to \(x\), Lagrangian mean continuity equation is deduced, to 2nd order in wave amplitude, after somewhat tedious manipulation, as follows:

\[ \frac{\partial \bar{V}}{\partial y} + \frac{\partial \bar{U}}{\partial z} = \frac{\partial}{\partial t} \left[ \frac{1}{2} \frac{\partial^2 \bar{y}}{\partial y^2} + \frac{\partial^2 \bar{y}}{\partial y \partial z} \right] + \frac{1}{2} \frac{\partial^2 \bar{y}}{\partial z^2} \]
\[ + \frac{\partial}{\partial y} \frac{\partial^2 \bar{y}}{\partial y \partial z} \frac{\eta \partial \bar{y}}{\partial y} \]  
\tag{4-18} 

\(^{(1)}\) Considering \(DX/DT = \bar{U}\), etc., in which \(D/DT\) is Lagrangian time differentiation, we can show that \(DJ/DT\) is equal to the left-hand side of (4-15') (cf. Tomotika, ibid.).

\(^{(2)}\) In their general theory of Lagrangian mean dynamics, introducing Lagrangian mean operator

\[ \bar{D} = \frac{\partial}{\partial t} + \bar{V}_L \cdot \bar{\nabla}, \]  

Andrews and McIntyre (1978) have shown that

\[ \bar{D}J = J(\bar{\nabla}_x \bar{V} - \bar{\nabla} \cdot \bar{V}_L) \]

where \(\nabla_x \cdot \bar{V}\) is the left-hand side of eq. (4-15). If \(\nabla_x \cdot \bar{V} = 0\), equation (4-15'') is equivalent to this but extremely complicated.
where we have made use of the kinematic relationships (4-1) to (4-3) and solenoidality of Lagrangian displacement to 1st order in wave amplitude.

Eq. (4-18) is the same as that obtained by operating divergence operator (in a meridional plane) to $\overrightarrow{V}_L$ and $\overrightarrow{W}_L$, i.e., (4-12) and (4-13). It is no paradox that in spite of Boussinesq assumption Lagrangian mean velocity field is divergent. We concern the (Lagrangian) mean state associated with (Lagrangian) mean trajectories around which actual trajectories of particles are fluctuating, and hence eq. (4-18) states that incompressibility condition cannot be satisfied along mean trajectories unless the second order effect of fluctuations of individual paths is included. In other words, a mean cross-sectional area of an undulating material tube projected onto a meridional plane is not constant along a mean trajectory. We note here that such a divergent property of Lagrangian mean velocity field is caused by time-dependence of wave amplitude.

The derivation procedure of eq. (4-18) is straightforward but somewhat tedious. An elegant proof has been given by Andrews and McIntyre (1978). They have introduced a 'mean flow density' which is defined by $\rho(X, Y, Z)$ in our notation and have proved that the mean flow density satisfies Lagrangian mean analogue of Eulerian mass conservation equation, and also that in the Boussinesq limit, eq. (4-18) holds (see also Footnote on p. 11).

(d) Solenoidal part of Lagrangian mean velocity field

As is seen from eqs. (4-12) and (4-13) or (4-18), the sink-source terms of the divergence of Lagrangian mean velocity field are originated from the mixing terms such as $(\partial/\partial y)\overline{v_0 v_0}$, $(\partial/\partial y)\overline{v_1 v_0}$ and so on. Then, eliminating these terms from $\overrightarrow{V}_L$ and $\overrightarrow{W}_L$, we can take out the solenoidal part of Lagrangian mean velocity field as follows.

$$\overrightarrow{V}_L^* = \overrightarrow{V} - \frac{\partial}{\partial y} \overline{v_0 v_0} - \frac{\partial}{\partial y} (\overline{v_0 v_1} + \overline{v_1 v_0})$$

$$= \overrightarrow{V} + \frac{\partial}{\partial z} \overline{\zeta_0 v_0}$$

(1) The mean flow density is utterly different from the Lagrangian mean density $\overline{\rho v}$, which is defined as $\overline{\rho v} = \overline{\rho} + (\partial/\partial y)\overline{\rho v_0}$ in the present case. $\overline{\rho v}$ is conserved along the Lagrangian mean trajectory determined by $(\overrightarrow{V}_L, \overrightarrow{W}_L)$ under adiabatic condition (Andrews and McIntyre, 1978; Nakamura, 1978).

$$\overrightarrow{W}_L^* = \overrightarrow{W} - \frac{\partial}{\partial y} \overline{v_0 v_0} - \frac{\partial}{\partial y} (\overline{v_0 v_1} + \overline{v_1 v_0})$$

(4-19)

It is easily seen that $(\overrightarrow{V}_L^*, \overrightarrow{W}_L^*)$ is a solenoidal vector. Then, we can define a stream function $\overline{\zeta_L^*}$ which is connected with $\overrightarrow{V}_L^*$ and $\overrightarrow{W}_L^*$ as

$$\overline{\zeta_L^*} = -\frac{\partial \overline{\zeta_L^*}}{\partial z}, \quad \overline{W}_L^* = \frac{\partial \overline{\zeta_L^*}}{\partial y},$$

(4-20)

Recalling the definition of Eulerian mean stream function $\overline{\zeta_E}$, (3-15), we see that $\overline{\zeta_L^*}$ is related to $\overline{\zeta_E}$ as follows;

$$\overline{\zeta_L^*} = \overline{\zeta_E} - \overline{\zeta_0 v_0},$$

(4-21)

Thus, substituting $\overline{\zeta_E}$ from this equation into eq. (3-17) and making use of (4-9), we obtain equation for $\overline{\zeta_L^*}$;

$$f^2 \frac{\partial^2 \overline{\zeta_L^*}}{\partial z^2} + N^2 \frac{\partial^2 \overline{\zeta_L^*}}{\partial y^2} = -f \frac{\partial}{\partial y} \overline{\zeta_0 v_0}$$

(4-22)

It is noticed that, as seen from the second form of the right-hand side of eq. (4-22), the source term for $\overline{\zeta_L^*}$ is originated from the distributions of force due to radiation stress, while the Eulerian mean meridional circulation is induced by the buoyancy flux as shown by eq. (3-17).

In case of stationary wave, $\overline{\zeta_0 v_0}$ identically vanishes, and eq. (4-22) becomes same as that derived by Matsuno and Nakamura (1978). In such a case, $\overline{\zeta_L^*}$ is exactly equal to the Lagrangian mean meridional stream function. Further, we note that if there is no critical line in case of stationary wave, the right-hand side of eq. (4-22) is identically zero because the buoyancy flux (and therefore $\overline{\zeta_0 v_0}$) is independent of height (e.g., Eliassen and Palm, 1961), and consequently the Lagrangian mean meridional circulation is not induced by stationary wave (Uryu, 1974b;
Matsuno and Nakamura, 1978, etc.). This is the verification of the so-called C-D theorem (Charney and Drazin, 1961) that a conservative stationary wave cannot change the second order zonal mean field unless there is a critical line, from Lagrangian mean dynamical view point. When there is a critical line somewhere in the fluid layer, as has been shown by Matsuno and Nakamura (1978), radiation stress \( \rho_0 \partial \bar{z}_0/\partial x \) or buoyancy flux \( (g/\rho_0)\rho_0 \bar{v}_0 \) changes discontinuously there, and hence the source term of eq. (4-22) shows a distribution like \((d/dz) \delta(z)\), to induce the Lagrangian mean meridional circulation.

In the present case, as already mentioned above, \( \bar{z}_L \) does not show exactly the Lagrangian mean velocity field, but gives the solenoidal part of it. The distribution of \( \bar{z}_L \) is presented in Fig. 4. In this figure, we see that the solenoidal part of Lagrangian mean velocity field is somewhat similar to the Eulerian mean meridional circulation (Fig. 1). The direct circulations appear near the vertical walls, while the indirect circulation covers almost entire region. The former are mainly caused by the Eulerian mean motion and the latter reflects large horizontal Stokes drifts which are directed to south in the upper layer and to north in the lower. It is noted that, according to Matsuno and Nakamura (1978, Fig. 5 in the paper), when a stationary wave is incident on a critical line, a similar pattern to Fig. 4 is obtained above the critical line. This is because a critical line absorbs a wave and hence has an effect on the wave like a dissipation which plays a role similar to growing-in-time behavior of wave as far as linear friction law acts on (e.g., Andrews and McIntyre, 1976; Boyd, 1976). The present result is attributed to the elimination of the mixing terms from the components of Stokes drift. Further, we note that different from the case of stationary wave as discussed by Matsuno and Nakamura (1978), the distribution of acceleration of Lagrangian mean zonal flow has not an exact correspondence to Fig. 4, because \( \partial/\partial x \bar{z}_0 v_0 \) cannot contribute to the change in mean zonal motion (see subsection (e)).

(e) Acceleration of Lagrangian mean zonal flow

Lagrangian mean zonal flow \( \bar{U}_L \) can be written, to the second order in wave amplitude, as

\[
\bar{U}_L = \bar{U} + \frac{\partial}{\partial y} \bar{v}_0 \bar{u}_0
\]

(4-23)

where higher order terms in Rossby number are omitted.

Making use of geostrophic relationship between \( u_0 \) and \( \bar{v}_0 \) and (4-2'), we obtain the functional form of Stokes drift \( \bar{U}_S \) as follows;

\[
\bar{U}_S = \frac{\partial}{\partial y} \bar{v}_0 \bar{u}_0 = \frac{-F \cos 2\varphi \cdot (U - C_0)}{2f \rho_0 v_0^2 |U - C_0|^2} \times |\beta_0|^2 e^{2\alpha \xi_0} x,
\]

(4-24)

Thus, since \( U>C_0 \) in the upper layer, \( \bar{U}_S \) is positive in the central region of the channel and negative near the vertical walls, while in the lower layer (\( U<C_0 \)), the distribution of \( \bar{U}_S \) is reversed. This result means that the Lagrangian mean zonal flow has a jet-like profile between the vertical walls. A similar phenomenon occurs even in case of stationary wave. As is well known in water waves, fluid particles move in the direction of wave propagation, while Eulerian mean flow vanishes (cf. Phillips, 1968). The present result is essentially similar to this.

As is seen from time-differentiation of \( \bar{U}_S \), the distribution of \( \partial \bar{U}_S/\partial t \) is similar to that of \( \bar{U}_S \). On the other hand, the acceleration of Eulerian mean zonal flow \( \partial \bar{U}/\partial t \) is negative (positive) in the upper (lower) layer in the central region due to the indirect meridional circulation, while near the vertical walls it is reversed due to the direct circulations (see Fig. 1). Thus, \( \partial \bar{U}/\partial t \) and \( \partial \bar{U}_S/\partial t \) counteract with each other. The functional form of \( \partial \bar{U}_L/\partial t \) is written as follows;

(1) Basic zonal flow is not included in this expression. “Actual” Lagrangian mean zonal flow is obtained by adding \( U \) to (4-23). (see (4-4))
Fig. 5 Distribution of the acceleration of Lagrangian mean zonal flow, $\partial \vec{U}_L / \partial t$. In the right part, the latitudinal distributions of $\partial \vec{U}_L / \partial t$, $\partial \vec{U}_S / \partial t$ and $\partial \vec{U} / \partial t$ at two representative levels are shown.

Andrews and McIntyre (1978) under more general consideration. The right-hand side is the divergence of radiation stress.

In the present case, eq. (4-26) is simplified as

$$\frac{\partial \vec{U}_L}{\partial t} = \frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{U}_s}{\partial t} = \frac{1}{N\rho_0} \sum \frac{8}{m^2 - 4} \gamma_{nm}^2 \sin \frac{m\pi}{2} H - \sinh \lambda_m \cosh \lambda_m (z - H)$$

$$- \frac{2k\rho_0 (U - c_v)}{f^2 \rho_0} \left| \rho_0 \right|^2 \cos 2\gamma y \cdot e^{\text{eigen} t},$$  

(4-25)

and in Fig. 5, the distribution of $\partial \vec{U}_L / \partial t$ is shown.

It is seen from this figure that the acceleration of $\vec{U}_S$ overcomes that of $\vec{U}$. Thus, the direction of acceleration of mean zonal flow in Lagrangian mean sense is completely reverse to that in Eulerian mean.

We consider here the origin of force which accelerates fluid particles in the zonal direction. Averaging the first of (4-17) with respect to $x$ and substituting $\vec{U}$ from the result into Eulerian mean zonal component of equation of motion, we can obtain that

$$\frac{\partial \vec{U}_L}{\partial t} + \vec{V}_L \frac{\partial \vec{U}}{\partial y} + \vec{W}_L \frac{\partial \vec{U}}{\partial z} - f\vec{V}_L$$

$$= \frac{\partial}{\partial y} \left[ \frac{p}{\rho_0} \frac{\partial \gamma}{\partial x} \right] + \frac{\partial}{\partial z} \left[ \frac{p}{\rho_0} \frac{\partial \gamma}{\partial x} \right],$$  

(4-26)

where only quasi-Boussinesq assumption is used. This equation has been already obtained by

because $U$ is assumed to be homogeneous in $y$-direction and $\vec{W}_L$-term is negligibly small under quasi-geostrophic assumption. Then, to the leading order in Rossby number, we have

$$-f\vec{V}_L = \frac{\partial}{\partial y} \left[ \frac{p}{\rho_0} \frac{\partial \gamma}{\partial x} \right] - \frac{\partial}{\partial y} \gamma \nu_0,$$

(4-27)

where the last form is derived by using geostrophic relation between $p_0$ and $\nu_0$.

Eq. (4-27) means that, to the leading order in Rossby number, Coriolis force acting on latitudinal mixing motion is balanced with a force due to a part of radiation stress which is caused by systematic correlation between pressure disturbance and latitudinally undulating material surface. As is seen from eq. (4-12), the second order (in wave amplitude) mass source is set up near the vertical walls and sink in the central region of the channel, as a consequence of the development of baroclinic wave. Then, the force
due to a part of radiation stress $\frac{\partial}{\partial y}(\rho_0' \rho_0 \frac{\partial \eta}{\partial x})$ counteracts Coriolis force acting on the meridional flow from the source to the sink so as to restore the mean zonal flow to geostrophic balance. Thus, to the leading order in Rossby number, the change in mean zonal flow does not occur. In other words, the leading order Stokes drift $(\frac{\partial}{\partial y})\bar{\eta}\bar{v}_0$ which shows large horizontal mixing of particles does not contribute to the acceleration of mean zonal flow, as already mentioned in subsection (b).

To the second order in Rossby number, we have the following equation from eq. (4-26).

$$\frac{\partial \bar{U}_L}{\partial t} - f\bar{V} = \frac{\partial}{\partial y} \left[ \frac{p_0}{\rho_0} \frac{\partial \bar{T}_1}{\partial x} + \frac{p_1}{\rho_0} \frac{\partial \bar{T}_0}{\partial x} \right] + \frac{\partial}{\partial z} \left[ \frac{p_0}{\rho_0} \frac{\partial \bar{u}}{\partial x} \right],$$ (4-28)

By considering that

$$\frac{\partial}{\partial y} \bar{T}_1; \bar{v}_0 = - \frac{1}{f} \frac{\partial}{\partial y} \left[ \frac{p_0}{\rho_0} \frac{\partial \bar{T}_1}{\partial x} \right]$$

$$\frac{\partial}{\partial z} \bar{T}_0; \bar{v}_0 = - \frac{1}{f} \frac{\partial}{\partial z} \left[ \frac{p_0}{\rho_0} \frac{\partial \bar{T}_0}{\partial x} \right]$$ (4-29)

eq (4-28) is reduced to

$$\frac{\partial \bar{U}_L}{\partial t} - f\bar{V} + \frac{\partial}{\partial y} \left( \bar{v}_0 \bar{v}_1 \right) = \frac{\partial}{\partial y} \left[ \frac{p_1}{\rho_0} \frac{\partial \bar{T}_0}{\partial x} \right],$$ (4-30)

Eqs. (4-29) states that Coriolis forces acting on meridional flows $f(\partial \bar{T}_1; \bar{v}_0)$ and $f(\partial \bar{T}_0; \bar{v}_0)$ are balanced with forces due to the first order radiation stresses $\frac{\partial}{\partial y}(\rho_0' \rho_0 \frac{\partial \bar{T}_1}{\partial x})$ and $\frac{\partial}{\partial z}(\rho_0' \rho_0 \frac{\partial \bar{T}_0}{\partial x})$, respectively. Since $\partial \bar{T}_1; \bar{v}_0/ \partial y$ is a part of the first order latitudinal mixing, the first equation of (4-29) is a correction term to the leading order balance equation (4-27). The direction of the force $\frac{\partial}{\partial y}(\rho_0' \rho_0 \frac{\partial \bar{T}_1}{\partial x})$ is westward in the central region and eastward near the vertical walls in the lower layer, while in the upper layer it is reversed (see Fig. 3(b)).

It should be noted here that in the present problem, different from the case of stationary wave incident on a critical level treated by Matsuno and Nakamura (1978), force $\frac{\partial}{\partial z}(\rho_0' \rho_0 \frac{\partial \bar{T}_0}{\partial x})$ does not include the buoyancy flux term; in fact, making use of eq. (4-9), we can obtain that

$$\frac{p_0}{\rho_0} \frac{\partial \bar{T}_0}{\partial x} = - f \frac{fL}{N^2} \left[ \frac{\partial}{\partial t} \frac{\bar{u}}{\bar{v}_0^2} + \frac{g}{\rho_0} \frac{\bar{v}_0}{\bar{v}_0} \right],$$ (4-31)

Since $\bar{v}_0$ is independent of height in the present case, it follows that

$$\frac{\partial}{\partial z} \left[ \rho_0 \frac{\partial \bar{T}_0}{\partial x} \right] = - \frac{f^2 \bar{A}}{2N^2} \frac{\partial}{\partial \bar{z}} \bar{\bar{z}} \bar{\bar{z}} - \frac{g}{\rho_0} \frac{\bar{v}_0}{\bar{v}_0},$$

$$= - \frac{k \rho_0 \bar{c}}{N^2 \rho_0^2} \sin^2 \bar{y} \cdot \frac{d}{dz} \left[ \frac{\bar{c}}{|U - c_0|^2} \right] \cdot e^{i\omega \bar{c} \bar{c}},$$ (4-32)

Thus, the force is determined only by the particle mixing term. $\bar{\bar{z}}$ attains a maximum value at the mid-level and decreases toward the top and the bottom, and hence $\frac{\partial}{\partial \bar{z}}(\rho_0' \rho_0 \frac{\partial \bar{T}_0}{\partial x})$ is negative (directed westward) in the lower layer and positive (directed eastward) in the upper layer. On the other hand, in case of stationary wave, $\frac{\partial}{\partial \bar{z}}(\rho_0' \rho_0 \frac{\partial \bar{T}_0}{\partial x})$ is zero and therefore $\frac{\partial}{\partial \bar{z}}(\rho_0' \rho_0 \frac{\partial \bar{T}_0}{\partial x})$ is determined by buoyancy flux which varies discontinuously at a critical level. Thus, if a wave is incident on the critical level from below, a net westward force acts on the fluid above the level, and a strong northward Stokes drift is induced there, on which Coriolis force acts to balance with the westward force (Matsuno and Nakamura, 1978).

Eq. (4-30) shows that among various components of radiation stress and Stokes drift, only $\frac{\partial}{\partial \bar{z}}(\rho_0' \rho_0 \frac{\partial \bar{T}_0}{\partial x})$ and $(\frac{\partial}{\partial \bar{y}}) \bar{\bar{z}} \bar{v}_1$ can contribute to changing the Lagrangian mean zonal flow. As already seen in Fig. 5, except near the vertical walls, $\frac{\partial \bar{U}_L}{\partial t}$ is almost cancelled by but slightly larger than $\frac{\partial \bar{U}_L}{\partial t}$ in magnitude. According to eq. (4-30), this means that Coriolis force is almost in balance with force due to radiation stress $\frac{\partial}{\partial \bar{y}}(\rho_0' \rho_0 \frac{\partial \bar{T}_1}{\partial x})$ but the former is slightly larger than the latter. Thus, the direction of $\frac{\partial \bar{U}_L}{\partial t}$ is reverse to that of $\frac{\partial \bar{U}_L}{\partial t}$. In other words, the force due to radiation stress $\frac{\partial}{\partial \bar{y}}(\rho_0' \rho_0 \frac{\partial \bar{T}_1}{\partial x})$ is not a cause of the zonal flow acceleration but plays a role of a kind of restoring force. This result can be attributed to the fact that there appears the divergence in Lagrangian mean flow field even in the first order in Rossby number. As is shown in Fig. 6 which presents the distribution of velocity vectors $(\vec{V} + (\bar{\bar{z}} \bar{v}_0) \bar{v}_1; \vec{W}_L \times 500)$, there are northward flows from source to sink in the upper layer of the central region, while southward flows in the lower layer. It is Coriolis force acting on these flows that causes the eastward (westward) acceleration in the upper (lower) layer in the central region.

The above discussions emphasize the role of
Fig. 6 Lagrangian mean meridional velocity field responsible to the mean zonal acceleration.

mass source and sink so as to keep an analogy to the role of heat source and sink in Eulerian mean problem. An alternative interpretation can be performed, based on the balance of forces in the meridional direction, as follows.

Applying a similar manipulation deriving eq. (4-26) to meridional component of zonal mean equation of motion, we can obtain that, to the leading order in Rossby number,

$$f \bar{U}_L = - \frac{1}{\rho_0} \frac{\partial \bar{p}_L}{\partial y} + \frac{\partial}{\partial y} \left( \frac{p_0}{\rho_0} \frac{\partial \bar{\eta}_0}{\partial y} \right),$$  

(4-33)

where

$$\bar{p}_L = \bar{p} + \frac{\partial}{\partial y} (\eta_0 p_0)$$

and

$$\frac{\partial}{\partial y} \left( \frac{p_0}{\rho_0} \frac{\partial \bar{\eta}_0}{\partial y} \right) = \frac{\rho \cos 2\eta}{2f \rho_0^2} \frac{|\bar{\rho}_0|^2}{|U - c_0|^2} \left( U - c_0 \right) e^{2\kappa x \alpha t},$$

$$\frac{\partial}{\partial y} (\eta_0 p_0) = \frac{\rho \sin 2\eta y}{2f \rho_0} \frac{|\bar{\rho}_0|^2}{|U - c_0|^2} \left( U - c_0 \right) e^{2\kappa x \alpha t},$$  

(4-34)

It is noted that there appear not only Lagrangian mean pressure gradient $\left( 1/\rho_0 \right) (\partial \bar{p}_L/\partial y)$ but also force due to radiation stress $\partial/\partial y \left[ (\rho_0/\rho_0) (\partial \bar{\eta}_0/\partial y) \right]$. As is seen from the functional forms of Stokes correction to pressure $\left( \partial/\partial y \right) \eta_0 p_0$ and radiation stress term, the Lagrangian mean pressure gradient increases in the upper layer of the central region, reflecting the particle mixing, while force due to radiation stress counteracts it. The former is, however, larger than the latter, and hence particles move toward north. Then, Coriolis force acts on the motion, to produce the eastward acceleration of Lagrangian mean zonal motion. This mechanism is reverse to that in Eulerian mean problem, in which mean meridional pressure gradient decreases over the entire depth of the fluid layer, as a result of heat transport, and hence southward flows are induced in the upper layer of the central region, decelerating mean zonal flow. Further, corresponding to the eastward (westward) acceleration in the upper (lower) layer in the central region, we can see a Lagrangian mean meridional motion like a direct circulation (Fig. 6).

Finally, we note that in Lagrangian mean problem, the so-called thermal wind relationship between $\bar{U}_L$ and $\bar{\rho}_L$ does not hold (Andrews and McIntyre, 1978). In the present problem, by a similar manipulation to derive eq. (4-26), we can obtain that, to the leading order in Rossby number,

$$0 = - \frac{1}{\rho_0} \frac{\partial \bar{p}_L}{\partial z} - \frac{g}{\rho_0} \bar{\rho}_L + \frac{\partial}{\partial y} \left( \frac{p_0}{\rho_0} \frac{\partial \bar{\eta}_0}{\partial z} \right),$$  

(4-35)

where

$$\bar{\rho}_L = \bar{\rho} + \frac{\partial}{\partial y} \eta_0 p_0$$

Then, we see that $\bar{p}_L$ and $\bar{\rho}_L$ are not in hydrostatic balance due to an additional force $\partial/\partial y \left[ (\rho_0/\rho_0) (\partial \bar{\eta}_0/\partial z) \right]$. Combining this with eq. (4-33), we have

$$f \frac{\partial \bar{U}_L}{\partial z} = \frac{g}{\rho_0} \frac{\partial \bar{\rho}_L}{\partial y} + \frac{1}{\rho_0} \frac{\partial}{\partial y} \left( \frac{p_0}{\rho_0} \frac{\partial \bar{\eta}_0}{\partial z} \right) \left( \eta_0 \frac{\partial \bar{\eta}_0}{\partial y} \right) - \frac{\partial}{\partial y} \left[ (\rho_0/\rho_0) (\partial \bar{\eta}_0/\partial z) \right],$$  

(4-36)

5. Eddy diffusivity due to baroclinic wave

Based on the results mentioned so far, we can estimate the so-called eddy diffusivities due to baroclinic wave.

Since the dispersion of air particles is given by $\left( \partial/\partial t \right) \frac{1}{2} \bar{\eta} \bar{p} = \frac{1}{2} \bar{\eta} \bar{p}$ in the latitudinal direction and that by $\left( \partial/\partial z \right) \bar{\xi} \bar{v} = \frac{1}{2} \bar{\xi} \bar{v}$ in the vertical, we write eddy diffusivities $K_H$ and $K_V$ as follows.

$$K_H = \left< \frac{\eta_0 \bar{v}_0}{\zeta} \right> = \frac{1}{DH} \int_0^R \int_0^D \frac{\eta_0 \bar{v}_0 dy dz}{\zeta_0 \bar{v}_0 dy dz},$$

$$K_V = \left< \frac{\bar{v}_0 \bar{w}_0}{\xi} \right> = \frac{1}{DH} \int_0^R \int_0^D \frac{\bar{v}_0 \bar{w}_0 dy dz}{\xi_0 \bar{w}_0 dy dz},$$  

(5-1)

where we retain only the leading order term in
Rossby number.

Substituting the numerical values given in section 3, we obtain that

\[ K_H = 9.6 \times 10^9 \text{ cm}^2/\text{sec} \]
\[ K_V = 8.1 \times 10^9 \text{ cm}^2/\text{sec} \],

where we omit time factor \( e^{2t} \).

The above values agree, at least in their orders in magnitude, with those used in the usual parameterization. Since we omit \( e^{2t} \), the present results can be regarded as eddy diffusivities due to a baroclinic wave which grows to a sufficient amplitude with surface pressure 11.3 mb.

Further, we can estimate buoyancy (heat) flux as follows. Integrating eq. (4-9) with respect to \( y \) and \( z \), we have

\[ \langle -\frac{g}{\rho_0} \rho_0 \rangle = f A \langle \overline{\gamma_0 v_0} \rangle - N^2 \langle \overline{\zeta_0 v_0} \rangle , \]

where the second term on the right is calculated by

\[ \langle \overline{\zeta_0 v_0} \rangle = \frac{1}{DH} \int_0^H \int_0^D \overline{\zeta_0 v_0} \, dy \, dz \],

Substituting the numerical values, we obtain

\[ f A \langle \overline{\gamma_0 v_0} \rangle \approx 2.9 \times 10^3 \text{ cm}^2/\text{sec}^3 \]
\[ N^2 \langle \overline{\zeta_0 v_0} \rangle \approx 6.0 \times 10^3 \text{ cm}^2/\text{sec}^3 \]
\[ \langle -\frac{g}{\rho_0} \rho_0 \rangle \approx 2.3 \times 10^3 \text{ cm}^2/\text{sec}^3 \]

It should be emphasized that, as already mentioned (see section 4 and foot-note on p. 8), latitudinal buoyancy (heat) flux consists of down-gradient transport term \( f A \langle \overline{\gamma_0 v_0} \rangle \) and transverse-gradient transport term \( N^2 \langle \overline{\zeta_0 v_0} \rangle \). In other words, we should not relate the buoyancy (heat) flux straightforwardly with transport due to the particle mixing \( f A \langle \overline{\gamma_0 v_0} \rangle \). If done so, such an estimation will include an error of about 20%, which could be neglected practically (cf. Green, 1970).

6. Conclusions

In the present paper, using the solutions of Eady type problem of baroclinic instability including non-geostrophic effects, we have discussed the Lagrangian mean motion induced by a growing baroclinic wave. The results are summarized as follows.

A baroclinically unstable wave induces, as its second order effect in wave amplitude, the Lagrangian mean meridional motion associated with strong southward (northward) flows and weak downward (upward) flows in the northern (southern) half region of the fluid layer (Fig. 2(a, b, c)). This means that, as is well known, baroclinic instability causes an almost horizontal overturning of the fluid layer. The present results agree qualitatively well with that by Kida's numerical experiment (1977) concerning the behaviors of tropospheric air particles, except for the effect of the tropical Hadley circulation and of the radiative cooling. However, the reason for the radiative cooling. However, the reason for the agreement is not clear at present. Further, we note that our results also agree with that by Riehl and Fultz (1958) in a rotating fluid annulus experiment, so far as the distribution of Lagrangian mean vertical motion is concerned.

The Lagrangian mean meridional motion can be decomposed into two parts by ordering in Rossby number. To the leading order, the meridional component of Lagrangian mean velocity is given by large horizontal Stokes drift \( (\partial/\partial y) \overline{\gamma_0 v_0} \) which means particle mixing or dispersion. To the second order, air particles move downward near the northern wall and upward near the southern one, while they move southward in the upper layer and northward in the lower layer in the central region of the fluid layer, except weak reverse flows near the top and the bottom (Fig. 2(c) and 3). It should be noted that even in case of incompressible fluid, the Lagrangian mean velocity field is divergent not only to the leading order but also to the second order in Rossby number. This divergence is originated predominantly from horizontal mixing term \( (\partial/\partial y) \overline{\gamma_0 v_0} \) including non-geostrophic effect and a little from a part of transverse-gradient transport term \( (\partial/\partial t) \overline{\zeta_0 v_0} \). The divergent property may be attributed to the second order effect of fluctuation of each actual trajectory around the mean trajectory.

If we eliminate horizontal mixing term and a part of transverse-gradient transport from Stokes drifts, we can obtain the solenoidal part of Lagrangian mean velocity field (Fig. 4). This residual circulation is somewhat similar to the Eulerian mean meridional circulation, and may be regarded as an equivalence to the meridional circulation induced by a dissipating planetary wave.

Concerning the acceleration of Lagrangian mean zonal flow, it is noted that only a part of the second order field can be responsible to the
change in mean zonal motion, and emphasized that the direction of the acceleration is reverse to that of force due to radiation stress. For instance, in the upper layer in the central region of the fluid layer, the mean zonal flow is accelerated eastward while in the lower layer it is accelerated westward. This Lagrangian mean picture is completely reversed to that of the Eulerian mean. The above result can be attributed to the divergent property of Lagrangian mean flow; Coriolis force acting on a part of the meridional flow from the wave-induced mass source to the sink causes the zonal flow acceleration, while the force due to radiation stress produced by systematic correlation between non-geostrophic pressure disturbance and geostrophic component of latitudinal displacement of material surface plays a role of a kind of restoring force (see eq. (4-26')). In other words, Lagrangian mean pressure gradient increases, for example, in the upper layer of the central region, and northward flow is induced. On this flow, Coriolis force acts, to accelerate the mean zonal flow.

Based on the present results, we can estimate the so-called eddy diffusivity due to baroclinic wave. Under the assumed condition which can be regarded as a typical cyclone, we obtain that $K_H = 9.6 \times 10^9 \text{cm}^2/\text{sec}$ and $K_V = 8.1 \times 10^9 \text{cm}^2/\text{sec}$, which are in agreement with those usually used, at least, in their order of magnitude.

Finally, we point out that latitudinal buoyancy (heat) flux consists of two parts, i.e., down-gradient transport $fA\langle <y_0v_0> \rangle$ and transverse-gradient transport $N^2\langle <z_0v_0> \rangle$. The latter is about 20% of the former in magnitude. If one should relate the buoyancy (heat) flux straightforwardly with the down-gradient transport (or particle mixing) term, one might overestimate the flux about 20%.

The effect of $\beta$-term including the so-called critical level instability will be discussed in the next paper in preparation.

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Appendix

Following Derome and Dolph (1970), from the first and the second order equations, we can obtain that

$$ik(U-c_0)u_0 - f v_1 = \frac{ik}{\rho_0} p_1,$$  \hspace{1cm} (A-1)

$$ik v_1 - \frac{\partial u_1}{\partial y} = \frac{1}{f \rho_0} \left( \frac{\partial^2 p_1}{\partial y^2} - k^2 p_1 \right),$$  \hspace{1cm} (A-2)

$$ik(U-c_0) \left[ ik v_1 - \frac{\partial u_1}{\partial y} - f \frac{\partial w_1}{\partial z} \right] + \frac{\partial w_0}{\partial y} = ikc_1 \left( ik v_0 - \frac{\partial u_0}{\partial y} \right),$$  \hspace{1cm} (A-3)

$$ik(U-c_0) \frac{\partial p_1}{\partial z} - f \lambda p_{00} v_1 + N^2 \rho_{00} w_1 = 0,$$  \hspace{1cm} (A-4)

Making use of geostrophic relationships between $p_0$ and $(u_0, v_0)$ and adiabatic equation (2-2), we can eliminate $v_1$ and $w_1$ from the above set of equations and obtain that

$$\frac{\partial^2 p_1}{\partial z^2} + \frac{N^2}{f^2} \left( \frac{\partial^2 p_1}{\partial y^2} - k^2 p_1 \right) = -\frac{2A}{f} \frac{\partial^2 p_0}{\partial y \partial z},$$  \hspace{1cm} (A-5)

Or, equivalently, we can deduce equation for $v_1$ as follows:

$$\frac{\partial^2 v_1}{\partial z^2} + \frac{N^2}{f^2} \left( \frac{\partial^2 v_1}{\partial y^2} - k^2 v_1 \right) = -\frac{4A}{f} \frac{\partial^2 v_0}{\partial y \partial z},$$  \hspace{1cm} (A-6)

Derome and Dolph (1970) have solved (A-6) under appropriate boundary conditions for $v_1$. In the present paper, in order to keep a correspondence to eqs. (2-1) and (2-2), we solve eq. (A-5) with boundary conditions

$$v_1 = 0 \text{ at } y = 0, D$$
$$w_1 = 0 \text{ at } z = 0, H$$  \hspace{1cm} (A-7)

These conditions can be rewritten as\(^{(1)}\)

$$\left[ p_1(y, z) \right]_{y = 0, D} = \frac{U-c_0}{f} \left[ \frac{\partial p_0}{\partial y} \right]_{y = 0, D},$$  \hspace{1cm} (A-8)

$$\left[ (U-c_0) \frac{\partial p_1}{\partial z} - A p_1 \right]_{z = 0, H} = 0.$$

\(^{(1)}\) These boundary conditions are inconsistent with each other at the corners of the fluid channel (Derome and Dolph, 1970).
Multiplying eq. (A-5) by $p_0$ and integrating the result in the region $0 \leq y \leq D$ and $0 \leq z \leq H$ under the boundary condition (A-8), we can show that $C_1 = 0$ if the instability condition (2-8) is satisfied (cf. Derome and Dolph, 1970). Then, substituting (2-5) into eq. (A-5) and boundary condition (A-8), we obtain eq. (2-10) with boundary condition (2-11). As mentioned in section 2, eq. (2-10) can be reduced to eq. (2-14) by substituting (2-12) and (2-13). We note that eq. (2-14) is same as that obtained by substituting $v_1 = \Sigma v_m \sin m Ly$ into eq. (A-6).

Considering $v_m = i k_p m / f$, we can readily write down the solution as follows.

\[
v_m = C_m e^{x \pm i y} + D_m e^{-x \mp i y} \cdot \frac{1}{f} \int_{0}^{H} \frac{4A}{\pi (m^2 - 1)} \cdot \frac{\mu}{\mu^2 - \mu_m^2} \cdot \left( \begin{array}{c} A e^{x + i y} - B e^{-x + i y} \\ A e^{-x + i y} - B e^{x - i y} \end{array} \right) \quad (2-16)
\]

where

\[
C_m = \frac{ik}{f} \cdot \frac{1}{2Q} \left[ \frac{F(A + \mu_m (Uz - C_0))}{G(A + \mu_m C_0)} \right] e^{-x \mp i y} - G(A - \mu_m C_0)
\]

\[
D_m = \frac{ik}{f} \cdot (1/2Q) \left[ - F(A - \mu_m (Uz - C_0)) e^{x \pm i y} + G(A + \mu_m C_0) \right]
\]

\[
Q = A^2 \sinh \mu_m H - \mu_m A Uz \cosh \mu_m H + \mu_m^2 C_0 \pm \mu_m C_0 (Uz - C_0) \sinh \mu_m H
\]

\[
F = 2A \int_{0}^{H} \frac{4m}{\pi (m^2 - 1)} \cdot \frac{1}{\mu_m^2 - \mu_m^2} \cdot \left\{ A^2 + B^2 \right\} C_0
\]

\[
G = 2A \int_{0}^{H} \frac{4m}{\pi (m^2 - 1)} \cdot \frac{1}{\mu_m^2 - \mu_m^2} \cdot \left\{ A e^{x \pm i y} - B e^{-x \mp i y} \right\}
\]

\[
\times \left\{ (Uz - C_0) \mu_m^2 - \mu_m^2 - 2A^2 \right\}
\]

where we set $U_1 = 0$.

Using the solution above and the following kinematic relationship

\[
v_1 = i k (U - C_0) \eta_1,
\]

we can obtain that

\[
v_1 \eta_1 = - \frac{e^{2kz} \sin ly}{2f \rho_0 |U - C_0|^2} \cdot \sum_{m: \text{even}}^{\infty} \text{Re}[(U - C_0) \rho_0 v_m] \sin mLy,
\]

\[
v_1 \eta_0 = \frac{e^{2kz} \sin ly}{2f \rho_0 |U - C_0|^2} \cdot \sum_{m: \text{even}}^{\infty} \text{Re}[(U - C_0) \rho_0 v_m^*] \sin mLy.
\]


傾圧不安定に伴うラグランジュ平均運動

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Eady 型の傾圧不安定論の解を用いて、時間的に増幅する傾圧波動に伴うラグランジュ平均運動をしらべた。

ロスビー数の最低次の運動として、傾圧不安定波の発達と共に、空気粒子は水平混合するが、次のオーダーで南北境界壁付近に上昇・下降運動がおこり、中緯度付近では上層で南向き、下層で北向きの運動がおこる（但し上面と下面付近には、ゆるやかな逆向きの流れがある）。これは通常のオイラー平均の子午面内循環とは大きく異っているが、Kida (1977) の数値実験における対流圏空気粒子の動きおよび Riehl・Fultz (1957) の回転水槽実験の解析結果と定性的に一致している。

ラグランジュ平均の子午面運動が Boussinesq 近似の下でも発散（収束）を示す (Andrews・McIntyre (1978)) という結果は、主として水平混合の効果に起因し一部 transverse-gradient 輸送の項も寄与している。

それゆえ、ストークス速度からこれらの項を差引くとラグランジュ平均子午面運動は非発散となり、その循環は非逸しつつある波によるもの（Matsumoto・Nakamura, 1978）と等価である。

最後に以上の結果に基づいて、いわゆる渦拡散係数を見積ると、典型的な低気圧の条件下で, $K_h=9.6 \cdot 10^7 \text{cm}^2/\text{sec}$, $K_v=8.1 \cdot 10^4 \text{cm}^2/\text{sec}$ がえられる。

なお、傾圧不安定波による南北の熱輸送は down-gradient 輸送（粒子の混合）の項と transverse-gradient 輸送の項から成り立っているが、後者は前者の20%位の大きさである。