NOTES AND CORRESPONDENCE

A Note on the Behavior of Waves around the Inertio-Frequency

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The purpose of this short note is to discuss the behaviors of waves on rotating system when their intrinsic frequency $\hat{\omega}$ is Doppler-shifted towards the inertio-frequency f . Recently, Kitchen and McIntyre (1980) discussed the problem based on the ray-tracing equation and showed that the singular absorption of the inertio-gravity wave (Jones, 1967) occurs only when rotation rate is uniform, and the absorption found by Miyahara (1976) is not due to the singularity but due to finite dissipations.

The discussion presented in this note, based on the theory of partial differential equations of the second order with two independent variables, gives consistent result with that obtained by Kitchen and McIntyre (1980).

We consider an inviscid rotating Boussinesq fluid. Its undisturbed state is in a geostrophic and hydrostatic balance, and the basic zonal wind is assumed to vary only with altitude with positive constant shear Λ . Assuming quasi-hydrostatic approximation, we get perturbation equations as follows;

$$i\hat{\omega}u + \Lambda w - fv = -ik\phi, \quad (1)$$

$$i\hat{\omega}v + fu = -\frac{\partial\phi}{\partial y}, \quad (2)$$

$$\frac{\partial\phi}{\partial z} = -g\frac{\delta\rho}{\rho_0}, \quad (3)$$

$$i\hat{\omega}\frac{g\delta\rho}{\rho_0} - N^2w + f\Lambda v = 0, \quad (4)$$

$$iku + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (5)$$

where $\hat{\omega}$ is Doppler-shifted frequency ($=\omega + ku_0(z)$), N , Brunt-Väisälä frequency which is assumed to be constant, $\phi = \delta p / \rho_0$, and other notations used are conventional.

Under the assumption that the variation of $\hat{\omega}$ in a distance of one vertical wavelength is sufficiently small, these equations can be combined into the following partial differential equation of second order,

$$N^2\frac{\partial^2 v}{\partial y^2} + 4f\Lambda\frac{\partial^2 v}{\partial y\partial z} + (f^2 - \hat{\omega}^2)\frac{\partial^2 v}{\partial z^2} + \Lambda\left(\frac{df}{dy} - 2k\hat{\omega}\right)\frac{\partial v}{\partial z} - kN^2\left\{k - \frac{df/dy}{\hat{\omega}}\left(1 - \frac{\Lambda^2}{N^2}\right)\right\}v = 0. \quad (6)$$

Equation (6) is of hyperbolic type or elliptic type according as $4f^2\Lambda^2 - N^2(f^2 - \hat{\omega}^2)$ is positive or negative, respectively. The curve $4f^2\Lambda^2 - N^2(f^2 - \hat{\omega}^2) = 0$ corresponds to the so-called turning point in WKB method, and inertio-gravity waves are reflected at this curve. The curves given by $f^2 - \hat{\omega}^2 = 0$ on which our present interest is focused lie in the hyperbolic domain. In the hyperbolic domain, equation (6) is transformed into the normal form

$$\begin{aligned} & \frac{\partial^2 v}{\partial \xi \partial \eta} + \frac{\left\{N^2\frac{\partial^2 \xi}{\partial y^2} + 4f\Lambda\frac{\partial^2 \xi}{\partial y\partial z} + (f^2 - \hat{\omega}^2)\frac{\partial^2 \xi}{\partial z^2} + \Lambda\left(\frac{df}{dy} - 2k\hat{\omega}\right)\frac{\partial \xi}{\partial z}\right\}}{4\frac{\partial \xi}{\partial z}\frac{\partial \eta}{\partial z}\left(f^2 - \hat{\omega}^2 - \frac{4f^2\Lambda^2}{N^2}\right)}\frac{\partial v}{\partial \xi} \\ & + \frac{\left\{N^2\frac{\partial^2 \eta}{\partial y^2} + 4f\Lambda\frac{\partial^2 \eta}{\partial y\partial z} + (f^2 - \hat{\omega}^2)\frac{\partial^2 \eta}{\partial z^2} + \Lambda\left(\frac{df}{dy} - 2k\hat{\omega}\right)\frac{\partial \eta}{\partial z}\right\}}{4\frac{\partial \xi}{\partial z}\frac{\partial \eta}{\partial z}\left(f^2 - \hat{\omega}^2 - \frac{4f^2\Lambda^2}{N^2}\right)}\frac{\partial v}{\partial \eta} \\ & - \frac{kN^2\left\{k - \frac{df}{dy}\left|\frac{\hat{\omega}}{\hat{\omega}}\right|(1 - \Lambda^2/N^2)\right\}}{4\frac{\partial \xi}{\partial z}\frac{\partial \eta}{\partial z}\left(f^2 - \hat{\omega}^2 - \frac{4f^2\Lambda^2}{N^2}\right)}v = 0, \end{aligned} \quad (7)$$

where the transformation function ξ and η are determined by the differential equations

$$\frac{\partial \xi}{\partial y} = \frac{1}{N^2} \{ \sqrt{4f^2 A^2 - N^2(f^2 - \omega^2)} - 2fA \} \frac{\partial \xi}{\partial z}, \quad (8)$$

$$\frac{\partial \eta}{\partial y} = -\frac{1}{N^2} \{ \sqrt{4f^2 A^2 - N^2(f^2 - \omega^2)} + 2fA \} \frac{\partial \eta}{\partial z}. \quad (9)$$

The curves $\xi = \text{constant}$ and $\eta = \text{constant}$ are the characteristic curves of equation (6). The inclinations of the characteristic curves are given by the following equations;

$$\left(\frac{dz}{dy} \right)_{\xi} = -\frac{1}{N^2} \{ \sqrt{4f^2 A^2 - N^2(f^2 - \omega^2)} - 2fA \}, \quad (10)$$

$$\left(\frac{dz}{dy} \right)_{\eta} = \frac{1}{N^2} \{ \sqrt{4f^2 A^2 - N^2(f^2 - \omega^2)} + 2fA \}. \quad (11)$$

It is noticeable that the inclination of the characteristic curves $\xi = \text{constant}$ is negative in the domain of $\omega^2 > f^2$ and positive for $\omega^2 < f^2$, and that it is zero on the curve $f^2 - \omega^2 = 0$. On the other hand the inclination of the characteristic curves $\eta = \text{constant}$ is always positive.

In case of uniform rotation ($f = \text{constant}$), the curve $f^2 - \omega^2 = 0$ is horizontal. Equation (8) is easily integrated by making use of Lagrange-Charpit method as follows;

$$\xi = ay + aN^2 \int \frac{1}{\sqrt{4f^2 A^2 - N^2(f^2 - \omega^2)} - 2fA} dz + C, \quad (12)$$

where a and c are arbitrary constants. The behaviors of characteristic curves are shown schematically in Fig. 1. The curves $\xi = \text{constant}$

approach to the curve $f^2 - \omega^2 = 0$ asymptotically, and the derivatives of ξ with respect to z become infinite at $\omega^2 = f^2$ as is seen from equation (8). However, the coefficients of equation (7) are regular at $\omega^2 = f^2$.

According to the theory of Riemann's representation of solution of a second order hyperbolic differential equation, a solution V_n at a point P_n is represented in terms of the initial data* on a noncharacteristic curve AB_n cut out by the two characteristic curves through the point P_n . Thus, if the initial data is continuous on the noncharacteristic curve, the solution is continuous around the point P_n . When the characteristic curves have an asymptotic curve as in the present case, the vertical derivatives of the solution become large around the asymptotic curve and go to infinity on the curve, even if the initial data is smooth. This property reminds us of a singular behavior of the wave at the curve $f^2 - \omega^2 = 0$. It is noticeable that the characteristic curves $\eta = \text{constant}$ can intersect the curve $f^2 - \omega^2 = 0$ in contrast to the curves $\xi = \text{constant}$. These behaviors seem to be consistent with the result obtained by Jones (1967) that the one series solution is regular and the another independent solution is singular at $\omega^2 = f^2$.

On the other hand, if f varies with y , the curve $f^2 - \omega^2 = 0$ is not horizontal but inclined in the meridional plane, and the inclination of the characteristic curve $\xi = \text{constant}$ becomes small with approaching to the curve $f^2 - \omega^2 = 0$ and becomes zero at $\omega^2 = f^2$. Thus, the curve is not the asymptotic curve of the characteristic curves in contrast to the case of uniform rotation. This property shows that the solution is regular at $\omega^2 = f^2$, when f varies with y .

In order to see these behaviors of the char-

Fig. 1 The behaviors of characteristic curves in case of uniform rotation ($f = \text{constant}$).

* Since the wave frequency ω is given in the present problem, it is not an initial value problem but a boundary value problem. In general, the boundary value problem of a hyperbolic differential equation is not well-posed. However, if a physically reasonable boundary condition (for instance, radiation condition) is adopted, the problem may have a unique solution, and the resulting wave solution may represent the steady state solution of the initial value problem. In this case, the term "initial data" is used in the meaning to be the value of the resulting wave solution and its derivative on the noncharacteristic curve, and we can use characteristic curves to diagnose which part of the initial data determines the wave solution at a point P_n in the hyperbolic domain.

Fig. 2 The behaviors of characteristic curves of Yanai wave in the configuration used by Lindzen (1970).

acteristic curves, we solve equations (10) and (11) by a standard Runge-Kutta method. Fig. 2 shows the behaviors of the characteristic curves of a Yanai wave in the configuration used by Lindzen (1970). In this case, both characteristic curves $\xi = \text{constant}$ and $\eta = \text{constant}$ can intersect the curve $f^2 - \omega^2 = 0$ in contrast to the case of uniform rotation, and the solution is regular at $\omega^2 = f^2$, and the wave is reflected at the curve $4f^2A^2 - N^2(f^2 - \omega^2) = 0$. This result is consistent with the numerical result obtained by Lindzen (1970).

However, when the variation of rotation rate df/dy is small enough, the situation is considerably different from the above case. As an example, the behaviors of the characteristic curves of an inertio-gravity wave in the configuration used by Miyahara (1976) are shown in Fig. 3. In this case, since the inclination of the curve $f^2 - \omega^2 = 0$ is smaller than that of the characteristic curves $\xi = \text{constant}$, the characteristic curves crowd to the curve $f^2 - \omega^2 = 0$, but do not approach asymptotically to the curve, and they seem to be refracted at the curve, and derivatives of ξ with respect to z become large around the curve but remain finite. Hence, the behavior of the solution becomes to resemble to that in case of uniform rotation, but the solution remains to be regular. Thus, the effect of the viscosity and conductivity becomes very important around the curve $f^2 - \omega^2 = 0$, because of large (but finite) derivatives of the solution.

Fig. 3 The behaviors of characteristic curves in the configuration used by Miyahara (1976).

The present discussion shows that the wave solution is regular at $\omega^2 = f^2$, and the singular absorption of the wave does not occur except the case of uniform rotation. However, when the inclination of the curve $f^2 - \omega^2 = 0$ is small (for instance, in case of small df/dy), the behavior of the wave solution resembles to that of the singular solution, and the effect of dissipations becomes very important around the curve $f^2 - \omega^2 = 0$. This result is consistent* with that obtained by Kitchen and McIntyre (1980) by using ray tracing method that the singular absorption of the wave occurs only when rotation rate is uniform, and the absorption found by Miyahara (1976) is not due to the singularity but due to finite dissipations.

To close the note, it may worth mentioning that the behavior of waves around the critical level $\omega = 0$ in case of no rotation ($f = 0$). In this case, equation (6) becomes as follows;

$$N^2 \frac{\partial^2 v}{\partial y^2} - \omega^2 \frac{\partial^2 v}{\partial z^2} - 2kA\omega \frac{\partial v}{\partial z} - k^2 N^2 v = 0. \quad (13)$$

Equation (13) is of hyperbolic type, except the case $\omega = 0$, and characteristic curves are given by the following equations,

$$\xi = ay + aN \int \frac{1}{\omega} dz + c, \quad (14)$$

* In their model Kitchen and McIntyre use local dispersion relation, so that the turning point coincides with the curve $f^2 - \omega^2 = 0$. On the other hand, in the present model the turning point is located at $4f^2A^2 - N^2(f^2 - \omega^2) = 0$. Thus, the details of the wave behavior around the turning point are somewhat different.

$$\eta = by - bN \int \frac{1}{\hat{\omega}} dz + d, \quad (15)$$

where, a , b , c and d are arbitrary constants. The both characteristic curves $\xi = \text{constant}$ and $\eta = \text{constant}$ approach to the curve $\hat{\omega} = 0$ asymptotically, and derivatives of ξ and η with respect to z become infinite at $\hat{\omega} = 0$. Thus, the derivatives of the wave solution become large around the curve $\hat{\omega} = 0$ and go to infinity on the curve. This result is consistent with the behavior of the wave solution that the both independent solutions are singular around the critical level obtained by Booker and Bretherton (1967).

In case of uniform rotation, the curve $\hat{\omega} = 0$ is not the singularity of equation (6), and the solution is regular at $\hat{\omega} = 0$ as shown by Jones (1967).

In case where f varies with y , the curve $\hat{\omega} = 0$ is again the singularity of equation (6). Since equation (6) is of elliptic type around the curve $\hat{\omega} = 0$, the real characteristic curve does not exist, and we cannot discuss the wave behavior by means of the present method.

These properties of waves around the critical level $\hat{\omega} = 0$ in rotating system are easily shown in quasi-geostrophic system.

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慣性周波数付近での波動の振舞についてのノート

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