A Generalization of "Eliassen-Palm Relation"

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Abstract

Eliassen and Palm (1961) has pointed out that two dimensional steady internal gravity waves do not accelerate the zonal mean flow if there is no external forces or dissipation, and Andrews and McIntyre (1978 a, b) have generalized this theorem. It is pointed out that the so-called "Eliassen-Palm relation" is the direct result from the conservations of energy and momentum of material layers by using the fact that energy and momentum fluxes through a material surface are connected by the phase velocity. From Lagrangian view point, it is shown that the difference between the Reynolds stress and the radiation stress results from the Lagrangian transversal flow due to the waves, and the "Eliassen-Palm relation" means in general that steady waves do not accelerate the mean flow in Lagrangian meaning.

In cases of three-dimensional rotational flow and quasi-geostrophic flow, this fact explains the meaning of Uryu's results (1973) and Charney and Drazin's results (1961), respectively. We can also obtain the mean flow acceleration when there are external forces and dissipation from the Lagrangian view point.

The theorem proved by Eliassen and Palm (1960) and generalized by Andrews and McIntyre (1976a) and also by Boyd (1976), is discussed from a somewhat different viewpoint from that so far.

1. Introduction

When an airstream flows over a surface topography like a mountain, some atmospheric waves, such as internal gravity waves, inertia gravity waves or quasi-geostrophic waves are excited. These waves propagate upward and at the same time transport energy and momentum or angular momentum. When the fluid is inviscid and non-rotating and there is no external forces and the basic flow depends only on height, Eliassen and Palm (1961) showed that the Reynolds stress $\overline{uw}$ (u and w are the velocity perturbations in zonal and vertical directions, respectively, and the bar means zonal average) is independent of height except at a critical level where the phase velocity of waves coincides with the velocity of the basic flow. This relation is sometimes called "Eliassen-Palm relation". When the fluids are rotating, $\overline{(u-f\eta)w}$ (η is the meridional displacement of a fluid particle due to wave, and f is the Coriolis' parameter) instead of Reynolds stress $\overline{uw}$ is independent of height (Jones, 1967). This term

$\overline{(u-f\eta)w}$ is interpreted as the angular momentum flux (Bretherton, 1967). When the fluid is quasi-geostrophic, Charney and Drazin (1961) showed that under the same assumption as Eliassen and Palm, the quasi-geostrophic waves do not interact with the basic flow. When the zonal current depends not only on height but also on meridional direction, Uryu (1973) showed that the 'wave momentum flux' is solenoidal in the meridional plane. Boyd (1976) also showed that the mean zonal flow does not interact with waves even on the spherical surface when the mean flow is geostrophic. These results show that the interaction between wave and mean flow does not occur when the waves are inviscid and stationary and there is no external forces. Andrews and McIntyre have discussed the "Eliassen-Palm relation" from a view point of Lagrangian mean flow (1978 a, b), and applied the Lagrangian mean flow concept to the mean flow acceleration by equatorial waves (1976 a, b). Their theory is very general, but somewhat hard to follow. In this paper we point out that if we restrict the discussion to the

second order accuracy, the “Eliassen-Palm relation” can be simply derived from energy and momentum conservations along a material surface, which are derived in Andrews and McIntyre's paper from the phase invariance of waves.

2. Outline of the derivation

In this section we will give a rough outline of the derivation of the “Eliassen-Palm relation”. To simplify the problem we discuss about waves in a homogeneous fluid with a shear of the mean flow. Suppose that there is an infinite inviscid fluid with a stationary wave which has the wave-number k and the horizontal phase velocity c . We think of the energy and the momentum flux through a material surface in the fluids. If this material surface is an undulating solid surface, this surface experiences the wave drag D which is called as the radiation stress (Bretherton, 1971), and momentum is transported to the upper fluid at the rate D from this surface. On the other hand, since the surface is pulled at the velocity c , the machine which is pulling this solid surface may work on the fluid at the rate cD . The time change of the momentum in x -direction M and energy E along the surface may be given by momentum and energy flux D and cD as

$$\frac{\partial M}{\partial t} + \frac{\partial D}{\partial z} = 0 \quad (2-1)$$

$$\frac{\partial E}{\partial t} + c \frac{\partial D}{\partial z} = 0. \quad (2-2)$$

On the other hand, since the wave is stationary, the time change of energy along the surface should be due to the time change of the mean flow, and (2-2) may be rewritten as

$$\rho_0 U \frac{\partial U}{\partial t} + c \frac{\partial D}{\partial z} = 0, \quad (2-3)$$

where ρ_0 is the density of fluid, and U is the velocity of the mean flow. Since the mean momentum M along the surface is $\rho_0 U$, (2-1) becomes as

$$\rho_0 \frac{\partial U}{\partial t} + c \frac{\partial D}{\partial z} = 0. \quad (2-4)$$

Equation (2-3) and (2-4) are energy and momentum equations, respectively, and have a similar form. The key point is that the energy and the momentum flux are connected each other by the phase velocity c .

We multiply (2-4) by the phase velocity c and subtract from (2-3), and we have

$$\rho_0(U-c) \frac{\partial U}{\partial t} = 0. \quad (2-5)$$

This equation says that the mean flow is steady except at the critical level where c coincides with U .

Since the foregoing derivation is a very rough one, we will check the precedent argument more precisely. Before the argument we must explain the material surface. The material surface is a surface consisted of fluid particles which is originally on a plane surface. If zonal and vertical displacements of a particle which was originally at the position (x, z) are denoted by $\xi(x, z, t)$ and $\zeta(x, z, t)$, respectively, the undulation of the surface is expressed as $(x + \xi(x, z, t), z + \zeta(x, z, t))$. If we divide the fluids by this material surface, the interaction between upper and lower fluids must be due to only the pressure drag at the surface, because the fluid is inviscid. After dividing the field into mean and perturbation parts, the pressure at the material surface p_s should be expressed as

$$p_s = P + p, \quad (2-6)$$

where P is the mean pressure which is assumed to be constant at present, and p is the perturbation pressure. Hereafter we restrict the discussion to the second order problem in perturbation amplitude.

The momentum flux D is the horizontal component of the pressure drag at the surface due to the waves and is written as

$$D = -\bar{p} \frac{\partial \bar{\zeta}}{\partial x}, \quad (2-7)$$

where ζ is the first order perturbation displacement and the bar means the time average or space average, or in more general an average with respect to particle phases. Hereafter we expand the mean quantity by wave amplitude, for example, as

$$U = U^{(0)} + U^{(2)} + \dots,$$

where the first term in the right hand side is the basic quantity which is the quantity when wave is absent, and the second term is the second order difference from the basic quantity due to waves.

The energy flux F is the product of pressure and velocity in normal direction to the surface and becomes

$$F = -U^{(0)} \bar{p} \frac{\partial \bar{\zeta}}{\partial x} + \bar{p} w. \quad (2-8)$$

To know the time change of momentum and

energy along a material surface, let us consider the fluids between two surfaces which are separated infinitesimally in vertical direction. A particle which was originally at a position (x, z) is now at a position $(x + \xi(x, z, t), z + \zeta(x, z, t))$ and have the velocity u_s which is written as,

$$u_s = U^{(0)} + U^{(2)} + \frac{dU^{(0)}}{dz} \zeta + \frac{1}{2} \frac{d^2 U^{(0)}}{dz^2} \zeta^2 + \frac{\partial u}{\partial x} \xi + \frac{\partial u}{\partial z} \zeta, \quad (2-9)$$

correctly to the second order in wave amplitude.

Averaging this equation we have the mean momentum Mdz between two material surfaces as

$$Mdz = \rho_0 \bar{u}_s = \rho_0 (U^{(0)} + U^{(2)} + U_s) dz, \quad (2-10)$$

where

$$U_s = \frac{\partial \bar{u}}{\partial x} \xi + \frac{\partial \bar{u}}{\partial z} \zeta + \frac{1}{2} \frac{d^2 U^{(0)}}{dz^2} \zeta^2, \quad (2-11)$$

which is the difference between the averaged particle velocity $\bar{u}_s$ and the mean flow U and is the so-called Stokes' drift.

The mean energy between two surfaces Edz or is also given as

$$Edz = \frac{1}{2} \rho_0 (\bar{u}_s^2 + \bar{w}^2) dz, \quad (2-12)$$

which can be rewritten with aid of (2-11) as

$$\begin{aligned} Edz &= \frac{1}{2} \rho_0 \{ (U^{(0)} + U^{(2)} + U_s + u_i)^2 + \bar{w}^2 \} dz \\ &= \left\{ \frac{1}{2} \rho_0 (U^{(0)} + U^{(2)} + U_s)^2 + E_i \right\} dz, \end{aligned} \quad (2-13)$$

where

$$u_i = u + \frac{dU^{(0)}}{dz} \zeta, \quad (2-14)$$

and

$$E_j = \frac{1}{2} \rho_0 (\bar{u}_i^2 + \bar{w}^2). \quad (2-15)$$

Equation (2-13) is corresponding to the mean energy Ee in Eulerian meaning which is expressed as

$$Ee = \frac{1}{2} \rho_0 U^2 + E_{ei}, \quad (2-16)$$

where

$$E_{ei} = \frac{1}{2} \rho_0 (\bar{u}_i^2 + \bar{w}^2),$$

From this analogy we may interpret E_i as the

perturbation energy along material surface.

The time change of energy E and momentum M must be due to the vertical divergence of energy flux F and momentum flux D through the material surface as

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial z} = 0 \quad (2-17)$$

$$\frac{\partial M}{\partial t} + \frac{\partial D}{\partial z} = 0. \quad (2-18)$$

Since the waves are stationary, U_s and E_i do not vary with time and the above equations are reduced to

$$\rho_0 U^{(0)} \frac{\partial U^{(2)}}{\partial t} + \frac{\partial F}{\partial z} = 0 \quad (2-19)$$

$$\rho_0 \frac{\partial U^{(2)}}{\partial t} + \frac{\partial D}{\partial z} = 0. \quad (2-20)$$

On the other hand, since ζ and w are connected as

$$\left(\frac{\partial}{\partial t} + U^{(0)} \frac{\partial}{\partial x} \right) \zeta = w, \quad (2-21)$$

$$(U^{(0)} - c) \frac{\partial \zeta}{\partial x} = 0, \quad (2-22)$$

w and ζ are out of phase, and $\bar{w}\zeta$ vanishes, and from (2-7) and (2-8) we can see that energy flux F is related to momentum flux D in such a way as

$$F = cD, \quad (2-23)$$

Then, from (2-19), (2-20) and (2-23) we have equations (2-1) and (2-2).

However, more things must be said. In the foregoing argument momentum and energy along a material surface are discussed. On the other hand fluid particles drift vertically in general, and the material surface also drifts. Then the time change may include the effect of the drift of the surface, which is a second order quantity, and equations (2-1) and (2-2) may be rewritten as

$$\rho_0 \left(U^{(0)} \frac{\partial U^{(2)}}{\partial t} + W_s \frac{\partial}{\partial z} \frac{1}{2} U^{(0)2} \right) + c \frac{\partial D}{\partial z} = 0 \quad (2-24)$$

$$\rho_0 \left(\frac{\partial U^{(2)}}{\partial t} + W_s \frac{\partial U^{(0)}}{\partial z} \right) + \frac{\partial D}{\partial z} = 0, \quad (2-25)$$

where W_s is the drift current. These equations may also be rewritten as

$$(U^{(0)} - c) \left(\frac{\partial}{\partial t} + W_s \frac{\partial}{\partial z} \right) U = 0 \quad (2-26)$$

$$(U^{(0)} - c) \frac{\partial D}{\partial z} = 0. \quad (2-27)$$

These energy and momentum equations say that the mean flow does not change by waves in Lagrangian meaning.

In the above argument the effect of vertical drift is introduced physically, not mathematically from Eulerian equations. In the next section we will demonstrate a simple example in which mean flow does not change in Lagrangian meaning from Eulerian equations. In the third section we will extend this result to rotational and stratified fluids.

3. Sound wave case

In this section we will demonstrate a typical example of the effect of the Stokes' drift. The derivation follows Eliassen and Palm, and it may not be systematic.

The fact that stationary sound waves propagating in a shear flow have the second order mass flux which corresponds to the Stokes' drift suggests us that the shear flow may be modified by the drift. We will demonstrate that the mean shear flow pattern also drifts, but does not change its shape, that is, the Lagrangian time change of the mean flow is vanished.

First we will discuss the simplest case that inviscid fluids flow in x -direction with a shear in z -direction and a steady sound wave is propagating in z -direction. In this case the discussion in section 2 must be modified a little, because there is no undulating material surface. A material surface which is initially along a line in x -direction oscillates in z -direction. The momentum flux D through this surface may be $(\partial p / \partial z) \xi$, and the energy flux F may be expressed as $\overline{pw}$, and w is $-c(\partial \xi / \partial z)$. Then F and D are also connected as (2-23), and the result in section 1 may hold.

The perturbation equations are as follows,

$$\frac{\partial u}{\partial t} + \frac{\partial U^{(0)}}{\partial z} w = 0 \quad (3-1)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} \quad (3-2)$$

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial w}{\partial z} = 0 \quad (3-3)$$

$$p = c_s^2 \rho, \quad (3-4)$$

where c_s is the sound velocity. Here, all symbols have conventional meanings.

In this case u is only a passive quantity, and w , ρ and p satisfy the plane sound wave equations, and c_s becomes the phase velocity, *e.g.*,

$$\frac{\partial^2 w}{\partial t^2} = c_s^2 \frac{\partial^2 w}{\partial z^2}.$$

The mean flow equation is expressed as

$$\frac{\partial U^{(2)}}{\partial t} + \overline{w \frac{\partial u}{\partial z}} = 0 \quad (3-5)$$

where a bar means time average. Here, we have assumed that there is no secondary Eulerian mean flow in z -direction due to waves; $W^{(2)} = 0$. Since the wave is assumed to be steady and sinusoidal in z -direction, the local time derivative is expressed as

$$\frac{\partial}{\partial t} = -c \frac{\partial}{\partial z}. \quad (3-6)$$

Then the second term of (3-5) is rewritten by using (3-1) as

$$\overline{w \frac{\partial u}{\partial z}} = \frac{dU^{(0)}}{dz} \frac{\overline{w^2}}{c} \quad (3-7)$$

On the other hand, from (3-3) referring to (3-6) we have

$$-c \frac{\partial \rho}{\partial z} + \rho_0 \frac{\partial w}{\partial z} = 0, \quad (3-8)$$

or

$$c\rho = \rho_0 w,$$

because of sinusoidality. Then, multiplying by w and averaging the result, we obtain

$$\frac{\rho_0}{c} \overline{w^2} = \overline{\rho w}. \quad (3-9)$$

Since $\overline{\rho w}$ is the rate of mass flux, it may become the Stokes' drift. For, the Stokes' drift may be expressed as

$$W_s = \frac{\partial w}{\partial z} \xi = -w \frac{\partial \xi}{\partial z} \quad (3-10)$$

where use is made of sinusoidal property of the wave in z -direction. From (3-3) and referring to $w = -c(\partial \xi / \partial z)$, we have

$$\frac{\partial \xi}{\partial z} = -\frac{\rho}{\rho_0}, \quad (3-11)$$

Substituting (3-11) into (3-10) we see that mass flux $\overline{\rho w}$ is due to the Stokes' drift W_s , and (3-9) can be rewritten as

$$\frac{\rho_0}{c} \overline{w^2} = \rho_0 W_s, \quad (3-12)$$

From (3-5), (3-7) and (3-12), the expected equation is obtained as

$$\frac{\partial U^{(2)}}{\partial t} + W_s \frac{\partial U^{(0)}}{\partial z} = 0. \quad (3-13)$$

This equation says that the mean shear flow only drifts by waves and does not change its shape.

Next, we discuss a more general case that waves propagate obliquely to z -direction. The perturbation equations are as follows,

$$(U^{(0)} - c) \frac{\partial u}{\partial x} + \frac{dU^{(0)}}{dz} w = -\frac{c_s^2}{\rho_0} \frac{\partial \rho}{\partial x} \quad (3-14)$$

$$(U^{(0)} - c) \frac{\partial w}{\partial x} = -\frac{c_s^2}{\rho_0} \frac{\partial \rho}{\partial z} \quad (3-15)$$

$$(U^{(0)} - c) \frac{\partial \rho}{\partial x} + \rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) = 0, \quad (3-16)$$

where c is the phase velocity in x -direction. We have already eliminated p and the time derivative by using (3-4) and an relation analogous to (3-6) as

$$\frac{\partial}{\partial t} = -c \frac{\partial}{\partial x}. \quad (3-17)$$

The mean flow equation is given by

$$\frac{\partial U^{(2)}}{\partial t} + w \frac{\partial u}{\partial z} = 0. \quad (3-18)$$

From the perturbation equations we have the same relation as Eliassen and Palm's one,

$$(U^{(0)} - c) \frac{\partial}{\partial z} \overline{uw} = 0, \quad (3-19)$$

following the same derivation.

Then, the Reynolds stress is independent of height if $U^{(0)} \neq c$ everywhere in the fluid layer. From the following identity

$$w \frac{\partial u}{\partial z} = \frac{\partial}{\partial z} \overline{uw} - u \frac{\partial w}{\partial z}, \quad (3-20)$$

and equation (3-18), it follows that if the fluid is incompressible mean flow does not change except at the critical level, because in (3-20) vanishes. In case of compressible fluid, however, this conclusion does not hold. The second term on the right of (3-20) can be written, with use of (3-16), as

$$-u \frac{\partial w}{\partial z} = u \left(\frac{U^{(0)} - c}{\rho_0} \frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial x} \right). \quad (3-21)$$

Then, using (3-19), (3-20) and (3-21), we have

$$w \frac{\partial u}{\partial z} = \frac{U^{(0)} - c}{\rho_0} u \frac{\partial \rho}{\partial x} = -\frac{U^{(0)} - c}{\rho_0} \rho \frac{\partial u}{\partial x}. \quad (3-22)$$

Multiplying (3-14) by ρ and averaging we obtain

$$(U^{(0)} - c) \rho \frac{\partial u}{\partial x} + \frac{dU^{(0)}}{dz} \overline{\rho w} = 0. \quad (3-23)$$

From this equation and (3-22), we have

$$\overline{w \frac{\partial u}{\partial z}} = \frac{1}{\rho_0} \frac{dU^{(0)}}{dz} \overline{\rho w}. \quad (3-24)$$

The right hand side of this equation includes the vertical mass flux $\overline{\rho w}$ which can also be represented by the Stokes' drift W_s . Using equation (3-16) and equations

$$(U^{(0)} - c) \frac{\partial \xi}{\partial x} = u + \frac{dU^{(0)}}{dz} \zeta \quad (3-25)$$

$$(U^{(0)} - c) \frac{\partial \zeta}{\partial x} = w, \quad (3-26)$$

we have

$$-\rho_0 \left(\frac{\partial \xi}{\partial x} + \frac{\partial \zeta}{\partial z} \right) = \rho. \quad (3-27)$$

Then we obtain

$$\frac{\partial w}{\partial x} \xi + \frac{\partial w}{\partial z} \zeta = -w \left(\frac{\partial \xi}{\partial x} + \frac{\partial \zeta}{\partial z} \right) = \frac{1}{\rho_0} \overline{w \rho}. \quad (3-28)$$

Here, the fact that w and ζ are out of phase is used. Since the left hand side is identical with the Stokes' drift W_s , equation (3-24) may be reduced to

$$\overline{w \frac{\partial u}{\partial z}} = \frac{dU^{(0)}}{dz} W_s. \quad (3-29)$$

This equation and (3-18) indicate that

$$\frac{dU^{(2)}}{dt} + W_s \frac{dU^{(0)}}{dz} = 0, \quad (3-30)$$

which is the same equation as (3-13) we have expected.

4. Case of three-dimensional stratified flow with rotation

In this section we will demonstrate a systematic derivation of "Eliassen-Palm relation" in a case of three-dimensional stratified flow with rotation under the Boussinesq approximation. This method can be extended to sound waves, and may be to other steady waves.

Before calculation we must re-examine the energy conservation relation.

The basic equations with Boussinesq approximation are

$$\frac{Du}{Dt} - fv = -\frac{\partial p}{\partial x} \quad (4-1)$$

$$\frac{Dv}{Dt} + fu = -\frac{\partial p}{\partial y} \quad (4-2)$$

$$\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - \frac{g}{\rho_{00}}\rho \quad (4-3)$$

$$\frac{D\rho}{Dt} = 0, \quad (4-4)$$

where D/Dt is the Lagrangian derivative, f is the Coriolis parameter, ρ_{00} is the mean reference density, g is the acceleration of gravity, and p is the pressure divided by ρ_{00} .

We introduce a reference stratification $\rho_0(z)$, which may vary vertically, to define clearly the available potential energy, and we modify the pressure and density as

$$\rho^* = \rho - \rho_0(z) \quad (4-5)$$

$$p^* = p - p_0(z), \quad (4-6)$$

where

$$p_0(z) = -\frac{g}{\rho_{00}} \int^z \rho_0(z) dz,$$

The modification $p_0(z)$ is due to the buoyancy from the background reference stratification $\rho_0(z)$. Then (4-3) and (4-4) are reduced to

$$\frac{Dw}{Dt} = -\frac{\partial p^*}{\partial z} - \frac{g}{\rho_{00}}\rho^* \quad (4-7)$$

$$\frac{D\rho^*}{Dt} + w \frac{d\rho_0}{dz} = 0. \quad (4-8)$$

Using (4-1), (4-2), (4-7) and (4-8), we have the energy equation as

$$\begin{aligned} \frac{D}{Dt} \frac{1}{2} (u^2 + v^2 + w^2) - \alpha \frac{D}{Dt} \frac{\rho^{*2}}{2} \\ = - \left(\frac{\partial}{\partial x} \overline{u p^*} + \frac{\partial}{\partial y} \overline{v p^*} + \frac{\partial}{\partial z} \overline{w p^*} \right), \end{aligned} \quad (4-9)$$

where

$$\alpha = \frac{g}{\rho_{00}} \left(\frac{d\rho_0}{dz} \right)^{-1},$$

which may depend on z . If α is constant we have the available potential energy $(-\alpha/2)\rho^{*2}$, and the total energy $1/2(u^2 + v^2 + w^2 - \alpha\rho^{*2})$. It is noticeable that α includes the gradient of the reference stratification, not the real mean stratification. When the flow is geostrophic and have a vertical shear, the mean density varies not only in vertical but also in horizontal direction, and we cannot make reference density stratification coincide with the mean density.

The perturbation equations are as follows

$$\begin{aligned} (U^{(0)} - c) \frac{\partial u}{\partial x} - f v + \frac{\partial U^{(0)}}{\partial y} v + \frac{\partial U^{(0)}}{\partial z} w \\ = - \frac{\partial p^*}{\partial x} \end{aligned} \quad (4-10)$$

$$(U^{(0)} - c) \frac{\partial v}{\partial x} + f u = - \frac{\partial p^*}{\partial y} \quad (4-11)$$

$$(U^{(0)} - c) \frac{\partial w}{\partial x} = - \frac{\partial p^*}{\partial z} - \frac{g}{\rho_{00}}\rho \quad (4-12)$$

$$(U^{(0)} - c) \frac{\partial \rho}{\partial x} + \frac{\partial \bar{\rho}^{(0)}}{\partial y} v + \frac{\partial \bar{\rho}^{(0)}}{\partial z} w = 0 \quad (4-13)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (4-14)$$

where small letters mean perturbation quantities, and capital letters or bar mean quantities. Since the perturbation of density ρ^* coincides with ρ , the asterisk is omitted in the buoyancy term in (4-12).

The equations of the mean state are also derived as follows,

$$\begin{aligned} \frac{\partial U^{(2)}}{\partial t} + \frac{\partial}{\partial y} \overline{u v} + \frac{\partial}{\partial z} \overline{u w} + \frac{\partial U^{(0)}}{\partial y} V^{(2)} \\ + \frac{\rho U^{(0)}}{\partial z} W^{(2)} - f V^{(2)} = 0 \end{aligned} \quad (4-15)$$

$$f U^{(0)} = - \frac{\partial P^{*(0)}}{\partial y} \quad (4-16)$$

$$0 = - \frac{\partial P^{*(0)}}{\partial z} - \frac{g}{\rho_{00}} \bar{\rho}^{*(0)} \quad (4-17)$$

$$\begin{aligned} \frac{\partial \bar{\rho}^{(2)}}{\partial t} + \frac{\partial}{\partial y} \overline{\rho v} + \frac{\partial}{\partial z} \overline{\rho w} + \frac{\partial \bar{\rho}^{(0)}}{\partial y} V^{(2)} \\ + \frac{\partial \bar{\rho}^{(0)}}{\partial z} W^{(2)} = 0. \end{aligned} \quad (4-18)$$

In the above equations for the perturbations and the mean quantities the Coriolis' parameter f may depend on y but not on z . The course of the calculation process is to express the momentum and energy change due to waves by the Stokes' drift and radiation stress instead of Reynolds stress and energy flux $\overline{p v}$.

To obtain the drag or the radiation stress, which are the key quantities in our discussion, we multiply (4-10) by ζ , as Uryu (1973) or McIntyre and Andrews (1976), and average the equation. Then we have

$$\begin{aligned} (U^{(0)} - c) \frac{\partial \overline{u \zeta}}{\partial x} - f \overline{v \zeta} + \frac{\partial U^{(0)}}{\partial y} \overline{v \zeta} + \frac{\partial U^{(0)}}{\partial z} \overline{w \zeta} \\ = - \frac{\partial \overline{p \zeta}}{\partial x}. \end{aligned} \quad (4-19)$$

The right hand side of (4-19) is rewritten as

$$- \frac{\partial \overline{p \zeta}}{\partial x} \zeta = \overline{p \frac{\partial \zeta}{\partial x}} = - D^x \quad (4-20)$$

which is a part of radiation stress caused by

systematic correlation between pressure perturbation and vertical displacement of material surface. This can be regarded as a pressure drag caused by the undulating material surface. The first term of (4-19) may be rewritten as

$$(U^{(0)} - c) \frac{\partial \bar{u}}{\partial x} \zeta = -(U^{(0)} - c) \frac{\partial \bar{\zeta}}{\partial x} u. \quad (4-21)$$

Since ζ and w are connected by the relation

$$w = (U^{(0)} - c) \frac{\partial \bar{\zeta}}{\partial x}, \quad (4-22)$$

we obtain

$$(U^{(0)} - c) \frac{\partial \bar{u}}{\partial x} \zeta = -uw \quad (4-23)$$

$$\bar{w}\zeta = -(U^{(0)} - c) \frac{\partial \bar{\zeta}}{\partial x} \zeta = 0. \quad (4-24)$$

Hence, from (4-19), (4-20), (4-22), (4-23) and (4-24), we obtain a relation between the Reynolds stress and the radiation stress as

$$uw = D^z - f\bar{v}\zeta + \frac{\partial U^{(0)}}{\partial y} v\zeta. \quad (4-25)$$

Similarly, multiplying (4-10) by η and using

$$v = (U^{(0)} - c) \frac{\partial \eta}{\partial x}, \quad (4-26)$$

and averaging the result zonally we have

$$uv = D^y + \frac{\partial U^{(0)}}{\partial z} w\eta \quad (4-27)$$

where

$$D^y = -p \frac{\partial \eta}{\partial x},$$

which is the drag or the radiation stress in y -direction.

If we use the Lagrangian perturbations u_l , v_l , w_l , as Uryu (1973), which are defined as

$$\begin{aligned} u_l &= u + \frac{\partial U^{(0)}}{\partial z} \eta + \frac{\partial U^{(0)}}{\partial z} \zeta, \\ v_l &= v, \quad w_l = w, \end{aligned} \quad (4-28)$$

and using the equation

$$\begin{aligned} \bar{w}\eta &= (U^{(0)} - c) \frac{\partial \bar{\zeta}}{\partial x} \eta = -(U^{(0)} - c) \frac{\partial \eta}{\partial x} \zeta \\ &= -\bar{v}\zeta \end{aligned} \quad (4-29)$$

(4-25) and (4-27) become as

$$\overline{(u_l - f\eta)v_l} = D^y \quad (4-30)$$

$$\overline{(u_l - f\eta)w_l} = D^z. \quad (4-31)$$

To obtain the change of the mean flow, we

must calculate the divergence of the Reynolds stress. From (4-25) and (4-27), we have

$$\begin{aligned} \frac{\partial}{\partial y} \bar{u}v + \frac{\partial}{\partial z} \bar{u}w &= \nabla \cdot \mathbf{D} + \left(\frac{\partial U^{(0)}}{\partial y} - f \right) \frac{\partial}{\partial z} \bar{v}\zeta \\ &+ \frac{\partial U^{(0)}}{\partial z} \frac{\partial}{\partial y} \bar{w}\eta + \frac{\partial^2 U^{(0)}}{\partial y \partial z} (\bar{w}\eta + \bar{v}\zeta), \end{aligned} \quad (4-32)$$

where

$$\mathbf{D} = (D^y, D^z),$$

and

$$\nabla = \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right).$$

The last term on the right hand side of (4-32) vanishes by (4-29). The second and third terms can be expressed in terms of the Stokes' drift V_s and W_s as follows. Using that

$$V_s = \frac{\partial \bar{v}}{\partial x} \xi + \frac{\partial \bar{v}}{\partial y} \eta + \frac{\partial \bar{v}}{\partial z} \zeta, \quad (4-33)$$

and for a Boussinesq fluid

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0, \quad (4-34)$$

V_s is expressed as

$$V_s = \frac{\partial}{\partial y} \bar{v}\eta + \frac{\partial}{\partial z} \bar{v}\zeta = \frac{\partial}{\partial z} \bar{v}\zeta. \quad (4-35)$$

Similarly we have

$$W_s = \frac{\partial}{\partial y} \bar{w}\eta. \quad (4-36)$$

Notice that the Stokes' drift velocities V_s and W_s are solenoidal from (4-29), (4-35) and (4-36), i.e.,

$$\frac{\partial V_s}{\partial y} + \frac{\partial W_s}{\partial z} = 0.$$

They are not in general solenoidal unless waves are steady (cf. Andrews and McIntyre, 1978a). Using (4-29), (4-35) and (4-36), equation (4-32) is rewritten as

$$\begin{aligned} \frac{\partial}{\partial y} \bar{u}v + \frac{\partial}{\partial z} \bar{u}w &= \nabla \cdot \mathbf{D} + \left(\frac{\partial U^{(0)}}{\partial y} - f \right) V_s \\ &+ \frac{\partial U^{(0)}}{\partial z} W_s. \end{aligned} \quad (4-37)$$

This equation shows that the difference between the divergence of the Reynolds stress and the radiation stress is due to the Stokes' drift. This is because the Reynolds stress is concerned with the Eulerian mean field, while the radiation stress

is concerned with the material surface which drift with fluid particles.

Substituting (4-37) into (4-15), we can obtain the change of the mean flow as

$$\frac{\partial U^{(2)}}{\partial t} + \left(\frac{\partial U^{(0)}}{\partial y} - f \right) V_l + \frac{\partial U^{(0)}}{\partial z} W_l + \nabla \cdot \mathbf{D} = 0, \quad (4-38)$$

where

$$V_l = V_s + V^{(2)} \quad (4-39)$$

$$W_l = W_s + W^{(2)} \quad (4-40)$$

This corresponds to the expected equation (2-25) for three dimensional rotating case.

A similar equation regarding to the change of the mean density can be obtained. From (4-13), (4-22) and (4-26), density perturbation is written as

$$\rho = -\frac{\partial \bar{\rho}^{(0)}}{\partial y} \eta - \frac{\partial \bar{\rho}^{(0)}}{\partial z} \zeta. \quad (4-41)$$

After almost the same process as before we have the time variation of the mean density as

$$\frac{\partial \bar{\rho}^{(2)}}{\partial t} + \frac{\partial \bar{\rho}^{(0)}}{\partial y} V_l + \frac{\partial \bar{\rho}^{(0)}}{\partial z} W_l = 0. \quad (4-42)$$

This equation is reasonable, because ρ is conserved for any fluid particles in Boussinesq approximation.

Another equation which corresponds to (2-24) is the energy equation. Multiplying (4-10)-(4-13) by u , v and w , respectively, and adding them, we have the perturbation kinetic energy equation of waves with the aid of (4-14) as follows,

$$\begin{aligned} \frac{\partial U^{(0)}}{\partial y} \overline{uv} + \frac{\partial U^{(0)}}{\partial z} \overline{uw} = & - \left(\frac{\partial}{\partial x} \overline{up} + \frac{\partial}{\partial y} \overline{vp} \right. \\ & \left. + \frac{\partial}{\partial z} \overline{wp} \right). \end{aligned} \quad (4-43)$$

Multiply (4-15) and (4-16) by $U^{(0)}$ and $V^{(2)}$, respectively, and adding them, we have

$$\begin{aligned} U^{(0)} \frac{\partial U^{(2)}}{\partial t} + U^{(2)} \frac{\partial}{\partial y} \overline{uv} + U^{(0)} \frac{\partial}{\partial z} \overline{uw} \\ + U^{(0)} \frac{\partial U^{(0)}}{\partial y} V^{(2)} + U^{(0)} \frac{\partial U^{(0)}}{\partial z} W^{(2)} \\ = - \frac{\partial P^{*(0)}}{\partial y} V^{(2)}. \end{aligned} \quad (4-44)$$

This is the horizontal kinetic energy equation for the mean flow correctly to second order of the perturbation amplitude. After adding (4-43) and (4-44) and substituting (4-25) and (4-27) the following equation can be obtained with the expression of the Stokes' drift (4-35) and (4-36).

$$\begin{aligned} U^{(0)} \left(\frac{\partial}{\partial t} \right)_l U + \nabla \cdot (U^{(0)} \mathbf{D}) - U^{(0)} f V_s \\ - f \frac{\partial U^{(0)}}{\partial z} v \bar{\zeta} = - \left(\frac{\partial}{\partial y} p v + \frac{\partial}{\partial z} p w \right) \\ - \frac{\partial P^{*(0)}}{\partial y} V^{(2)} - \frac{g}{\rho_{00}} \overline{\rho w}, \end{aligned} \quad (4-45)$$

where

$$\begin{aligned} \left(\frac{\partial}{\partial t} \right)_l () = \frac{\partial}{\partial t} ()^{(2)} + V_l \frac{\partial}{\partial y} ()^{(0)} \\ + W_l \frac{\partial}{\partial z} ()^{(0)}. \end{aligned} \quad (4-46)$$

The last term of the left hand of this equation can be rewritten by using (4-41) as

$$\frac{g}{\rho_{00}} \overline{\rho w} = - \frac{g}{\rho_{00}} \frac{\partial \bar{\rho}^{(0)}}{\partial y} \overline{\eta w}. \quad (4-47)$$

Using the thermal wind relation for the Eulerian mean flow,

$$f \frac{\partial U^{(0)}}{\partial z} = \frac{g}{\rho_{00}} \frac{\partial \bar{\rho}^{(0)}}{\partial y}, \quad (4-48)$$

and referring to (4-29) we see that the last terms of the left and right hand sides of (4-45) are cancelled.

By using (4-20), $\overline{pv}$ and $\overline{pw}$ are written as

$$(\overline{pv}, \overline{pw}) = -(U^{(0)} - c) \mathbf{D}. \quad (4-49)$$

Then we can rewrite the second term of the left hand side and the first term of the right hand side in (4-45) as

$$\nabla \cdot (U^{(0)} \mathbf{D}) + \nabla \cdot (\overline{pv}, \overline{pw}) = c(\nabla \cdot \mathbf{D}) \quad (4-50)$$

And using the geostrophic relation (4-16), (4-45) becomes as

$$U^{(0)} \frac{\partial U^{(2)}}{\partial t} + c \nabla \cdot \mathbf{D} = - \frac{\partial P^{*(0)}}{\partial y} V_l, \quad (4-51)$$

which is the kinetic energy equation for the mean flow which corresponds to (2-24).

Though Equations (4-38) and (4-51) are sufficient to obtain the "Eliassen-Palm relation", it may be convenient to obtain the potential and the total energy equations. To obtain the change of the potential energy of the mean field, we rewrite (4-42) by (4-5) as

$$\frac{\partial \bar{\rho}^{*(2)}}{\partial t} + \frac{\partial \bar{\rho}^{*(0)}}{\partial y} V_l + \frac{\partial \bar{\rho}^{*(0)}}{\partial z} W_l + \frac{d\rho_0}{dz} W_l = 0. \quad (4-52)$$

Then we multiply (4-17) and (4-52) by $W^{(2)}$ and $-\alpha \bar{\rho}^{*(0)}$, respectively, and add them. The result is written as follows:

$$\begin{aligned}
 & -\alpha \bar{\rho}^{*(0)} \left(\frac{\partial}{\partial t} \right)_i \bar{\rho}^* - \frac{g}{\rho_{00}} \rho^{*(0)} W_s \\
 & = -W^{(2)} \frac{\partial P^{*(0)}}{\partial z}.
 \end{aligned} \quad (4-53)$$

Reminding (4-17), this equation can be rewritten as

$$-\alpha \bar{\rho}^{*(0)} \left(\frac{\partial}{\partial t} \right)_i \bar{\rho}^* = -W_l \frac{\partial P^{*(0)}}{\partial z}. \quad (4-54)$$

This is the potential energy equation which says that the change of potential energy is due to the work of the vertical flow against the vertical mean pressure.

Adding (4-51) and (4-54) and using the incompressibility condition and the fact that the Stokes' drift is solenoidal, we have

$$\begin{aligned}
 & \left(\frac{\partial}{\partial t} \right)_i \frac{1}{2} U^2 - \alpha \left(\frac{\partial}{\partial t} \right)_i \frac{1}{2} \rho^{*2} + c \nabla \cdot \mathbf{D} \\
 & = -\nabla \cdot (\mathbf{P}^{*(0)} \mathbf{V}_l),
 \end{aligned} \quad (4-55)$$

where

$$\mathbf{V}_l = (V_l, W_l).$$

This is the total energy equation. This indicates that the energy change of fluid particles is due to two parts, one is from the divergence of the radiation stress, and the other is from the work of the second order transversal Lagrangian flow against the zero order mean pressure.

If the fluids are non-rotating, we can choose the reference stratification to coincide with the basic mean stratification. Though it may be impossible to make it coincide to second order accuracy, it is sufficient to make it coincide to zero order accuracy. In this case the horizontal gradient of the zero order pressure vanishes and the potential energy of the mean field becomes forth order and can be neglected. Then the kinetic energy equation (4-51) and total energy equation (4-55) coincide with each other, and we obtain the momentum and energy equations in the form

$$\left(\frac{\partial}{\partial t} \right)_i U + \nabla \cdot \mathbf{D} = 0 \quad (4-56)$$

$$U^{(0)} \left(\frac{\partial}{\partial t} \right)_i U + c \nabla \cdot \mathbf{D} = 0. \quad (4-57)$$

After eliminating $\nabla \cdot \mathbf{D}$ or $(\partial/\partial t)_i U$, we have "Eliassen-Palm relation" as

$$\begin{aligned}
 & (U^{(0)} - c) \left(\frac{\partial}{\partial t} \right)_i U = 0 \\
 & (U^{(0)} - c) \nabla \cdot \mathbf{D} = 0.
 \end{aligned} \quad (4-58)$$

Moreover, if the fluid is two-dimensional, that is, v and η vanish, we see from (4-35) and (4-36) that the Stokes' drift vanishes. Then, the divergence of the radiation stress coincides with that of the Reynolds stress as we see from (4-37). This means that the divergence of the Reynolds stress vanishes, which is equivalent with the original "Eliassen-Palm relation".

The almost same method can be applied to deep internal gravity waves, that is, ρ_{00} depends on z (see Nakamura 1976), and then almost the same result is easily obtained as

$$\left(\frac{\partial}{\partial t} \right)_i \rho U + \nabla \cdot \mathbf{D} = 0 \quad (4-59)$$

$$\left(\frac{\partial}{\partial t} \right)_i \bar{\rho} = 0 \quad (4-60)$$

$$\left(\frac{\partial}{\partial t} \right)_i \left(\frac{1}{2} \bar{\rho} U^2 \right) + c \nabla \cdot \mathbf{D} = 0, \quad (4-61)$$

and

$$(U^{(0)} - c) \left(\frac{\partial}{\partial t} \right)_i U = 0. \quad (4-62)$$

Here, we have made $\rho_{00}(z)$ and $\rho_0(z)$ coincide to $\bar{\rho}^{(0)}(z)$.

In a rotating case, eliminating the term of the radiation stress from (4-38) and (4-51) and replacing $\partial p^{*(0)}/\partial y$ by $-fU^{(0)}$ from geostrophic relation, we have

$$(U^{(0)} - c) \left\{ \left(\frac{\partial U}{\partial t} \right)_i - f V_l \right\} = 0. \quad (4-63)$$

The divergence of the radiation stress also vanishes except at the critical level such as

$$(U^{(0)} - c) \nabla \cdot \mathbf{D} = 0. \quad (9-64)$$

This is the "Eliassen-Palm relation" for the case of three-dimensional rotating flow.

When quasi-hydrostatic approximation is applied, the left hand side of (4-12) is omitted. Uryu (1973) pointed out that in three-dimensional rotating flow under quasi-hydrostatic approximation, the divergence of 'wave momentum flux' vanishes, *i.e.*,

$$\frac{\partial}{\partial y} (\overline{u_l - f\eta} \overline{v_l}) + \frac{\partial}{\partial z} (\overline{u_l - f\eta} \overline{w_l}) = 0. \quad (4-65)$$

Referring to (4-30) and (4-31) we see immediately that this is equivalent to (4-58) except at the critical level.

Using (4-41) and the definition of u_l , (4-28), we see that equation (4-65) is also equivalent to the one that Eliassen and Palm (1961) have obtained which is rewritten in our notation as

$$\begin{aligned} & \frac{\partial}{\partial y} \left\{ \left(\frac{\partial \bar{\rho}^{(0)}}{\partial z} \right)^{-1} \frac{\partial U^{(0)}}{\partial z} v \rho - uv \right\} \\ & + \frac{\partial}{\partial z} \left\{ \left(\frac{\partial \bar{\rho}^{(0)}}{\partial z} \right)^{-1} \left(f - \frac{\partial U^{(0)}}{\partial y} \right) v \rho - uw \right\} = 0. \end{aligned} \quad (4-66)$$

In Eliassen and Palm's paper the equation corresponding to (4-10) in case of stationary gravity waves,

$$U^{(0)} \frac{\partial u}{\partial x} + \frac{\partial U^{(0)}}{\partial z} w = - \frac{\partial p}{\partial x}, \quad (4-67)$$

is multiplied by $(U^{(0)}u + p)$. By integrating this equation by x we have

$$U^{(0)}u + p = - \frac{\partial U^{(0)}}{\partial z} \int^x w dx. \quad (4-68)$$

On the other hand, the right hand side of this equation can be rewritten as

$$\frac{\partial U^{(0)}}{\partial z} \int^x w dx = \frac{\partial U^{(0)}}{\partial z} \frac{\zeta}{U^{(0)}}, \quad (4-69)$$

Then we see that their procedure is to multiply (4-67) by $\partial U^{(0)}/\partial z \zeta/U^{(0)}$, which is almost the same process as ours used when we obtain the relation between the Reynolds stress and radiation stress.

The "Eliassen-Palm relation" is deduced from momentum and energy equations which have simple form like (2-24) and (2-25). If there exists dissipation or external forces, these may also affect the time change of energy and momentum and it seems that the "Eliassen-Palm relation" no more holds. We can get the momentum and energy equations for internal gravity waves from the same procedure as before;

$$\left(\frac{\partial}{\partial t} \right)_l U + \nabla \cdot \mathbf{D} = \bar{F}_m \quad (4-70)$$

$$U^{(0)} \left(\frac{\partial}{\partial t} \right)_l U + c \nabla \cdot \mathbf{D} = \bar{F}_e, \quad (4-71)$$

where

$$\bar{F}_m = \overline{\frac{\partial X}{\partial x} \xi} + \overline{\frac{\partial X}{\partial y} \eta} + \overline{\frac{\partial X}{\partial z} \zeta} \quad (4-72)$$

$$\begin{aligned} \bar{F}_e &= \overline{Xu} + \overline{Yv} + \overline{Zw} - \alpha Q \rho + \frac{\partial}{\partial y} (U^{(0)} X \bar{\eta}) \\ &+ \frac{\partial}{\partial z} (U^{(0)} X \bar{\zeta}), \end{aligned} \quad (4-73)$$

and X , Y , Z and Q are external forces, dissipation or heat source which we assume to be first order quantities of wave amplitude like Rayleigh damping or Newtonian cooling. We have also assumed

that waves are also steady.

A fluid particle which is at position $(x + \xi, y + \eta, z + \zeta)$ may feel forces F_m which is expressed as follows;

$$F_m = \begin{pmatrix} X + \frac{\partial X}{\partial x} \xi + \frac{\partial X}{\partial y} \eta + \frac{\partial X}{\partial z} \zeta \\ Y + \frac{\partial Y}{\partial x} \xi + \frac{\partial Y}{\partial y} \eta + \frac{\partial Y}{\partial z} \zeta \\ Z + \frac{\partial Z}{\partial x} \xi + \frac{\partial Z}{\partial y} \eta + \frac{\partial Z}{\partial z} \zeta \end{pmatrix} \quad (4-74)$$

We see from this form that the right hand side of (4-72) is the zonal component of the mean external force which a fluid particle feels.

Since a fluid particle has velocity V_l as

$$V_l = \begin{pmatrix} u + U^{(0)} + \frac{\partial U^{(0)}}{\partial y} \eta + \frac{\partial U^{(0)}}{\partial z} \zeta \\ v \\ w \end{pmatrix} \quad (4-75)$$

The energy production by external forces for a particle may be written as

$$\begin{aligned} \overline{F_m \cdot V_l} &= \overline{Xu} + \overline{Yv} + \overline{Zw} \\ &+ U^{(0)} \left(\overline{\frac{\partial X}{\partial x} \xi} + \overline{\frac{\partial Y}{\partial y} \eta} + \overline{\frac{\partial Z}{\partial z} \zeta} \right) \\ &+ X \left(\overline{\frac{\partial U^{(0)}}{\partial y} \eta} + \overline{\frac{\partial U^{(0)}}{\partial z} \zeta} \right), \end{aligned} \quad (4-76)$$

Using (4-34) and adding the energy production by heat source Q we have the right hand side of (4-73). Then the right hand side of (4-71) may be interpreted as the energy production by external forces or heat source for a fluid particle as Andrews and McIntyre (1978a, b) have pointed out.

Eliminating the term of radiation stress from (4-70) and (4-71) we have

$$\left(\frac{\partial}{\partial t} \right)_l U = \frac{1}{U^{(0)} - c} (\bar{F}_e - c \bar{F}_m), \quad (4-77)$$

which gives how the mean flow varies when external forces or heat source are present.

5. Quasi-geostrophic flow case

If the time scale of the change of the mean flow due to waves are large compared with the rotation of the earth, the mean field varies under thermal wind relation. Using this fact we can obtain a valuable information about the Lagrangian mean meridional circulation.

From (4-38), (4-42) and (4-64) we have, except at the critical level,

$$\frac{\partial U^{(2)}}{\partial t} + V_l \left(\frac{\partial U^{(0)}}{\partial y} - f \right) + \frac{\partial U^{(0)}}{\partial z} W_l = 0 \quad (5-1)$$

$$\frac{\partial \bar{\rho}^{(2)}}{\partial t} + V_l \frac{\partial \bar{\rho}^{(0)}}{\partial y} + W_l \frac{\partial \bar{\rho}^{(0)}}{\partial z} = 0. \quad (5-2)$$

We eliminate the time derivatives by differentiating (5-1) and (5-2) by z and y , respectively, and using the thermal wind relation we have

$$\begin{aligned} & \frac{\partial}{\partial z} \left\{ V_l \left(\frac{\partial U^{(0)}}{\partial y} - f \right) + W_l \frac{\partial U^{(0)}}{\partial z} \right\} \\ & - \frac{g}{\rho_{00}f} \frac{\partial}{\partial y} \left\{ V_l \frac{\partial \bar{\rho}^{(0)}}{\partial y} + W_l \frac{\partial \bar{\rho}^{(0)}}{\partial z} \right\} = 0. \end{aligned} \quad (5-3)$$

Since Lagrangian meridional circulation is solenoidal, there exists a stream function such that

$$\frac{\partial \chi}{\partial y} = -W_l, \quad \frac{\partial \chi}{\partial z} = V_l. \quad (5-4)$$

After substituting these equations to (5-3) and using thermal wind relation and neglecting the y -dependency of f we have so-called ω -equation as

$$\begin{aligned} & \left(\frac{\partial U^{(0)}}{\partial y} - f \right) \frac{\partial^2 \chi}{\partial z^2} - 2 \frac{\partial U^{(0)}}{\partial z} \frac{\partial^2 \chi}{\partial y \partial z} \\ & + \frac{g}{\rho_{00}f} \frac{\partial \bar{\rho}^{(0)}}{\partial z} \frac{\partial^2 \chi}{\partial y^2} = 0. \end{aligned} \quad (5-5)$$

Boyd (1976) has gotten similar equation for stream function which gives Eulerian mean meridional circulation over spherical surface (equation (2-36) in his paper). In his equation there are forcing terms which may express the effect of the Stokes' drift.

The equation (5-5) is elliptic type in inertially stable case (Charney 1973) and can be solved if the values on the boundary enclosing the whole area are specified. When the fluids are bounded in north-south direction by vertical walls and waves are generated by topographical cause like mountains, the stream function may be constant at lateral and lower boundaries, since fluids cannot penetrate the wall or the ground. If we can also assume that it has the same constant at the upper boundary, it becomes constant at all boundaries. Then the inhomogeneous solution vanishes, because there is no forcing term. This means that there is no Lagrangian mean meridional circulation induced by waves. The time change of the mean flow also vanishes from (5-1). Charney and Drazin (1961) have stated that steady geostrophic waves do not interact with the mean flow. Since we have not assumed that waves are geostrophic, these results may be a little wider than Charney

and Drazin's result. If there is no rigid upper boundary, there may be in general horizontal Lagrangian mean flow at the lower boundary, and there may be a Lagrangian mean circulation.

6. Some simplification of the arguments

In the preceding arguments, the procedure is somewhat tedious. However, if we choose suitable co-ordinates or reference field, the procedure becomes easier. For example, in non-rotating case, if we choose the reference stratification to coincide to the mean stratification, zero order potential energy vanishes, and its time change becomes higher order quantity and can be neglected. Similar procedure may be applied to the mean velocity field by Galilean transformation. In case of non-rotating fluids, if we move the co-ordinates in zonal direction at the basic velocity $U^{(0)}$ of the level or line concerned, the phase velocity c changes to $c - U^{(0)}$ in this new co-ordinates, and the energy change at the level becomes forth order quantity. From the energy equation (4-57), we directly have

$$(c - U^{(0)}) \nabla \cdot D = 0. \quad (6-1)$$

In words, there is a divergence of the energy flux $(c - U^{(0)}) \nabla \cdot D$, and this divergence causes the energy change of the fluid particles. On the other hand the energy is only the Lagrangian mean perturbation energy which is steady, then the divergence of the energy flux must vanishes.

When the fluids are rotating, Galilean transformation also changes the pressure field such that

$$P \rightarrow P - fU^{(0)}.$$

After the Galilean transformation there is no horizontal gradient of the mean pressure, and we can choose the reference stratification coincide with the the mean stratification at the interested line. Then the energy change of the mean field becomes higher order quantity, and (4-51) gives the same equations (6-1).

7. Remarks

In the preceding arguments, it is remarkable that energy and momentum have explicit simple meanings when we concern fluid particles, and the radiation stress becomes a key concept instead of the Reynolds stress. This is reasonable, because the Reynolds stress is concerning to the Eulerian mean flow, but the radiation stress is to the material line or sheat. The Reynolds stress is used only in interpreting the radiation stress to have Eulerian meanings. Eulerian description

is a method to treat variables as field variables, while energy or momentum of fluids can be treated explicitly by Lagrangian meanings. The reason may be that fluid dynamics is an extension of the mechanics of particles, not of the field theory. However, since the Eulerian description is much simpler than Lagrangian descriptions, we made a picture that fluid particles move in a field described by Eulerian variables and the displacements is also described as Eulerian variables. This is a useful method as far as to second order quantities, but it may be too complicated to treat higher order quantities.

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“Eliassen-Palm の関係” の一般化

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二次元定常内部重力波は非粘性で外力の無い場合には、クリティカル・レベル以外では平均流を加速しない事が Eliassen と Palm (1961) によって示されている。又最近 Andrews と McIntyre らによって一般化されている。ここではこれが、物質面を通過するエネルギーと運動量とが位相速度によって結びつけられている事を使って、ラグランジュ的に見たエネルギー及び運動量の保存から直接に導き出される事を示した。又いわゆる radiation stress と Reynolds stress との差は波によるドリフトであり、Eliassen と Palm の結果はラグランジュ的に見た時の平均流の加速が無くなる、と一般化された。これは三次元成層回転流体については Uryu (1973) の結果を、準地衡風近似のもとでは Charney と Drazin (1961) の結果を導く。