A Review of Some Analytical Studies of Finite Amplitude Baroclinic Waves, Including a New Algorithm for the Saturation Effects of Static Stability and Baroclinicity Variations

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Abstract

A brief review is given of a series of analytical studies by the authors demonstrating the effects of second order nonlinear and nongeostrophic processes on the evolution of an amplifying two-level baroclinic wave. It is shown that many of the finite amplitude properties of the waves observed in the atmosphere and ocean can be deduced. However, in these studies the feedback between the altered basic state and the lower order (primary) wave field is neglected so that the waves grow in an unrealistic exponential manner without reaching saturation (i.e., occlusion). In order to demonstrate the saturation effects of feedbacks between the static stability and baroclinicity (vertical wind shear) variations and the growing primary wave, a heuristic semi-numerical algorithm is developed that, in principle, can be coupled with the previous analytical studies. It is found that for a typical initial basic state, and in the absence of friction and barotropic wave-wave and wave-mean flow interactions, the time scale for the development of an occluded state is about 10 days for the atmosphere and 2 months for the ocean. In both cases the effects of baroclinicity variations are much more important than static stability variations in bringing the wave to saturation. Because friction is neglected the ultimate decay of the occluded wave cannot be represented.

1. Introduction

The literature on the theory of finite-amplitude baroclinic waves is extensive (see, e.g., reviews by Kuo 1973, LeBlond and Mysak 1978, Hart 1979, and Pedlosky 1979) and contributions to this subject are continuing with vigor (e.g., Blumen 1980, Boville 1980, Pedlosky and Frenzen 1980, and Simmons and Hoskins 1980). In addition, Tang (1980) has reconsidered the

method of Arakawa (1962) and Gambo (1966) in order to study the roles of the time variation of static stability and vertical wind shear on the development of baroclinic waves, a topic that will be considered further here from a different point of view.

In a series of previous studies to be reviewed in Section 2 (Saltzman and Tang, i.e., S-T 1972, 1974, 1975a, b) we showed analytically how the inclusion of second order nongeostrophic effects accounts for many features of developing cyclone waves observed on weather maps (cf., also, Williams, 1965). However, in our studies the

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primary, first-order, wave was allowed to grow exponentially with no feedback from the changing basic state, so that saturation and an occlusion stage could not be represented. In Section 3 we shall use a heuristic, semi-numerical method to calculate a first approximation to this feedback (*i.e.*, the finite amplitude correction to the growth of the primary wave due to the increase in static stability and decrease of the vertical wind shear with time), but we shall neglect the non-geostrophic effects that were considered previously as well as non-adiabatic, and frictional effects. In treating only the single primary wave we also neglect barotropic wave-wave and wave-mean flow interactions that are of known importance in the real atmospheric system. However, the method developed can be coupled with the previous analytical calculation and supplemented by the inclusion of non-adiabatic frictional, and barotropic nonlinear processes to yield a more complete model of cyclone growth and decay.

The component parts of the complete problem discussed in our previous studies and in our present extension are shown schematically in Fig. 1. The symbols used in this figure are the same as were used previously—*i.e.*, ψ is the geostrophic stream function, χ is the velocity potential for divergent flow, ω is the individual pressure change dp/dt (p =pressure, t =time) which is strongly related to the vertical motion w , $s = -\alpha \partial \ln \theta / \partial p$ is the “static stability” (α =specific volume, θ =potential temperature), the bar ($\bar{}$) denotes a zonal average along a latitude, the wavy-bar ($\bar{\sim}$) denotes an area average over

the entire horizontal domain, and the superscript numbers in parentheses denote the properties of the initial basic state (0), the primary baroclinic wave and its modification of the basic state (1), and the second order nongeostrophic modifications of the waves (2); *i.e.*, $\phi = \bar{\phi} + \phi'$, $\bar{\phi} = \bar{\phi}^{(0)} + \bar{\phi}^{(1)}$, $\phi' = \phi'^{(1)} + \phi'^{(2)}$, where the prime denotes the wave departure from the zonal average. The light arrowed-lines denote the processes treated in the previous studies (S-T, 1972, 1975a, b), the bold arrowed-lines denote the feedbacks we shall discuss in Section 3, while the dashed arrow denotes effects not discussed previously or in the present study.

2. Review of the model; previous results

a. System of equations

Let us now set down the equations governing the primary wave, forced zonal state, and secondary fields, which are the three main components treated in the previous studies as shown in Fig. 1. These quantities are to be determined for the two-layer vertical resolution portrayed in Fig. 2, and for a fundamental harmonic region of a horizontal β -plane of meridional width D taken to be the same as the zonal wavelength L of the most unstable baroclinic wave. Similar equations and boundary conditions are described by Moen (1974) and Paegle and MacDonald (1974).

First, we define the following symbols, some of which have already been introduced in Section 1:

x : distance eastward
 y : distance northward
 z : height above sea level

Fig. 2 Two-layer representation of baroclinic flow showing at the left the levels for the atmosphere (p) or ocean (z) at which the dependent variables are defined, and at the right the subscript notation.

Fig. 1 Flow diagram showing component fields treated in previous studies (light arrows) and present study (heavy arrows). See text.

- p : pressure
 t : time
 u : eastward wind speed
 v : northward wind speed
 ω : dp/dt
 g : acceleration of gravity
 c_p : specific heat at constant pressure
 R : gas constant for air
 f : Coriolis parameter, $2\Omega \sin \varphi$
 Ω : angular speed of earth's rotation
 φ : latitude
 β : df/dy
 ϕ : geopotential, gz
 ψ : geostrophic streamfunction, $f^{-1}\phi$
 χ : velocity potential
 T : temperature, $-p/R^{-1} \partial\phi/\partial p$
 α : specific volume
 θ : potential temperature, $R^{-1}p_4^*p_1^{-\kappa}\alpha$
 s : static stability, $-\alpha \partial \ln \theta / \partial p$
 κ : R/c_p
 i : unit vector eastward
 j : unit vector northward
 k : unit vector upward
 ∇ : $i\partial/\partial x + j\partial/\partial y$
 $\mathbf{v}$: vector horizontal wind, $ui + vj = \mathbf{v}_\phi + \mathbf{v}_\chi$
 $\mathbf{v}_\phi$: nondivergent (geostrophic) wind component, $\mathbf{k} \times \nabla\phi = u_\phi i = v_\phi j$ ($u_\phi = -\partial\phi/\partial y$, $v_\phi = \partial\phi/\partial x$)
 $\mathbf{v}_\chi$: irrotational wind component, $\nabla\chi = u_\chi i + v_\chi j$ ($u_\chi = \partial\chi/\partial x$, $v_\chi = \partial\chi/\partial y$)
 ζ : vertical component of vorticity, $\mathbf{k} \cdot \nabla \times \mathbf{v} = \nabla^2\phi$
 δ : horizontal divergence, $\nabla \cdot \mathbf{v} = \nabla^2\chi$

With reference to Fig. 1 we also define

$$\phi_2 = \frac{\phi_1 + \phi_3}{2}$$

$$\phi_T = \frac{\phi_1 - \phi_3}{2}$$

$$T = T_2 = \frac{2f}{R} \phi_T$$

$$u_T = -\frac{\partial\phi_T}{\partial y}$$

$$v_T = \frac{\partial\phi_T}{\partial x}$$

Using these definitions, we may write the three sets of equations in the form (S-T 1972):

- I. Equations for the primary wave, $[\phi^{(1)}, \omega^{(1)}, \chi^{(1)}]$

$$\left. \begin{aligned} \frac{\partial}{\partial t} \nabla^2 \phi_{1,3}^{(1)} + \hat{u}_{1,3} \frac{\partial}{\partial x} \nabla^2 \phi_{1,3}^{(1)} \\ + \beta \frac{\partial \phi_{1,3}^{(1)}}{\partial x} \mp \frac{f \omega_2^{(1)}}{p_2} = 0 \\ \frac{\bar{s}}{f} \nabla^2 \omega_2^{(1)} - \frac{2f}{p_2^2} \omega_2^{(1)} + \frac{2\beta}{p_2} \frac{\partial \phi_T^{(1)}}{\partial x} \\ + \frac{4}{p_2} \hat{u}_T \frac{\partial}{\partial x} \nabla^2 \phi_2^{(1)} = 0 \\ \nabla^2 \chi_{1,3}^{(1)} = \mp \frac{\omega_2^{(1)}}{p_2} \end{aligned} \right\}$$

(1a, b, c)

- II. Equations for the forced zonal state, $[\bar{\phi}^{(1)}, \bar{\omega}^{(1)}, \bar{\chi}^{(1)}]$

$$\left. \begin{aligned} \frac{\bar{s}}{f} \frac{\partial^2 \bar{\omega}_2^{(1)}}{\partial y^2} - \frac{2f}{p_2} \bar{\omega}_2^{(1)} = A(\phi^{(1)}) \\ \frac{\partial}{\partial t} \left(\frac{\partial^2 \bar{\phi}_{1,3}^{(1)}}{\partial y^2} \right) = -\mathbf{v}_{\phi_{1,3}^{(1)}} \cdot \nabla (\nabla^2 \phi_{1,3}^{(1)}) \\ \pm \frac{f}{p_2} \bar{\omega}_2^{(1)} \\ \frac{\partial^2 \bar{\chi}_{1,3}^{(1)}}{\partial y^2} = \mp \frac{\bar{\omega}_2^{(1)}}{p_2} \end{aligned} \right\}$$

(2a, b, c)

- III. Equations for the secondary fields, $[\phi^{(2)}, \omega^{(2)}, \bar{\omega}^{(2)}]$

$$\left. \begin{aligned} \frac{\partial}{\partial t} \nabla^2 \phi_{1,3}^{(2)} + \hat{u}_{1,3} \frac{\partial}{\partial x} \nabla^2 \phi_{1,3}^{(2)} \\ + \beta \frac{\partial \phi_{1,3}^{(2)}}{\partial x} \mp \frac{f}{p_2} \omega_2^{(2)} \\ = B(\phi^{(1)}, \omega^{(1)}, \chi^{(1)}, \phi^{(1)}, \bar{\omega}^{(1)}, \bar{\chi}^{(1)}) \\ \bar{s} \nabla^2 \omega_2^{(2)} - \frac{2f^2}{p_2^2} \omega_2^{(2)} = C(\phi^{(1,2)}, \bar{\phi}^{(1)}) \\ \bar{s} \frac{\partial^2 \bar{\omega}_2^{(2)}}{\partial y^2} - \frac{2f^2}{p_2^2} \bar{\omega}_2^{(2)} = E(\phi^{(1,2)}, \bar{\phi}^{(1)}) \end{aligned} \right\}$$

(3a, b, c)

In these equations the upper sign refers to level 1 (250 mb) and the lower sign to level 3 (750 mb); $\hat{u}_{1,3}$ and $\bar{s}$ are assumed to be horizontally uniform values of the zonal current and static stability, respectively, representative of conditions prevailing near the wave center (*i.e.*, near $y=D/2$); and A , B , C , and E represent non-homogeneous forcing due to the lower order pre-calculated fields, the expressions for which are given in S-T (1972, 1975a).

b. Solutions

The eigenvalue solution of the lowest order system governed by (1a, b, c) determines the primary baroclinically unstable wave. This solution, which was obtained first by Phillips (1954)

for a bounded 2-layer channel, can be written for our harmonic region in the compact form (c.f., S-T, 1972),

$$\left. \begin{aligned} \phi_2^{(1)} &= |\Psi_2| e^{\nu t} \cos(kX + \gamma) \sin ly \\ \phi_T^{(1)} &= |\Psi_T| e^{\nu t} \cos(kX + \sigma) \sin ly \\ \omega_2^{(1)} &= |\Omega| e^{\nu t} \cos(kX + \vartheta) \sin ly \\ \chi_{1,3}^{(1)} &= \pm \frac{\omega_2^{(1)}}{\mu^2 p_2} \end{aligned} \right\} \quad (4a, b, c, d)$$

where

$$k = 2\pi/L$$

$$L = \text{wavelength in } x\text{-direction}$$

$$l = \pi/D$$

$$D = \text{meridional width}$$

$$\mu^2 = k^2 + l^2$$

$$X = x - c_r t$$

$$\nu = k c_i \text{ (amplification factor)}$$

$$c_r = \bar{u}_2 - b(1-r/2) \text{ (phase speed)}$$

$$c_i = [\bar{u}_T^2(2r-1) - (r^2/4)b^2]^{1/2}$$

$$b = \beta/\mu^2$$

$$r = 2/(2 + p_2^2 \mu^2 \bar{s} f^{-2})$$

$$|\Psi_2| = \text{initial amplitude of } \phi_2\text{-perturbation}$$

$$|\Psi_T| = (2r-1)^{1/2} |\Psi_2|$$

$$|\Omega| = \frac{p_2 \mu^2 k |\Psi_2|}{f \left[\frac{(1-r)}{r} + 1 \right]} \left[\left(2\bar{u}_T - \frac{rb^2}{2\bar{u}_T} \right)^2 + \left(\frac{b\nu}{k\bar{u}_T} \right)^2 \right]^{1/2}$$

$$\gamma = -\sigma = \arctan \left\{ - \left[\frac{2\bar{u}_T(2r-1)^{1/2} - rb}{2\bar{u}_T(2r-1)^{1/2} + rb} \right]^{1/2} \right\}$$

and

$$\vartheta = \arctan \left\{ \frac{- \left(2\bar{u}_T - \frac{rb^2}{2\bar{u}_T} + \frac{b\nu}{k\bar{u}_T} \tan \gamma \right)}{\left[\left(2\bar{u}_T - \frac{rb^2}{2\bar{u}_T} \right) \tan \gamma - \frac{b\nu}{k\bar{u}_T} \right]} \right\}$$

The solution of the zonal mean equations (2a, b, c) determines the modifications of the initial basic state $[\bar{u}_T = \bar{u}_T^{(0)}, \bar{s} = \bar{s}^{(0)}]$ due to the growing primary baroclinic wave. This solution is of the form (c.f., Phillips, 1954; S-T, 1972),

$$\left. \begin{aligned} \bar{\omega}_2^{(1)} &= \left[\frac{2fl^3 p_4 \nu |\Psi_2|^2}{\bar{u}_T (\bar{s} l^2 p_4^2 + 2f^2)} \right] e^{2\nu t} \sin 2ly \\ \bar{\phi}_{1,3}^{(1)} &= \mp \left[\frac{f^2 l |\Psi_2|^2}{2\bar{u}_T (\bar{s} l^2 p_4^2 + 2f^2)} \right] e^{2\nu t} \sin 2ly \\ \bar{\chi}_{1,3}^{(1)} &= \pm \left[\frac{f\nu l |\Psi_2|^2}{\bar{u}_T (\bar{s} l^2 p_4^2 + 2f^2)} \right] e^{2\nu t} \sin 2ly \end{aligned} \right\} \quad (5a, b, c)$$

The solution of equations [3a, b, c], representing the second order modifications of the stream functions wave $\phi^{(2)}$ and the associated vertical motion field $[\omega_2^{(2)}, \bar{\omega}_2^{(2)}]$, are given in S-T (1972) and S-T (1975a), respectively. The

analytical expressions for these solutions are lengthy and will not be repeated here.

c. Comparisons with observations

Applying these solutions to the most unstable wave arising from a typical atmospheric initial basic state represented by $[\bar{u}_T^{(0)}, \bar{s}^{(0)}]$, and assuming continuous exponential growth with a constant amplification factor $\nu(\bar{u}^{(0)}, \bar{s}^{(0)})$, we obtained in S-T (1972) the composite solutions for

$$\begin{aligned} \phi_{1,3} &= \bar{\phi}_{1,3}^{(0)} + \bar{\phi}_{1,3}^{(1)} + \phi_{1,3}^{(1)} + \phi_{1,3}^{(2)}, \\ \omega_2 &= \bar{\omega}_2^{(1)} + \omega_2^{(1)}, \end{aligned}$$

and

$$\phi_T = \bar{\phi}_T^{(0)} + \bar{\phi}_T^{(1)} + \phi_T^{(1)} + \phi_T^{(2)}$$

shown in Fig. 3a, for 5.6 and 6.4 days after inception as an initial perturbation. These can be compared with the observed example of the evolution of a baroclinic wave presented by Newton (1956) reproduced in Fig. 3b. In Fig. 4a we show the more complete solution for the vertical motion field, $\omega_2 = \bar{\omega}_2^{(1)} + \bar{\omega}_2^{(2)} + \omega_2^{(1)} + \omega_2^{(2)}$, for 5.6 days after inception, obtained by S-T (1975a). This vertical motion field is in good agreement with the idealized numerical solution of Phillips (1956) reproduced in Fig. 4b, and with calculations of the ω -field in observed wave cyclones (one of which is reproduced in Fig. 4c).

More generally, we can see that many of the salient features of the observed development of cyclone waves are deduced from the analytic theory. In particular, we account for the following features itemized in S-T (1972):

1) the development in *low* levels of intense cyclonic circulation systems (lows) moving generally eastward and poleward, with weaker and more diffuse anti-cyclonic circulation systems (highs) drifting eastward and equatorward.

2) the simultaneous development in *high* tropospheric levels of large-amplitude waves, often culminating in the formation of "cutoff lows" that are displaced somewhat equatorward of the latitude in which they form and often accompanied by a "splitting" of the jet stream.

3) the development of a relatively deep S-shaped frontal zone separating cold air masses displaced equatorward and warm air masses displaced poleward (cf., also, S-T, 1974).

4) the intrusion of a region of descending motion just to the equatorward side of the surface low, leading to a split, bow-shaped, pattern of rising motion with maxima poleward and equatorward of the low center in the vicinities of the warm front and cold front, respectively. The pattern of vertical motions is associated with

Fig. 3a Theoretical fields of (top) $\psi_1 = \bar{\psi}_1^{(0)} + \bar{\psi}_1^{(1)} + \psi_1^{(1)} + \psi_1^{(2)}$ (solid curves: $10^7 \text{ m}^2 \text{ s}^{-1}$), and $\omega_2 = \bar{\omega}_2^{(1)} + \omega_2^{(1)}$ (dashed curves: $10^{-3} \text{ mb s}^{-1}$); and (bottom) $\psi_3 = \bar{\psi}_3^{(0)} + \bar{\psi}_3^{(1)} + \psi_3^{(1)} + \psi_3^{(2)}$ (solid curve: $10^7 \text{ m}^2 \text{ s}^{-1}$), and $2\psi_T = \psi_1 - \psi_3$ (dashed curves: $10^7 \text{ m}^2 \text{ s}^{-1}$), for $t = 5.6\text{d}$ (left) and $t = 6.4\text{d}$ (right). Reproduced from S-T (1972).

Fig. 3b (Top) Observed fields of 300 mb geopotential contour height (solid curves: 400 ft. intervals) and geostrophic absolute vorticity (dashed curves: 10^{-4} s^{-1}), and (bottom) surface fronts and 1,000 mb contour heights (solid curves: 200 ft. interval) and 500-1,000 mb thickness (dashed curves: 200 ft. intervals) for 1500 GCT Nov. 17, 1948 (left) and 1500 GCT Nov. 19, 1948 (right). In the top (300 mb) maps the heavy dots show locations of the surface cyclone centers; in bottom (1,000 mb) maps, the precipitation areas are shaded; the dotted lines are ridges of the principal mountain ranges. Reproduced from Newton (1956).

Fig. 4 (a) Theoretical fields of $\omega_2 = \bar{\omega}_2^{(1)} + \bar{\omega}_2^{(2)} + \omega_2^{(1)} + \omega_2^{(2)}$ (solid curves: 10^{-3} mbs^{-1}) and surface streamfunction $\psi_4 = \bar{\psi}_4^{(0)} + \bar{\psi}_4^{(1)} + \psi_4^{(1)} + \psi_4^{(2)}$ (dashed curves: $10^7 \text{ m}^2 \text{ s}^{-1}$), for $t = 5.6$ days after initial perturbation. Reproduced from S-T (1975a). (b) Phillips' (1956) numerical solution for the field of vertical motion at 500 mb in cm s^{-1} (solid lines) and 1000 mb contours at 200 ft. intervals (dashed lines). (c) Field of ω at 500 mb (10^{-3} mbs^{-1}) computed by Kung (1973) for observed North American conditions on 2 July 1962; the black dot indicates the position of the amplifying pressure center at the surface.

the characteristic "comma cloud" pattern (Carlson, 1980).

Thus, we have shown that many of the *finite amplitude* properties of baroclinic waves observed in nature and obtained by numerical integration

of the primitive equations (*e.g.*, Hinkelmann, 1959) can be obtained analytically, and in principle could have been deduced before the advent of the large-scale computer.

We note, also, that the above theory has been

applied to the formation of meanders in a baroclinic ocean current (S-T, 1975b) essentially by making the following substitutions in the system (1), (2) and (3): $p_2 \rightarrow H/2$ (H =depth of baroclinic ocean layer), $\tilde{s} \rightarrow \sigma = \varepsilon g \partial \tilde{T} / \partial z$ (ε =coefficient of thermal expansion for ocean water), $\omega \rightarrow -w$ (vertical velocity), and $\partial p \rightarrow -\partial z$. The results, for average conditions near the Gulf Stream, show that nongeostrophic effects can modify an exponentially growing meander in such a way that oceanic fronts and cutoff cyclonic cold pools form on the equatorward side of the current axis and cutoff anticyclonic warm pools form on the poleward side (Fig. 5).

d. Deficiencies

In spite of the above-mentioned capabilities of the model to account for observed features of finite-amplitude baroclinic waves in the atmosphere and ocean, there are obvious theoretical inadequacies of the model as a consequence of which several important features, clearly evident in Fig. 3b for example, cannot be deduced. For example, because the initial exponential growth is permitted to continue unabated, the waves cannot possibly "saturate" (*i.e.*, reach a maximum amplitude) and ultimately dissipate. From observations we know that such a saturation is characterized by a coming into phase of the streamlines and isotherms so that there is no longer a westward tilt of the trough and ridge lines with height. This represents the onset of "occlusion" in the atmospheric wave (*c.f.*, Fig. 3b), which probably bears some analogy to the state of concentric streamlines and isotherms

observed in oceanic "rings". This stage of the wave is also characterized by a stabilization of the vertical stratification and a reduction of the horizontal baroclinicity in response to the upward and poleward flux of heat by the growing baroclinic wave. During and beyond this stage, the entire wave system comes under the increasingly dominant influence of surface friction, which, as consequence of Ekman pumping, intensifies the cooling of the low pressure center at the same time the cyclone wave stagnates and gradually decays in amplitude from below.

With the exception of the decrease in baroclinicity of the basic state, none of the above important features of a mature baroclinic wave system were manifest in our previous solutions. Moreover, as noted by Williams (1965) and Pedlosky (1970), because the higher order terms grow at an even faster rate than the primary wave [*c.f.*, *e.g.*, (5)], there is a time limit to the applicability of the expansion used in the previous studies, governed by (1), (2) and (3), and the relative contribution of these higher order terms is bound to be somewhat exaggerated. For all these reasons, it is important to take into account the stabilizing effects of (i) increasing static stability, (ii) decreasing baroclinicity, (iii) barotropic wave-mean flow interactions and (iv) friction, which tend to arrest the growth of the wave. As indicated in the Introduction and in Fig. 1, we shall in the following explore the effects only of items (i) and (ii) above, continuing the neglect the role of friction and neglecting also all the secondary nongeostrophic and non-linear interactive effects between different wave modes (including the eddy momentum transport processes that help regulate and maintain the basic zonal state).

3. Effects of variation of static stability and baroclinicity on the primary wave

a. Formulation of the algorithm

As we have noted, in our previous studies the amplification factor for the exponential growth of the primary baroclinic wave, given by

$$\nu = k[\hat{u}_T^2(2r-1) - (b^2/4)r^2]^{1/2}, \quad (6)$$

was considered to be independent of time determined by the initially prescribed values of the basic state $\hat{u}_T = \hat{u}_T^{(0)}$ and $r = r(\tilde{s}^{(0)})$. We know, however, that both r and the baroclinicity across the wave at mid-latitudes measured by $\hat{u}_T$ decrease as the wave amplifies. Thus, since the second term (*i.e.*, the β -term) in (6) is typically

Fig. 5 Theoretical fields of the oceanic surface streamfunction ψ_0 (solid line), and the temperature function ψ_T (dashed line), representing a meander that evolved exponentially in 26 days from a small perturbation of a baroclinic mean current. Reproduced from S-T (1975b).

smaller than the first term, there is a negative feedback between ν and both r and $\hat{u}_T$ that can potentially lead to saturation of the wave even in the absence of friction. We next develop formulas for the rate of change of area-averaged or "gross" static stability (and hence of r), and of the representative mid-latitude vertical shear (or baroclinicity) $\hat{u}_T$. The potentially important effects of local horizontal variations of static stability and basic wind shear within the wave system will not be considered here.

1) Static stability

Following Arakawa (1962), Gambo (1966), or Tang (1980), we can express the time rate of change of the gross static stability $\bar{s}$ for the two-level model in the form,

$$\frac{d\bar{s}}{dt} = -\frac{4f}{p_2^3} \overline{\omega_2^{(1)} \phi_T^{(1)}}. \quad (7)$$

To evaluate the vertical flux of heat by the primary wave, appearing on the right hand side, we have from (1b) and (4),

$$\omega_2^{(1)} = \frac{p_2 \mu^2 r}{f} \left(-2\hat{u}_T v_2^{(1)} + \frac{\beta}{\mu^2} \frac{\partial \phi_T^{(1)}}{\partial x} \right), \quad (8)$$

which can be multiplied by $\phi_T^{(1)}$ and averaged over the fundamental horizontal domain to yield

$$\overline{\omega_2^{(1)} \phi_T^{(1)}} = -\frac{2p_2 \mu^2 r}{f} \overline{\hat{u}_T v_2^{(1)} \phi_T^{(1)}}. \quad (9)$$

Substituting for $v_2^{(1)} = \partial \phi^{(1)} / \partial x$ and $\phi_T^{(1)}$ from (4) we find

$$\overline{v_2^{(1)} \phi_T^{(1)}} = \frac{\nu}{4\hat{u}_T} |\Psi_2|^2 e^{2\nu t}, \quad (10)$$

which implies that

$$\frac{d\bar{s}}{dt} = \frac{2\mu^2 |\Psi_2|^2}{p_2^2} r \nu e^{2\nu t}. \quad (11)$$

Let us now define a time coordinate consisting of discrete intervals Δt , so that $t = t_{(n)} = t_{(0)} + n\Delta t$. If we integrate (11) over a time interval Δt , from $t_{(n)}$ to $t_{(n+1)}$, assuming that we can approximate ν and r by constant values $\nu_{(n)}$ and $r_{(n)}$ over this interval. We obtain

$$\bar{s}_{(n+1)} - \bar{s}_{(n)} = \frac{\mu^2 |\Psi_2|^2}{p_2} r_{(n)} e^{2\nu_{(n)} t_{(n)}} (e^{2\nu_{(n)} \Delta t} - 1), \quad (12)$$

or using the definition

$$r_{(n)} = \frac{2}{2 + (p_2^2 \mu^2 / f^2) \bar{s}_{(n)}} \quad (13)$$

we have

$$r_{(n+1)} = \frac{r_{(n)}}{1 + (r_{(n)} \mu^4 / 2f^2) |\Psi_2|^2 e^{2\nu_{(n)} t_{(n)}} (e^{2\nu_{(n)} \Delta t} - 1)} \quad (14)$$

2) Vertical wind shear

The first order modification of the basic zonal current is governed by (2). From the solution (5b) we see that the rate of change of $\bar{\phi}_{1,3}^{(1)}$ at any time t , when the wave is growing exponentially with an amplification factor ν , is

$$\frac{\partial \bar{\phi}_{1,3}^{(1)}}{\partial t} = \mp \frac{f^2 l |\Psi_2|^2 \nu}{\hat{u}_T (5l^2 p_4^2 + 2f^2)} e^{2\nu t} \sin 2ly \quad (15)$$

Introducing the definition of r , and noting that $p_4 = 2p_2$, $\bar{\phi}_T = (\bar{\phi}_1 - \bar{\phi}_3)/2 = \bar{\phi}_1^{(1)}$, and $\hat{u}_T^{(1)} = -\partial \bar{\phi}_T^{(1)} / \partial y$, we have

$$\frac{\partial \hat{u}_T^{(1)}}{\partial t} = \frac{l^2 |\Psi_2|^2 \nu e^{2\nu t}}{\hat{u}_T \left[\frac{4(1-r)l^2}{r\mu^2} + 1 \right]} \cos 2ly \quad (16)$$

If, as above, we integrate over a time interval $t_{(n)}$ to $t_{(n+1)}$, again assuming we can approximate r , $\hat{u}_T$, and ν by constant values ($r_{(n)}$, $\hat{u}_{T(n)}$, $\nu_{(n)}$) over this interval, we obtain

$$\hat{u}_{T(n+1)}^{(1)} = \hat{u}_{T(n)}^{(1)} + \frac{l^2 |\Psi_2|^2 e^{2\nu_{(n)} t_{(n)}} (e^{2\nu_{(n)} \Delta t} - 1)}{2\hat{u}_{T(n)} \left[\frac{4(1-r_{(n)})l^2}{\mu^2 r_{(n)}} + 1 \right]} \cos 2ly \quad (17)$$

Since $\hat{u}_{T(0)}^{(1)} = 0$, we can see that $\hat{u}_{T(n)}^{(1)}(y) \sim \cos 2ly$, which implies that the vertical shear decreases to a minimum at the mid-latitude $y = D/2$ and increases to a maximum value at the boundaries $y=0, D$ (cf., S-T, 1972). It is clear that the mean value of $\hat{u}_T^{(1)}(y)$ over all latitudes vanishes, and if we defined $\hat{u}_T$ to be of the form

$$\hat{u}_T = \bar{u}_T^{(0)} + \frac{1}{D} \int_0^D \hat{u}_T^{(1)}(y) dy \quad (18)$$

it would be equivalent to setting $\hat{u}_T(y) = \bar{u}_T^{(0)}$ (a constant). At the other extreme, we could set

$$\hat{u}_T = \bar{u}_T^{(0)} + \hat{u}_T^{(1)}(D/2) \quad (19)$$

which maximizes the influence of the vertical shear at the mid-latitude ($y=D/2$) on the growth of the wave. A more logical choice might be to weight $\hat{u}_T^{(1)}(y)$ by the primary wave amplitude, i.e.,

$$\hat{u}_T = \bar{u}_T^{(0)} + \frac{1}{D} \int_0^D \hat{u}_T^{(1)}(y) \sin ly dy \quad (20)$$

for which it follows that

$$\hat{u}_T = \bar{u}_T^{(0)} + \frac{2}{3\pi} \hat{u}_T^{(1)}(D/2). \quad (21)$$

More generally, let us set

$$\hat{u}_T = \bar{u}_T^{(0)} + \varepsilon \hat{u}_T^{(1)}(D/2) \quad (22)$$

where ε will be treated as a variable parameter that can be assigned arbitrary values between 0 and 1. Thus, combining (17) and (22) we

have,

$$\hat{u}_{T(n+1)} = \hat{u}_{T(n)} - \frac{\varepsilon l^2 |\Psi_2|^2 e^{2\nu_{(n)} t_{(n)}} (e^{2\nu_{(n)} \Delta t} - 1)}{2\hat{u}_{T(n)} \left[\frac{4(1-r_{(n)})l^2}{\mu^2 r_{(n)}} + 1 \right]} \quad (0 < \varepsilon < 1) \quad (23)$$

where, from (6)

$$\nu_{(n)} = k \left[\hat{u}_{T(n)}^2 (2r_{(n)} - 1) - \frac{b^2}{4} r_{(n)}^2 \right]^{1/2} \quad (24)$$

The values of $\bar{\phi}_{T(n)}^{(1)} = (\bar{\phi}_1^{(1)} - \bar{\phi}_3^{(1)})/2 = \bar{\phi}_1^{(1)}$ at any time $t_{(n)}$ can also be determined from (15) using the same procedure. We obtain,

$$\bar{\phi}_{T(n+1)}^{(1)} = \Pi_{(n+1)} \sin 2ly \quad (25)$$

where

$$\Pi_{(n+1)} = \Pi_{(n)} - \frac{l |\Psi_2|^2 e^{2\nu_{(n)} t_{(n)}} (e^{2\nu_{(n)} \Delta t} - 1)}{4\hat{u}_{T(n)} \left[\frac{4(1-r_{(n)})l^2}{r_{(n)}\mu^2} + 1 \right]} \quad (26)$$

To determine the phase velocity of the wave at time $t_{(n)}$ we can also set

$$c_{r(n)} = \bar{u}_2 - b \left(1 - \frac{r_{(n)}}{2} \right) \quad (27)$$

3) Wave evolution

With the use of (14), (23), and (24) we are in a position to numerically iterate the evolution of r , $\hat{u}_T$ and ν from the time of the inception of a baroclinic wave, and to calculate the corresponding evolution of the wave amplitude and phase and the structure of the total field of flow and temperature. Thus, if we express the primary wave solution in the form

$$\left. \begin{aligned} \phi_2^{(1)}(t) &= \Phi(t) \cos \{k[x - c_r(t)t] + \gamma(t)\} \sin ly \\ \phi_T^{(1)}(t) &= \Lambda(t) \cos \{k[x - c_r(t)t] - \gamma(t)\} \sin ly \\ \omega_2^{(1)}(t) &= \Xi(t) \cos \{k[x - c_r(t)t] + \vartheta(t)\} \sin ly \end{aligned} \right\} \quad (28a, b, c)$$

where

$$t = t_{(n)} = t_{(0)} + n\Delta t$$

and assume that the growth rate over any small interval $\Delta t = t_{(n)} - t_{(n-1)}$ is given by $(\nu_{(n-1)} + \nu_{(n)})/2$, the evolution of the wave structure can be determined from the following formulas for the amplitude and relative phases (c.f. Section 2b):

$$\begin{aligned} \Phi(t) &= \Phi(t_{(0)} + n\Delta t) \equiv \Phi_{(n)} \\ &= \Phi_{(n-1)} \exp \left\{ (\nu_{(n-1)} + \nu_{(n)}) \frac{\Delta t}{2} \right\} \\ n &= 1, 2, 3, \dots \end{aligned} \quad (29)$$

$$\begin{aligned} \Lambda(t) &= \Lambda(t_{(0)} + nt) \equiv \Lambda_{(n)} \\ &= \Phi_{(n)} (2r_{(n)} - 1)^{1/2} \end{aligned} \quad (30)$$

$$\Xi(t) = \Xi(t_{(0)} + n\Delta t) \equiv \Xi_{(n)}$$

$$\begin{aligned} &= \Phi_{(n)} \left\{ \frac{p_2 \mu^2 k}{f \left[\frac{(1-r_{(n)})}{r_{(n)}} + 1 \right]} \right. \\ &\quad \times \left[\left(2\hat{u}_{T(n)} - \frac{r_{(n)} b^2}{2\hat{u}_{T(n)}} \right)^2 + \left(\frac{b\nu_{(n)}}{k\hat{u}_{T(n)}} \right)^2 \right]^{1/2} \left. \right\} \end{aligned} \quad (31)$$

and

$$\begin{aligned} \gamma(t) &= \gamma(t_{(0)} + n\Delta t) \equiv \gamma_{(n)} \\ &= \arctan \left\{ - \left[\frac{2\hat{u}_{T(n)} (2r_{(n)} - 1)^{1/2} - b r_{(n)}}{2\hat{u}_{T(n)} (2r_{(n)} - 1)^{1/2} + b r_{(n)}} \right]^{1/2} \right\} \end{aligned} \quad (32)$$

$$\vartheta(t) = \vartheta(t_{(0)} + n\Delta t) \equiv \vartheta_{(n)}$$

$$\begin{aligned} &= \arctan \left\{ \frac{- \left(2\hat{u}_{T(n)} - \frac{r_{(n)} b^2}{2\hat{u}_{T(n)}} + \frac{b\nu_{(n)}}{k\hat{u}_{T(n)}} \tan \gamma_{(n)} \right)}{\left[\left(2\hat{u}_{T(n)} - \frac{r_{(n)} b^2}{2\hat{u}_{T(n)}} \right) \tan \gamma_{(n)} - \frac{b\nu_{(n)}}{k\hat{u}_{T(n)}} \right]} \right\} \end{aligned} \quad (33)$$

4) Initial conditions

If $t_{(0)} = 0$ is taken to be the time of introduction of the initial perturbation into a basic state characterized by $\bar{s}_{(0)} = \bar{s}^{(0)}$ and $\bar{\phi}_{1,3(0)} = \bar{\phi}_{1,3}^{(0)}$ (i.e., $\hat{u}_{T(0)} = \bar{u}_{T(0)}$), we can specify the following as initial conditions:

$$r_{(0)} = \frac{2}{2 + (p_2^2 \mu^2 \bar{s}^{(0)} / f^2)} \quad (34)$$

$$\hat{u}_{T(0)} = \bar{u}_{T(0)} \quad (35)$$

$$\Pi_{(0)} = 0 \quad (36)$$

$$\Phi_{(0)} = |\Psi_2| \quad (37)$$

b. Results

Let us now apply the formulas developed above to the atmosphere and ocean, using the initial conditions (34)-(37).

1) Atmospheric model

We adopt the following reference values of the physical parameters, most of which were specified by S-T (1972):

$$\left. \begin{aligned} f &= 1.03 \times 10^{-4} \text{ s}^{-1} \\ \beta &= 1.62 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1} \end{aligned} \right\} \quad \varphi = 45^\circ$$

$$p_2 = 500 \times 10^2 \text{ Pa}$$

$$r_{(0)} = 0.7071$$

$$\bar{s}^{(0)} = 137.77 \times 10^{-8} \text{ m}^4 \text{ s}^2 \text{ kg}^{-2}$$

$$\bar{u}_2^{(0)} = 20 \text{ m s}^{-1}$$

$$\bar{u}_{T(0)} = 8 \text{ m s}^{-1}$$

$$|\Psi_2| = 3.496 \times 10^6 \text{ m}^2 \text{ s}^{-1}$$

$$L = D = 4393.17 \text{ km}$$

$$k = 2\pi/L = 1.4302 \times 10^{-6} \text{ m}^{-1}$$

$$l = \pi/D = 0.7151 \times 10^{-6} \text{ m}^{-1}$$

$$\begin{aligned}\mu^2 &= k^2 + l^2 = 2.5568 \times 10^{-12} \text{ m}^{-2} \\ b &= \beta/\mu^2 = 6.3281 \text{ m s}^{-1} \\ c_{T(0)} &= 15.909 \text{ m s}^{-1} \\ \nu_{(0)} &= kc_{i(0)} = 6.6321 \times 10^{-6} \text{ s}^{-1} \\ \varepsilon &= 0.5\end{aligned}$$

Although we arbitrarily take $\varepsilon=0.5$ as our reference value, we shall discuss the sensitivity of the results (namely, the time of saturation τ , and amplitude at saturation $\Phi(\tau)$) to this parameter, as well as to the initial baroclinicity $\bar{u}_T^{(0)}$ and perturbation amplitude $\Phi_{(0)} = |\Psi_2|$. The reference value we have chosen for $\bar{u}_T^{(0)}$ and $\bar{s}^{(0)}$ seem reasonable from climatic data (e.g., Oort and Rasmusson, 1971). Similarly, the initial perturbation value $\Phi_{(0)} = |\Psi_2| = 3.496 \times 10^6$ (corresponding to a perturbation velocity of 5 ms^{-1}) seems reasonable since departures from zonal mean conditions of at least this magnitude are always present, if only in the form of standing waves (cf., e.g., Saltzman and Ashe, 1977).

Using these values, and taking $\Delta t = 0.2 \text{ day} = 17280 \text{ s}$, we can iterate the system (14), (23),

Fig. 6 Evolution of (a) gross static stability $\bar{s}$ and vertical shear or baroclinicity $\bar{u}_T$, and (b) amplification factor ν , all normalized by their initial values, for reference values of parameters, cases (i)-(iv).

(24), (26), (27), (29), (30), (31), (32) and (33), subject to the initial conditions (34) to (37). The results are shown in Figs. 6 to 11.

The increase of the static stability $\bar{s}$, normalized by its initial value $\bar{s}^{(0)}$ is shown in Fig. 6a for $\bar{u}_T$ fixed at its initial value [dashed curve labelled (ii)] and for $\bar{u}_T$ allowed to vary also as a function of time [solid curve labelled (iv)]. We can see that when $\bar{u}_T$ is fixed, the ratio $(\bar{s}/\bar{s}^{(0)})$ reaches an asymptotic value of about 2.18 at $t \approx 15d$, while in the more general case when $\bar{u}_T$ is allowed to vary (thereby leading to a more rapid decrease of ν with time; cf., Fig. 6b), its maximum value is smaller (1.14) and achieved earlier ($t = 9.6 \text{ d}$).

From Fig. 6a we see that for $\bar{s}$ fixed [dashed curve labelled (iii)] the ratio $(\bar{u}_T/\bar{u}_T^{(0)})$ drops to a minimum value of 0.43 in 8.8 days, and for the more general case when $\bar{s}$ varies also [solid curve labelled (iv)] a slightly higher minimum value of 0.45 is achieved after 9.6 days.

The change with time of the ratio of the growth rate ν to its initial value $\nu_{(0)}$ is shown in Fig. 6b, for the following cases: (i) $\bar{u}_T = \bar{u}_T^{(0)}$ and $\bar{s} = \bar{s}^{(0)}$ [i.e., unabated exponential growth, as in the earlier studies S-T (1972, 1975)], (ii) $\bar{u}_T = \bar{u}_T^{(0)}$, but $\bar{s} = \bar{s}(t)$, (iii) $\bar{s} = \bar{s}^{(0)}$, but $\bar{u}_T = \bar{u}_T(t)$, and (iv) the general case where both $\bar{u}_T$ and $\bar{s}$ are allowed to vary according to (13) and (17). For this latter general case we see that the growth rate goes to zero at $t = \tau = 9.6 \text{ d}$, at which time the primary wave should achieve maximum amplitude (i.e., saturation). This saturation time, $\tau = 9.6 \text{ d}$, seems reasonable in view of the fact that we have neglected friction, which could plausibly reduce this value by a factor of two (cf., Haltiner and Caverly, 1965). From case (ii) we see that when static stability alone is varied the approach to saturation ($\nu \rightarrow 0$) is very slow, going well beyond the limit of our abscissa at 14 d.

In Fig. 7 we show the amplitude function $\Phi(t)$ for the above four cases (i) to (iv). As is implied by the $(\nu/\nu_{(0)})$ -curve for the general case (iv), zero growth occurs at $t = \tau = 9.6 \text{ d}$, with the amplitude $\Phi(\tau) = 32 \times 10^6 \text{ m}^2 \text{ s}^{-1}$ corresponding to a meridional velocity of about 50 ms^{-1} . It is clear from case (iii) that the baroclinicity (i.e., $\bar{u}_T$) variations are much more important than the $\bar{s}$ -variations in effecting saturation. That is, even if $\bar{s}$ -variations are neglected the saturation time and amplitude would be nearly the same. This result which was already apparent in Fig. 6b provides some justification for the neglect

of $\bar{\zeta}$ -variations in finite amplitude analyses of baroclinic waves (e.g., Pedlosky, 1970). On the other hand, it is worth noting that even if the baroclinicity (or vertical shear) was maintained

Fig. 7 Evolution of the amplitude of the mid-level streamfield wave Φ for the reference values, cases (i)-(iv).

Fig. 8 Evolution of amplitudes Π , Λ and Ξ and phases γ and θ for the reference case (iv) [see (25) and (28)].

somehow and therefore did not change as the wave matures, static stability changes alone could lead to saturation though at a much slower rate and at a much higher amplitude.

In Fig. 8 we show the variations of the amplitudes Π , Λ and Ξ , and phase angles γ and θ , corresponding to case (iv) which includes the combined effects of static stability and baroclinicity variations. From these quantities, along with Φ , we can construct the fields of ψ , ψ_T and ω at any time. Some examples of these are shown in Figs. 9 and 10.

In Fig. 9 we show maps of the mid level streamfield $\psi_2 = \bar{\psi}_2^{(0)} + \psi_2^{(1)}$ (full curves) and the 'temperature' field $\psi_T = \bar{\psi}_T^{(0)} + \bar{\psi}_T^{(1)} + \psi_T^{(1)}$ (dashed curves) at an intermediate time $t = 4$ d, for the unmodified exponential growth case (i)

Fig. 9 Maps of the mid-level streamfield ψ_2 (solid curve) and 'temperature' field ψ_T (dashed curve) at 4 days for reference parameters, for exponential case (i) at the top and general case (iv) at the bottom. Units are $10^7 \text{ m}^2 \text{ s}^{-1}$. Positions of troughs are denoted by heavy lines; and centers of rising and sinking motion are denoted by circles containing dot and cross, respectively, with value near center of sinking motion denoting the amplitude in 10^{-4} mbs^{-1} .

(top) and for the general case (iv) (bottom). The positions of the trough and ridges are shown by the heavy lines. The center of rising motion of the $\omega_2^{(1)}$ -wave is shown by the circle containing a dot, and the center of sinking motion by the circle containing a cross near which a number representing the $\omega_2^{(1)}$ -amplitude in $10^{-4} \text{ mb s}^{-1}$ is entered. As expected we find that the amplitudes of the fields are weaker and the phase lags smaller in the case of variable $\bar{s}$ and $\bar{u}_T$.

In Fig. 10 we show the fields at the time of saturation ($t = \tau = 9.6 \text{ d}$) when the streamfield reaches its maximum amplitude (see Fig. 7) and the streamline and isotherms are in phase implying no tilt of the wave with height. At this time the flow therefore attains an essentially "equivalent barotropic" condition which is typical of the global flow except in localized regions where active baroclinic growth is occurring (Charney and Eliassen, 1949). The locking in phase of the upper and lower waves is a characteristic feature of the onset of occlusion that has been noted by many investigators (*e.g.*, Petterssen and Smebye (1971), Rhines (1977), Thompson (1959)).

Sensitivity analysis—Let us denote the above case involving the reference values $\bar{u}_T^{(0)} = 8 \text{ ms}^{-1}$ and $\varepsilon = 0.5$ as Case I. In order to determine the sensitivity of the above results to changes in the initial baroclinicity $\bar{u}_T^{(0)}$ and to our choice of the parameter ε , let us consider the following two additional cases,

Case II: $\bar{u}_T^{(0)} = 8 \text{ ms}^{-1}$, $\varepsilon = 2/3\pi$

[corresponding to formulation (20, 21)]

Case III: $\bar{u}_T^{(0)} = 10 \text{ ms}^{-1}$, $\varepsilon = 2/3\pi$.

In Fig. 11a, b we show the saturation time τ , and amplitude at saturation $\Phi(\tau)$, for these three cases, as a function of the initial amplitude of the perturbation. Thus, the reference case

Fig. 10 Same as Fig. 9, but for case (iv) at the time of saturation (9.6 days).

Fig. 11 Saturation time τ and amplitude at saturation $\Phi(\tau)$ for Cases I, II and III (see text), as a function of initial amplitude of the perturbation $\Phi_{(0)}$.

described above is denoted by the large dot on the Case I curves where $\Phi_{(0)} = 3.5 \times 10^6 \text{ m}^2 \text{ s}^{-1}$. We can see that τ and $\Phi(\tau)$ are, in fact, highly dependent on $\Phi_{(0)}$, increasing sharply for small and decreasing values of the initial perturbation. The results for large $\Phi_{(0)}$ seem to be in qualitative agreement with the results of Yeh and Chen (1963). We note also that a decrease in ε (*cf.*, Case I *vs* Case II) gives rise to higher values of τ and $\Phi(\tau)$, as does an increase in the initial baroclinicity (*cf.*, Case II *vs* Case III).

2) Oceanic model

For the ocean the following parameter values specified previously by S-T (1975b) are appropriate.

$$\left. \begin{aligned} f &= 0.90 \times 10^{-4} \text{ s}^{-1} \\ \beta &= 1.80 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1} \end{aligned} \right\} \varphi = 38^\circ$$

$$H = 1000 \text{ m}$$

$$r_{(0)} = 0.7071$$

$$\sigma_{(0)} = 5 \times 10^{-5} \text{ s}^{-2}$$

$$\bar{u}_2^{(0)} = 0.30 \text{ m s}^{-1}$$

$$\bar{u}_T^{(0)} = 0.15 \text{ m s}^{-1}$$

$$|\Psi_2| = 24 \times 10^2 \text{ m}^2 \text{ s}^{-1}$$

$$L = D = 300 \text{ km}$$

$$k = 2\pi/L = 2.088 \times 10^{-5} \text{ m}^{-1}$$

$$l = \pi/D = 1.044 \times 10^{-5} \text{ m}^{-1}$$

$$\begin{aligned}\mu^2 &= k^2 + l^2 = 0.0545 \times 10^{-8} \text{ m}^{-2} \\ b &= \beta/\mu^2 = 3.3085 \times 10^{-2} \text{ m s}^{-1} \\ c_{T(0)} &= 0.2786 \text{ m s}^{-1} \\ \nu_{(0)} &= kc_{i(0)} = 2.0013 \times 10^{-6} \text{ s}^{-1}\end{aligned}$$

Because of the slower growth rate of a baroclinic wave in the ocean (engendered largely by the lower value of $\bar{u}_T^{(0)}$), we can take a larger time step than for the atmosphere. We take $\Delta t = 1 \text{ d} = 86400 \text{ s}$. The initial value $|\psi_2| = 24 \times 10^2 \text{ m}^2 \text{ s}^{-1}$ corresponds to a perturbation velocity of 5 cm s^{-1} . The results of the iteration are shown in Figs. 12 and 13.

In Fig. 12 we show the change with time of the static stability σ and baroclinicity $\bar{u}_T$ (normalized to their initial values) when both vary simultaneously as in atmospheric case (iv) above, and the corresponding changes in the amplitude Φ of the ϕ_2 -wave for this case. We can see that

Fig. 12 Evolution of static stability σ , vertical shear $\bar{u}_T$, both normalized by their initial values, and amplitude of the mid-level streamfield wave Φ , for the ocean.

Fig. 13 Map of the surface oceanic streamfield ψ_0 (solid curves) and ‘temperature’ field ψ_T , at the time of saturation ($\tau = 63 \text{ d}$). Units are $10^4 \text{ m}^2 \text{ s}^{-1}$.

whereas the static stability σ increases slightly by only about 2 percent to the time of saturation ($\tau = 63 \text{ d}$), the baroclinicity $\bar{u}_T$ is reduced by about 90 percent, and the amplitude increases over 30 fold to a value corresponding to about 1.8 ms^{-1} .

In Fig. 13 we show the stream function at the top surface of the ocean $\phi_0 = \bar{\phi}_0^{(0)} + \bar{\phi}_0^{(1)} + \phi_0^{(1)}$, and the ‘temperature’ field $\phi_T = \bar{\phi}_T^{(0)} + \bar{\phi}_T^{(1)} + \phi_T^{(1)}$, at $t = \tau = 63 \text{ d}$ which is near the time of maximum amplitude. We note that the solution shows the isotherms and streamlines to be more in phase, showing somewhat more of the character of concentrate ‘rings’ than the previous solution shown in Fig. 5.

4. Summary

After a review of our previous studies of the finite amplitude evolution of baroclinic waves in the atmosphere and ocean (S-T 1972, 1974, 1975a, b), we use a simple, semi-numerical method to demonstrate some important effects neglected in these previous studies. In particular, we show how the wave-induced modifications of the vertical wind shear (or baroclinicity) and, to a lesser extent, of gross static stability, can lead to saturation of the waves in the absence of friction, nonlinear wave-wave interactions, and nongeostrophic effects. This saturation is characterized by the onset of ‘occlusion’ for the low pressure region, and the approach to an ‘equivalent barotropic’ state for the flow as a whole. For typical initial states, the time scales for these developments are 10 days for the atmosphere and two months for the ocean and the mid-level amplitudes are roughly 50 ms^{-1} and 1.8 ms^{-1} , respectively. These time scales and amplitudes are larger than observed, but seem reasonable in view of the above-mentioned neglect of frictional, nonlinear, and nongeostrophic effects. We can expect that more realistic solutions showing the observed features of the decaying phase of mature systems, can be achieved when these latter factors as well as non-adiabatic effects are taken into account.

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若干の解析解による有限振幅傾圧波の研究のレビュー
(安定度と傾圧性の変化の効果を取り入れる
新しい計算方法を含む)

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二次の非線形過程および非地衡風過程が2層の増幅傾圧波の時間発展に及ぼす効果を調べた著者らの一連の研究をレビューする。大気、海洋中の傾圧波に見られる有限振幅効果による特性（高・低気の状態、前線の位置や形など）の多くは理論から導かれ得ることを示す。しかしながら、上記の研究においては基本状態の変化と低次（一次）の波動場の間のフィードバックが省略されており、そのため波動は飽和振幅に達する（つまり閉塞する）ことなく、不自然にどこまでも発達する。鉛直安定度および傾圧性（鉛直シア）の変化と発達する傾圧波の間の相互のフィードバックによる飽和の効果を示すため、新しい半数値的計算方法を開発した。この方法はこれまでの解析解による研究と結びつけることができる。初期の基本状態が典型的で、かつ異なる波数の間の相互作用や摩擦がない場合には、閉塞状態になるのに要する時間スケールは大気で10日ほど、海洋で2か月であることがわかる。いずれの場合も波の飽和をもたらすには傾圧性の変化の方が鉛直安定度の変化の効果よりも重要である。摩擦が省略されているので、閉塞した傾圧波の最終的衰弱はこの理論では表現されない。