Quasi-Resonant Rossby Wave

Part I: External Wave

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Abstract

Non-linear resonance of external Rossby wave in a continuously stratified atmosphere under cyclic condition in the meridional direction is studied by the perturbation expansion method. The feed-back effects of the mean zonal flow induced by the wave-topography (and/or heating) and wave-self interaction are included. As the forcings, a sinusoidal mountain and/or heating are assumed, while the Ekman damping, Rayleigh friction and Newtonian cooling are included as the dissipations.

In the case of topographic forcing alone, it is shown that for long (short) waves the amplitude response curves incline toward negative (positive) side of the detuning parameter δ (deviation of the basic zonal flow from the linear resonant condition), to give the so-called multiple (three) equilibrium solutions for a certain range of δ for small damping coefficients (e.g., Charney and DeVore, 1979; hereafter referred to as CD). The positions of maximum response on δ -axis shift in proportion to 2nd power of the magnitude of forcing and -2 nd power of the damping coefficients. At a 'critical' wave number, the amplitude response curve is similar to that of linear resonance as a result of the cancellation between non-linear terms; the multiplicity of equilibrium vanishes.

The stabilities of equilibrium solutions are examined by perturbation method, to show that the upper and the lower branches are stable, while the middle branch is unstable as has been pointed out by CD. All the equilibrium states outside the multiple equilibrium region are stable.

If thermal forcing is incorporated, the response becomes complicated, depending upon its magnitude and its phase relative to the topography; for example, the amplitude response curve forms a loop if plotted against δ and the largest amplitude state is shown to be unstable. This loop becomes larger and wider as the heating increases. It is noted that such a loop can be found also in CD's model if the mountain becomes high, though it seems to have been overlooked so far. Finally, we show the time-dependent behaviors of the solutions by numerical integrations of the equation obtained by the present method.

1. Introduction

In recent years, concerning the so-called tropospheric blocking phenomenon and the trigger of the stratospheric sudden warming, the problem of resonant Rossby waves has been studied by many authors. It is noted, however, that this problem can be traced back to Charney and Eliassen (1949), in the sense that they pointed out that a stationary, topo-

graphically forced planetary wave can be resonant if the basic zonal flow velocity is equal to the phase velocity of a free Rossby wave. It is further noted that Murakami (1967) found, using the linearized primitive equations independent of latitudes that wave number 3 or 4 is near resonant in the background zonal flows in winter of the northern hemisphere.

On the basis of linear theory, Tung and Lindzen (1979, a, b) have discussed the re-

sonance of large-scale planetary wave forced by the surface topography and/or heating, and suggested that the resonant growth (in strict sense, not resonance because of the inclusion of some damping effect) could cause a state like the blocking phenomenon and also be an essential factor initiating the sudden warming as assumed by Matsuno (1971). However, because of the linear theory, they have neglected the feed-back effect of the growing wave on the basic zonal flow, which is caused by the wave self-interaction, the wave-topography interaction and so on and can change the basic zonal flow, i.e., break down the linear resonant condition.

On the other hand, Charney and DeVore (1979), Vickroy and Dutton (1979), Hart (1979), Trevisan and Buzzi (1980), Pedlosky (1981)¹⁾ and others have discussed non-linear stationary wave forced by topography using barotropic models, and pointed out that multiple equilibrium solutions can exist; particularly, Charney and DeVore (1979 hereafter referred to as CD) have obtained two stable states with large and small amplitude waves, respectively, and one unstable state with moderate amplitude wave, and they have identified the stable solution with large wave amplitude near resonance with a blocking-type atmospheric flow. However, their severe truncated systems do not include wave self-interaction. As will be shown later in this paper, this identification is questionable or even misleading particularly if thermal forcing is incorporated.

In what follows, we shall discuss topographically and/or thermally driven non-linear stationary waves in a continuously stratified atmosphere with dissipation (Rayleigh friction, Newtonian cooling and Ekman damping) on a β -plane by using the perturbation expansion method, well-known in the studies of non-linear pendulum and so on (cf. Nayfeh and Mook, 1979) and also similar to that adopted by, for example, Pedlosky (1981).

The equation system for amplitude and phase can be obtained from the solvability conditions. As will be shown later, this equation system has multiple equilibrium solutions in some region of the external parameter space. The solution surface has the property

of the so-called cusp-type catastrophe, as in CD. The main purposes of the present paper are to examine the roles of various external parameters and the stability of the equilibrium solutions²⁾.

In section 2, the model and the basic equations are mentioned. Section 3 will briefly review the linear resonance theory for the convenience of non-linear discussions. In section 4, non-linear equation system for resonant Rossby waves will be derived. The equilibrium solutions and their stabilities will be discussed in section 5. In section 6, the time-dependent behaviors are briefly shown. Section 7 is devoted to the conclusion. Further, CD's model will be re-examined briefly in Appendix D.

2. Model and the basic equations

We consider a situation in which a stationary planetary wave is excited by the forcing due to a corrugated bottom over which a westerly, homogeneous wind blows, and/or by the thermal forcing which could be due to the continent-ocean temperature contrast. If the basic zonal flow satisfies the resonant condition, the planetary wave grows linearly in time. When some dissipation acts on the wave, the growth of wave is suppressed and the wave settles down to a steady state after sufficient time (cf. Tung and Lindzen, 1979). However, when the wave attains a sufficiently large amplitude, its non-linear effect becomes important through the wave self-interaction and the wave-topographic interaction, leading to change in the basic zonal flow and the break-down of resonant condition.

In what follows, we shall discuss such a non-linear behavior of resonant planetary wave, including some dissipation mechanisms such as Rayleigh friction, Newtonian cooling and

1) Pedlosky (1981) has discussed briefly also the case of a two layer baroclinic flow like Charney and Straus (1980). This problem will be studied extensively in the next paper in preparation (Wakata and Uryu, 1984a).

2) Particularly, we shall discuss the effects of thermal forcings, to show that the stability properties are remarkably different from those obtained in the case of topographical forcing alone as in one layer barotropic model of CD, Pedlosky and so on.

the Ekman pumping at the bottom.

The set of governing equations can be written as follows, with use of log-pressure coordinate in the vertical direction. The notations used are listed in Appendix A. In this set of equations, time, space coordinates and velocities are made dimensionless by using L/U , L and H as time, horizontal and vertical scales, respectively, and also geopotential, temperature and heating are made dimensionless by $2\Omega UL$, $2\Omega UL/R$ and $2\Omega U^2/\kappa$, respectively (cf. Holton, 1975);

$$Ro\left(\frac{du}{dt} + cu\right) + \sin(\varphi) \mathbf{k} \times \mathbf{u} = -\nabla \Psi, \quad (2.1)$$

$$Ro\left(\frac{d\Psi_z}{dt} + c\Psi_z\right) + Brw = RoJ, \quad (2.2)$$

$$u_x + v_y + w_z - w = 0, \quad (2.3)$$

$$\Psi_z = T, \quad (2.4)$$

where $d/dt \equiv \partial/\partial t + u \cdot \nabla_H$ and c is a Rayleigh friction and Newtonian cooling coefficients, respectively, in (2.1) and (2.2); here, we assume that these two parameters are of equal magnitude.

After some manipulation, we can obtain the following quasi-geostrophic potential vorticity equation on a β -plane;

$$\left(\frac{d}{dt} + c\right) \left[\nabla^2 + \frac{\sin^2(\varphi_0)}{Br} \left(\frac{\partial^2}{\partial z^2} - \frac{1}{4} \right) \right] \Phi + \beta \frac{\partial}{\partial x} \Phi = -\frac{\sin^2(\varphi_0)}{Br} \left(1 - \frac{\partial}{\partial z} \right) \hat{f} e^{-z/2}, \quad (2.5.1)$$

where

$$\Psi = \sin(\varphi_0) e^{z/2} \Phi, \quad (2.5.2)$$

$$J = \sin(\varphi_0) \hat{f}, \quad (2.5.3)$$

The geostrophic and hydrostatic relations are expressed as

$$\mathbf{k} \times \mathbf{u} = -e^{z/2} \nabla \Phi, \quad (2.6)$$

$$T = \sin(\varphi_0) \frac{\partial}{\partial z} (e^{z/2} \Phi), \quad (2.7)$$

The vertical 'velocity' in log-pressure coordinate is related to real vertical velocity $\hat{w}$ as follows;

$$w = \hat{w} + \left(\frac{2\Omega UL}{gH} \right) \sin(\varphi_0) e^{z/2} \frac{d\Phi}{dt} \quad (2.8)$$

On the other hand, by using thermodynamic equation (2.2), w can be written as

$$w = -e^{z/2} \frac{Ro \sin(\varphi_0)}{Br} \cdot \left[\left(\frac{d}{dt} + c \right) \left(\frac{\partial \Phi}{\partial z} + \frac{\Phi}{2} \right) - e^{-z/2} \hat{f} \right] \quad (2.9)$$

Then, substitution of (2.9) into (2.8) gives

$$\hat{w} = -e^{z/2} \frac{Ro \sin(\varphi_0)}{Br} \cdot \left[\left(\frac{d}{dt} + c \right) \left(\frac{\partial \Phi}{\partial z} + \frac{\Phi}{2} \right) - S \frac{d\Phi}{dt} - e^{-z/2} \hat{f} \right] \quad (2.10)$$

As the bottom boundary condition, we assume the Ekman pumping effect in addition to the topographic forcing;

$$\hat{w} = u \cdot \nabla_H h + \hat{E}^{1/2} \nabla^2 \Phi \quad \text{at } z=0. \quad (2.11)$$

Combining (2.10) with (2.11), we can rewrite the bottom boundary condition in terms of Φ as follows;

$$\frac{d}{dt} \left(\frac{\partial \Phi}{\partial z} + \left(\frac{1}{2} - s \right) \Phi \right) + c \left(\frac{\partial \Phi}{\partial z} + \frac{\Phi}{2} \right) = \hat{f} - u \cdot \nabla \hat{h} - E^{1/2} \nabla^2 \Phi \quad \text{at } z=0, \quad (2.12)$$

where $\hat{h} = Br/(Ro \sin(\varphi_0))h$ and $E^{1/2} = \hat{E}^{1/2} Br/(Ro \sin(\varphi_0))$. Since it is necessary for resonance that the wave energy is trapped, then we assume the external wave³⁾.

As the side boundary condition, we impose the cyclic condition, so that the linear wave solution is of $\sin(ly)$.

In what follows, we assume the following sinusoidal form

$$\hat{h} = h_0 \sin(ly) e^{ikx} + \text{comp. conj.} \quad (2.13)$$

for topography with amplitude h_0 and

$$\hat{f} = J_0 \sin(ly) e^{-z/d} e^{i(kx + \alpha)} + \text{com. conj.} \quad (2.14)$$

for thermal forcing with amplitude J_0 , where α is the phase difference between the bottom corrugation and the heating. As will be seen later, α is one of the important parameters

3) The problem of internal wave will be discussed in Part II (Wakata and Uryu, 1984b). In this case, some reflector like a strong westerly is necessary.

in this work. d is a real positive number that expresses the vertical distribution of thermal heating (cf. Charney, 1973).

3. A brief review of linear theory

Here, we shall briefly mention the resonance of small amplitude stationary wave forced by the bottom topography and heating, for the sake of later discussions on the non-linear results.

After linearizing eq. (2.5), we obtain that

$$\begin{aligned} (c+U_0 \frac{\partial}{\partial x}) \left[\nabla^2 \Phi' + \frac{\sin^2(\varphi_0)}{Br} \left(\frac{\partial^2 \Phi'}{\partial z^2} - \frac{\Phi'}{4} \right) \right] \\ + \beta \frac{\partial \Phi'}{\partial x} \\ = \frac{\sin^2(\varphi_0)}{Br} e^{-z/2} \left(\frac{\partial \hat{f}}{\partial z} - \hat{f} \right), \end{aligned} \quad (3.1)$$

where Φ' is the disturbance part of Φ , and U_0 is the basic zonal wind which is assumed to be homogeneous. The boundary condition for stationary waves becomes as

$$\begin{aligned} (c+U_0 \frac{\partial}{\partial x}) \left(\frac{\partial \Phi'}{\partial z} + \frac{\Phi'}{2} \right) \\ = -U_0 \frac{\partial \hat{h}}{\partial x} + \hat{f}(0) - E^{1/2} \nabla^2 \Phi' \quad \text{at } z=0, \end{aligned} \quad (3.2)$$

where $\hat{f}(0) = J_0 \sin(l y) e^{i(kx + \alpha)}$.

Assuming the following wave form

$$\Phi' = \varphi(z) \sin(l y) e^{i k x}, \quad (3.3)$$

At the limit of non dissipation $c = E^{1/2} = 0$, A becomes infinitely large as $a \rightarrow 1/2$, i.e., the resonance occurs. The topographic resonance condition $a = 1/2$ is rewritten in terms of U_0 as

$$U_0 = \frac{\beta}{k^2 + l^2} \equiv U_r, \quad (3.11)$$

(Charney and Eliassen, 1949). In other words, if the speed of the basic zonal flow is equal to the phase velocity of two dimensional free Rossby wave, the resonance occurs. As has been pointed out by Tung & Lindzen (1979, a),

we obtain that

$$\begin{aligned} \frac{d^2 \varphi}{dz^2} + \left[\frac{Br}{\sin^2(\varphi_0)} \frac{ik\beta}{c + ikU_0} - \frac{1}{4} - \frac{Br(k^2 + l^2)}{\sin^2(\varphi_0)} \right] \varphi \\ = -\frac{1 + 1/d}{c + ikU_0} J_0 e^{i\alpha} e^{-(1/2 + 1/d)z}, \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} \frac{d\varphi}{dz} + \frac{\varphi}{2} - \frac{E^{1/2}(k^2 + l^2)}{c + ikU_0} \varphi = \frac{-ikU_0 h_0 + J_0 e^{i\alpha}}{c + ikU_0} \\ \text{at } z=0. \end{aligned} \quad (3.5)$$

Then, it follows that

$$\varphi = A e^{-az} + C e^{az} + B e^{-(1/2 + 1/d)z + i\alpha}, \quad (3.6)$$

where a is given by

$$a^2 = \frac{1}{4} + \frac{Br(k^2 + l^2)}{\sin^2(\varphi_0)} - \frac{Br}{\sin^2(\varphi_0)} \frac{ik\beta}{c + ikU_0}, \quad (3.7)$$

and B is the coefficient of the particular solution of eq. (3.4), i.e.,

$$B = \frac{J_0}{c + ikU_0} \frac{1 + 1/d}{a^2 - (1/2 + 1/d)^2}. \quad (3.8)$$

On the other hand, A and C are determined by the boundary conditions. According to the sign of a^2 , we have two cases; one is 'external' mode, the other 'internal'. Here, we will deal with the external mode, and choose a solution which damps upward. φ becomes as

$$\varphi = A e^{-az} + B e^{-(1/2 + 1/d)z + i\alpha}, \quad (3.9)$$

with a positive, where

$$A = \frac{-ikU_0 h_0 + \frac{a^2 - 1/4 + (1 + 1/d) E^{1/2}(k^2 + l^2)}{a^2 - (1/2 + 1/d)^2} \frac{ikU_0 + c}{J_0 e^{i\alpha}}}{(ikU_0 + c)(1/2 - a) - E^{1/2}(k^2 + l^2)}, \quad (3.10)$$

such a topographically resonant wave is barotropic.

It is here noted that if there acts thermal forcing alone, i.e., $h_0 = 0$ and $J_0 \neq 0$, the wave amplitude remains finite even under the condition $U_0 = U_r$, since at $a = 1/2$ the free mode is isothermal everywhere, and hence the thermal contrast can not resonate the wave; although the denominator of the second term in A due to the thermal forcing becomes zero at $a = 1/2 + 1/d$, this term is cancelled out by the particular solution B . Thus the resonance

of external mode due to the thermal forcing is impossible⁴⁾.

As will be shown later, our non-linear dis-

cussion takes account of 'weak' dissipation near resonant condition. With the assumption that $E^{1/2}$, c and $U_0 - U_r$ are small a and A are approximated as

$$a \sim \frac{1}{2} + \frac{Br}{\sin^2(\varphi_0)} \frac{\beta}{U_r} \left[\frac{c}{ikU_r} + \frac{(U_0 - U_r)}{U_r} \right], \quad (3.12)$$

$$A \sim \frac{ih_0 U_0}{i\beta(U_0 - U_r) \frac{Br}{U_0 \sin^2(\varphi_0)} + \left(c \frac{\beta Br}{kU_r \sin^2(\varphi_0)} + \frac{E^{1/2}(k^2 + l^2)}{k} \right)} \quad (3.13)$$

4. Non-linear behaviors of resonant waves

4.1 The assumptions of scaling

As has been seen in the last section, when the basic zonal wind U_0 is equal to the speed of two dimensional free Rossby wave U_r , the resonance occurs, and as is well known, the resonant wave grows linearly in time (cf. Tung and Lindzen, 1979a). As the result, the wave-mountain interaction and/or wave-self interaction would be so effective that it might change the basic zonal flow from the initial magnitude U_r . Accordingly, the resonant condition can break down. Furthermore, dissipative mechanisms would also tend to suppress the growth of the wave. Thus, we would have some equilibrium state in which the growth due to the resonant forcing and the suppressions due to the break-down of the resonant condition by the nonlinear interaction and due to the dissipation are balanced (cf. CD).

In what follows, we shall discuss the above mentioned situation by using the perturbation expansion method which is well-known, in the discussion of, for example, non-linear pendulum, non-linear spring and so on (cf. Nayfe & Mook, 1979).

For this purpose, we assume that the forcing, the damping and the non-linear effect appear simultaneously in the perturbation scheme. Thus, using a small non-dimensional parameter ε , we assume that

$$h_0 = \tilde{h}_0 \varepsilon^3, \quad J_0 = \tilde{J}_0 \varepsilon^3, \quad E^{1/2} = \tilde{E}^{1/2} \varepsilon^2$$

$$c = \tilde{c} \varepsilon^2, \quad \delta (\equiv U_0 - U_r) = \tilde{\delta} \varepsilon^2, \quad (4.1)$$

where $\tilde{h}_0$, $\tilde{J}_0$, $\tilde{E}^{1/2}$, $\tilde{c}$ and $\tilde{\delta}$ are of order 1. It is noted that δ is a measure of the magnitude of the deviation from the resonant condition of the mean zonal flow.

The above scaling assumption is based on the following inspection. Let the forcing terms h_0 and J_0 be of order ε^l (hereafter written as $h_0, J_0 \sim O(\varepsilon^l)$), the dissipation terms $E^{1/2}$ and c be $O(\varepsilon^m)$ and the deviation δ of mean zonal flow from U_r be $O(\varepsilon^m)$. At the linear resonance condition $U_0 = U_r$, it is seen from eq. (3.10) that the order of wave amplitude at large time limit is given by the balance between the forcing and the dissipation terms, i.e., $|\varphi| \sim O(\text{forcings})/(\text{dissipation}) = O(\varepsilon^{l-m})$. As a result of the interaction of the wave and the topography, the mean zonal flow $\bar{u}$ could be induced; the magnitude $|\bar{u}|$ can be estimated from the balance between the forcing $O(h_0 \varphi) = O(\varepsilon^{2l-m})$ and the dissipation $O(E^{1/2})|\bar{u}| = O(\varepsilon^m)|\bar{u}|$, to give that $|\bar{u}| \sim O(\varepsilon^{2l-2m})$. Since $|\bar{u}|$ gives the deviation of U_0 from U_r , we have that $|\bar{u}| \sim \delta$, i.e., $O(\varepsilon^{2l-2m}) = O(\varepsilon^m)$. Then, it follows that $l = 3n$ and $m = 2n$ with arbitrary positive integer n . Without loss of generality, we can put $n = 1$, because the case with $n \neq 1$ can be reduced to the case $n = 1$ by transforming ε^n into, say, $\hat{\varepsilon}$. Thus, the above scaling assumption can be obtained. In this perturbation scheme, we can deal with all the effects of forcing, damping and non-linearity simultaneously. It should be noted here that, although $|\bar{u}| \sim O(\varepsilon^{2l-2m})$ would seem to be a direct result of wave-self interaction ($\bar{u} \sim O(|\varphi|^2) = O(\varepsilon^{2l-2m})$), this is not the case in the present situation as will be shown in subsection 4.2.

The basic equation (2.5) can be rewritten

4) This is completely different from the case of internal wave, which can be resonant to thermal forcing alone if $\sin(aZ_T) = 0$, where Z_T is an altitude of the upper lid (Wakata and Uryu, 1984b).

as follows

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + (U_r + u' + \varepsilon^2 \bar{\partial}) \frac{\partial}{\partial x} + v' \frac{\partial}{\partial y} + \varepsilon^2 c \right] \\ & \cdot \left(\nabla_h^2 + \frac{\sin^2(\varphi_0)}{Br} \left(\frac{\partial^2}{\partial z^2} - \frac{1}{4} \right) \right) \Phi + \beta \frac{\partial \Phi}{\partial x} \\ & = -\varepsilon^3 \frac{\sin^2(\varphi_0)}{Br} e^{-(1/2+1/d)z} \\ & \cdot (1+1/d) J_0 \sin(ly) e^{i(kx+\alpha)}, \quad (4.2) \end{aligned}$$

with the boundary condition

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + (U_r + u' + \varepsilon^2 \bar{\partial}) \frac{\partial}{\partial x} + v' \frac{\partial}{\partial y} + \varepsilon^2 c \right] \\ & \cdot \left(\frac{\partial}{\partial z} + \frac{1}{2} \right) \Phi - S \frac{\partial \Phi}{\partial t} \\ & = \varepsilon^3 J_0 \sin(ly) e^{i(kx+\alpha)} - (U_r + u' + \varepsilon^2 \bar{\partial}) \frac{\partial h}{\partial x} \\ & - v' \frac{\partial h}{\partial y} - \varepsilon^2 E^{1/2} \nabla_h^2 \Phi \quad \text{at } z=0. \quad (4.3) \end{aligned}$$

where the tildes for $\bar{\partial}$, c and J_0 are omitted for simplicity.

In the following, we shall use the method of multiple time-scale:

$$T^{(n)} = \varepsilon^n t, \quad n=0, 1, 2, \dots \quad (4.4)$$

Then, $\partial/\partial t$ can be expanded into the series of $\partial/\partial T^{(n)}$ as follows:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T^{(0)}} + \varepsilon \frac{\partial}{\partial T^{(1)}} + \varepsilon^2 \frac{\partial}{\partial T^{(2)}} + \dots \quad (4.5)$$

Geopotential Φ is also expanded into Fourier series and each amplitude is expanded by ε as follows:

$$\begin{aligned} \Phi &= \sum_{n=-\infty}^{\infty} \varphi_n(y, z, T^{(1)}, T^{(2)}, \dots) e^{in k x}, \\ \varphi_n &= \varepsilon^{1+|n-1|} \sum_{j=0}^{\infty} \varepsilon^j \varphi_{n,j}(y, z, T^{(1)}, T^{(2)}, \dots). \end{aligned} \quad (4.6)$$

Then, substitution of (4.5) and (4.6) into eq. (4.2) and the boundary condition (4.3) gives equations for $\varphi_{n,j}$ and we solve the equations under the corresponding boundary conditions. Here, it is noted that since we treat quasi-resonant wave, i.e., the wave does not propagate horizontally with large phase velocity, the solution is assumed reasonably

to be independent of $T^{(0)}$.

4.2 The set of time evolution equations

The complete set of equations of amplitude and phase of the wave and the zonal flow will be obtained as follows. Substituting (4.5) and (4.6) into eq. (4.2), we obtain to 1st order in ε that

$$M\varphi_{10}=0, \quad (4.7)$$

where M is a differential operator defined by

$$M \equiv U_r \left[\frac{\partial^2}{\partial y^2} - k^2 + \frac{\sin^2(\varphi_0)}{Br} \left(\frac{\partial^2}{\partial z^2} - \frac{1}{4} \right) \right] + \beta. \quad (4.8)$$

The boundary conditions become, to 1st orders in ε , as

$$\begin{cases} L\varphi_{10}=0 & \text{at } z=0, \\ \varphi_{10} \rightarrow 0 & \text{at } z \rightarrow \infty, \end{cases} \quad (4.9)$$

where L is a differential operator defined by

$$L \equiv U_r \left(\frac{\partial}{\partial z} + \frac{1}{2} \right). \quad (4.10)$$

Then, the solution to (4.7) can be written as follows

$$\begin{aligned} \varphi_{10} &= R(T^{(1)}, T^{(2)}, \dots) \\ & \cdot \exp(i\Theta(T^{(1)}, T^{(2)}, \dots)) F(z) \sin(ly), \end{aligned} \quad (4.11)$$

where $F(z) = e^{-z/2}$, which shows the same vertical structure as that of the linear resonant solution with $a=1/2$. The amplitude and phase of this wave is determined by higher order discussion. At this stage, we can merely know the spatial form of the wave; its amplitude does not depend on $T^{(0)}$, since the basic zonal flow is near the resonant speed; to this order, the wave is stationary with constant amplitude. The resonant growth occurs with longer time scale.

To 2nd order, it follows that

$$M\varphi_{11} = \frac{\beta}{ikU_r} \frac{\partial \varphi_{10}}{\partial T^{(1)}}, \quad (4.12)$$

and the boundary conditions are written as

$$\begin{cases} L\varphi_{11} = \frac{S}{ik} \frac{\partial \varphi_{10}}{\partial T^{(1)}} & \text{at } z=0, \\ \varphi_{11} \rightarrow 0 & \text{as } z \rightarrow \infty. \end{cases} \quad (4.13)$$

In order for the secular term to vanish (i.e., solvability condition), it must hold that

$$\frac{\partial \varphi_{10}}{\partial T^{(1)}} = 0. \quad (4.14)$$

Therefore, φ_{10} is independent of $T^{(1)}$. In order to see the time dependence of the wave, this time scale is not sufficient; further slower time is necessary. As is seen from (4.11) and (4.12), φ_{11} is of the same structure as φ_{10} ;

$$\begin{aligned} \varphi_{11} = & \hat{R}(T^{(1)}, T^{(2)}, \dots) \\ & \cdot \exp(i\hat{\Theta}(T^{(1)}, T^{(2)}, \dots)) e^{-z/2} \sin(\ell y). \end{aligned} \quad (4.15)$$

The time-behaviors of φ_{11} and φ_{10} are determined by 3rd order equations, which are as follows:

$$\frac{\partial}{\partial T^{(1)}} \left[\frac{\partial^2}{\partial y^2} + \frac{\sin^2(\varphi_0)}{Br} \left(\frac{\partial^2}{\partial z^2} - \frac{1}{4} \right) \right] \varphi_{00} = 0, \quad (4.16.1)$$

$$\begin{aligned} M\varphi_{12} = & (\delta - \varphi_{00,y} e^{z/2}) \frac{\beta}{U_r} \varphi_{10} \\ & + \frac{\beta}{ikU_r} \left(\frac{\partial}{\partial T^{(2)}} + c \right) \varphi_{10} \\ & - e^{z/2} \varphi_{10} \left[\frac{\partial^2}{\partial y^2} + \frac{\sin^2(\varphi_0)}{Br} \left(\frac{\partial^2}{\partial z^2} - \frac{1}{4} \right) \right] \varphi_{00,y} \\ & - \frac{\sin^2(\varphi_0)}{ikBr} (1 + 1/d) J_0 e^{i\alpha - (1/2 + 1/d)z} \sin(\ell y) \\ & + \frac{\beta}{ikU_r} \frac{\partial}{\partial T^{(1)}} \varphi_{11}. \end{aligned} \quad (4.16.2)$$

The corresponding boundary conditions are written as

$$\begin{cases} \frac{\partial}{\partial T^{(1)}} \left(\frac{\partial}{\partial z} + \frac{1}{2} \right) \varphi_{00} - S \frac{\partial}{\partial T^{(1)}} \varphi_{00} = 0 \\ \qquad \qquad \qquad \text{at } z=0, \\ \varphi_{00} \rightarrow 0 \qquad \qquad \qquad \text{as } z \rightarrow \infty, \end{cases} \quad (4.17.1)$$

$$\begin{cases} L\varphi_{12} = \frac{S}{ik} \frac{\partial}{\partial T^{(2)}} \varphi_{10} - \varphi_{10} \left(\frac{\partial}{\partial z} + \frac{1}{2} \right) \varphi_{00,y} \\ \qquad + \frac{J_0 e^{i\alpha}}{ik} \sin(\ell y) - U_r h_0 \sin(\ell y) \\ \qquad + \frac{E^{1/2}}{ik} (l^2 + k^2) \varphi_{10} + \frac{S}{ik} \frac{\partial \varphi_{11}}{\partial T^{(1)}} \\ \qquad \qquad \qquad \text{at } z=0, \\ \varphi_{12} \rightarrow 0 \qquad \qquad \qquad \text{as } z \rightarrow \infty. \end{cases} \quad (4.17.2)$$

From (4.16.1) and (4.17.1), it follows that φ_{00} is independent of $T^{(1)}$. This, in turn, together with $\partial \varphi_{10} / \partial T^{(1)} = 0$, gives that φ_{11} is independent of $T^{(1)}$. Thus, φ_{11} can be involved in φ_{10} or can be set to zero. Therefore, we know that the slow time variable to see the time change of the wave is $T^{(2)}$. Thus we replace $T^{(2)}$ by T , since no confusion occurs.

In order to eliminate the secular terms in the particular solution of (4.16.2) under (4.17.2), the following solvability condition should be satisfied;

$$\begin{aligned} & \iint \varphi_{10} M \varphi_{12} dy dz - \iint \varphi_{12} M \varphi_{10} dy dz \\ & = \frac{\sin^2(\varphi_0)}{Br} \int [\varphi_{10} L \varphi_{12}]_{z=0}^{\infty} dy. \end{aligned} \quad (4.18)$$

Here, for the convenience of later discussion, we separate φ_{12} into two parts;

$$\varphi_{12} = (\varphi_{12}^{(1)} + i\varphi_{12}^{(2)}) e^{i\theta}, \quad (4.19)$$

where the first term is in phase with φ_{10} , while the second is of $\pi/2$ out of phase.

Substituting (4.11) and (4.19) into (4.16.2) and (4.17.2), and eliminating φ_{11} , we obtain the following set of equations and boundary conditions for $\varphi_{12}^{(1)}$ and $\varphi_{12}^{(2)}$.

$$\begin{cases} M\varphi_{12}^{(1)} = (\delta - \varphi_{00,y} e^{z/2}) \frac{\beta}{U_r} R e^{-z/2} \sin(\ell y) \\ \qquad + \frac{\beta}{kU_r} R \frac{\partial \Theta}{\partial T} e^{-z/2} \sin(\ell y) - R \sin(\ell y) \\ \qquad \cdot \left[\frac{\partial^2}{\partial y^2} + \frac{\sin^2(\varphi_0)}{Br} \left(\frac{\partial^2}{\partial z^2} - \frac{1}{4} \right) \right] \varphi_{00,y} \\ \qquad - \frac{\sin^2(\varphi_0)}{kBr} (1 + 1/d) J_0 \sin(\alpha - \Theta) \\ \qquad \cdot e^{-(1/2 + 1/d)z} \sin(\ell y), \end{cases} \quad (4.20.1)$$

$$\begin{cases} L\varphi_{12}^{(1)} = \frac{S}{k} \frac{\partial \Theta}{\partial T} R \sin(\ell y) - R \sin(\ell y) \\ \qquad \left(\frac{\partial}{\partial z} + \frac{1}{2} \right) \varphi_{00,y} + \frac{J_0}{k} \sin(\alpha - \Theta) \sin(\ell y) \\ \qquad - U_r h_0 \sin(\ell y) \cos(\Theta) \text{ at } z=0, \\ \varphi_{12}^{(1)} \rightarrow 0 \qquad \qquad \qquad \text{as } z \rightarrow \infty, \end{cases} \quad (4.20.3)$$

$$\begin{cases} M\varphi_{12}^{(2)} = -\frac{\beta}{kU_r} \left(\frac{\partial}{\partial T} + c \right) R e^{-z/2} \sin(\ell y) \\ \qquad + \frac{\sin^2(\varphi_0)}{kBr} (1 + 1/d) J_0 \cos(\alpha - \Theta) \end{cases}$$

$$\left\{ \begin{aligned} & \cdot e^{-(1/2+1/d)z} \sin(l y), \quad (4.21.1) \\ & L\varphi_{12}^{(2)} = -\frac{S}{k} \frac{\partial R}{\partial T} \sin(l y) - \frac{J_0}{k} \cos(\alpha - \Theta) \sin(l y) \\ & \quad + U_r h_0 \sin(\Theta) \sin(l y) \\ & \quad - E^{1/2} \frac{k^2 + l^2}{k} R \sin(l y) \quad \text{at } z=0, \quad (4.21.2) \\ & \varphi_{12}^{(2)} \rightarrow 0 \quad \text{as } z \rightarrow \infty. \quad (4.21.3) \end{aligned} \right.$$

From the solvability condition for $\varphi_{12}^{(2)}$, the time evolution equation for amplitude R can be obtained as,

$$\begin{aligned} & \left(\frac{S}{k} + \frac{\beta Br}{k U_r \sin^2(\varphi_0)} \right) \frac{\partial R}{\partial T} \\ & = - \left(c \frac{\beta Br}{k U_r \sin^2(\varphi_0)} + E^{1/2} \frac{k^2 + l^2}{k} \right) R \\ & \quad + U_r h_0 \sin(\Theta). \quad (4.22) \end{aligned}$$

Phase Θ is determined by the solvability condition for $\varphi_{12}^{(1)}$. Since eq. (4.20) for $\varphi_{12}^{(1)}$ includes the induced zonal mean field φ_{00} , we must know the induced zonal flow before discussing the evolution of phase.

The zonal mean field φ_{00} is determined at 4th order as follows,

$$\begin{aligned} & \left(\frac{\partial}{\partial T} + c \right) \left[\frac{\partial^2}{\partial y^2} + \frac{\sin^2(\varphi_0)}{Br} \left(\frac{\partial^2}{\partial z^2} - \frac{1}{4} \right) \right] \varphi_{00} \\ & = -2k \left(\varphi_{10,y} + \varphi_{10} \frac{\partial}{\partial y} \right) \left[-l^2 - k^2 \right. \\ & \quad \left. + \frac{\sin^2(\varphi_0)}{Br} \left(\frac{\partial^2}{\partial z^2} - \frac{1}{4} \right) \right] \varphi_{12}^{(2)} e^{-i\theta} e^{z/2} \\ & \quad - \frac{2\beta k}{U_r} \left(\varphi_{12}^{(2)},_y + \varphi_{12}^{(2)} \frac{\partial}{\partial y} \right) \varphi_{10} e^{-i\theta} e^{z/2}, \quad (4.23.1) \\ & \left(\frac{\partial}{\partial T} + c \right) \left[\frac{\partial}{\partial z} + \frac{1}{2} \right] \varphi_{00} - S \frac{\partial \varphi_{00}}{\partial T} + E^{1/2} \frac{\partial^2}{\partial y^2} \varphi_{00} \\ & = -2k \left[\varphi_{10,y} \left(\frac{\partial}{\partial z} + \frac{1}{2} \right) \varphi_{12}^{(2)} e^{-i\theta} \right. \\ & \quad \left. + \varphi_{10} \left(\frac{\partial}{\partial z} + \frac{1}{2} \right) \varphi_{12}^{(2)},_y e^{-i\theta} \right] \\ & \quad - Re[2ik(\varphi_{10,y} h_0 \sin(l y) + \varphi_{10} h_0 l \cos(l y))] \\ & \quad \text{at } z=0, \quad (4.23.2) \\ & \varphi_{00} \rightarrow 0 \quad \text{as } z \rightarrow \infty. \quad (4.23.3) \end{aligned}$$

It is seen from this set of equations that the forcing to φ_{00} includes the nonlinear coupling between φ_{10} and $\varphi_{12}^{(2)}$, which shows the wave-self interaction. It is noted that the self coupling of φ_{10} vanishes. In this sense, what have already been mentioned in the scaling analysis (the beginning part of this section) is verified. The final term of (4.23.2) is attributed to the wave-mountain interaction. It is noted that this effect is included in *CD* model, but the wave-self interaction is excluded. In order to solve eq. (4.23), we must know $\varphi_{12}^{(2)}$ in advance. This is easily done, because eq. (4.23) does not include $\varphi_{12}^{(1)}$. The reason why we have divided φ_{12} into $\varphi_{12}^{(1)}$ and $\varphi_{12}^{(2)}$ lies in this point. Substituting eq. (4.22) into eq. (4.21), equation for $\varphi_{12}^{(2)}$ can be obtained, which can be solved as follows:

$$\begin{aligned} \varphi_{12}^{(2)} &= \frac{Br}{\sin^2(\varphi_0)} - \frac{\beta}{k U_r^2} \left(\frac{\partial}{\partial T} + c \right) R z e^{-z/2} \sin(l y) \\ & \quad + \frac{d}{k U_r} J_0 \cos(\alpha - \Theta) e^{-(1/2+1/d)z} \sin(l y). \quad (4.24) \end{aligned}$$

Substituting (4.24) into eq. (4.23), and considering the cyclic lateral condition, we can write φ_{00} as follows,

$$\begin{aligned} \varphi_{00} &= [D_1(T) e^{-z/2} + J_0 D_2(T) e^{-(1/2+1/d)z} \\ & \quad + D_3(T) \exp(-\sqrt{1/4 + Br l^2 / \sin^2(\varphi_0)} z)] \\ & \quad \cdot \sin(2ly), \quad (4.25) \end{aligned}$$

where D_1 and D_2 are determined by direct substitution of (4.25) into (4.23.1), and D_3 by the boundary condition (4.23.2).

Returning to eq. (4.20), and substituting (4.25), we can obtain equation for Θ from the solvability condition for $\varphi_{12}^{(1)}$ as follows,

$$\begin{aligned} & \left(\frac{S}{k} + \frac{Br}{\sin^2(\varphi_0)} \frac{\beta}{k U_r} \right) \frac{\partial \Theta}{\partial T} \\ & = - \frac{Br}{U_r \sin^2(\varphi_0)} \beta \delta + U_r h_0 \frac{\cos(\Theta)}{R} \\ & \quad + \frac{Br}{\sin^2(\varphi_0)} \left(4l^2 - \frac{\beta l}{U_r} \right) \left[D_1 + \frac{J_0 D_2}{1+1/d} \right. \\ & \quad \left. + D_3 / (1/2 + \sqrt{1/4 + 4Br l^2 / \sin^2(\varphi_0)}) \right]. \quad (4.26) \end{aligned}$$

Thus, summarizing the results above, we

have the following set of equations to determine the amplitude and phase of a resonantly growing wave.

$$\frac{\partial R}{\partial T} + P_1 R = P_2 h_0 \sin(\Theta), \quad (4.27.1)$$

$$\begin{aligned} \frac{\partial \Theta}{\partial T} = & -P_3 \delta + P_4 h_0 \cos(\Theta)/R \\ & + P_5 D_1 + J_0 P_6 D_2 + P_7 D_3, \end{aligned} \quad (4.27.2)$$

$$\left(\frac{\partial}{\partial T} + c\right) D_1 = P_8 R \left(\frac{\partial}{\partial T} + c\right) R, \quad (4.27.3)$$

$$\left(\frac{\partial}{\partial T} + c\right) D_2 = P_3 R \cos(\alpha - \Theta), \quad (4.27.4)$$

$$\begin{aligned} \frac{\partial D_3}{\partial T} + P_{10} D_3 = & P_{11} D_1 + P_{12} J_0 D_2 \\ & + P_{13} R \left(\frac{\partial}{\partial T} + c\right) R + P_{14} J_0 R \cos(\alpha - \Theta) \\ & + P_{15} h_0 R \sin(\Theta), \end{aligned} \quad (4.27.5)$$

where the coefficients are listed in Appendix B.

4.3 Various effects included in the set of equations (4.27)

The set of equations above include many physical processes, and then each effect can be known qualitatively by picking up separately.

(a) Effect of δ (the deviation of zonal flow from U_r)

By setting $h_0, J_0, E^{1/2}, c$ and nonlinear term D_n ($n=1, 2, 3$) to zero in eq. (4.27), it follows that

$$\frac{\partial R}{\partial T} = 0, \quad (4.28.1)$$

$$\frac{\partial \Theta}{\partial T} = -P_3 \delta. \quad (4.28.2)$$

The solution to this set of equations is given by $\varphi_{10} = R_0 \exp(-iP_3 \delta T)$; this is a free wave with phase velocity $P_3 \delta / k$ and with constant amplitude.

(b) Dissipation effect

If we retain the damping effects ($E^{1/2} \neq 0, c \neq 0$) in Case (a), eq. (4.28) is replaced by

$$\frac{\partial R}{\partial T} = -P_1 R, \quad (4.29.1)$$

$$\frac{\partial \Theta}{\partial T} = -P_3 \delta. \quad (4.29.2)$$

The wave propagates with phase velocity $P_3 \delta / k$ as in Case (a), but its amplitude decreases as $\exp(-P_1 T)$. As is seen from Appendix B (see P_1), this decay comes from the Ekman damping, Rayleigh friction and Newtonian cooling.

(c) Forcing effect

It is found from eq. (4.22) that the growth of the wave amplitude is only due to the topographic forcing. The thermal forcing does not appear in eq. (4.22). This result agrees with the linear theory discussed in section 3. Here, we shall consider the process of linear resonance.

By including the forcing effects ($h_0 \neq 0$) in Case (b), eq. (4.29) is replaced by

$$\frac{\partial R}{\partial T} + P_1 R = P_2 h_0 \sin(\Theta), \quad (4.30.1)$$

$$\frac{\partial \Theta}{\partial T} = -P_3 \delta + P_4 h_0 \cos(\Theta)/R. \quad (4.30.2)$$

These equations can be reduced to

$$\frac{\partial A}{\partial T} + (P_1 + iP_3 \delta) A = ih_0 P_2, \quad (4.31)$$

where $A = R e^{i\Theta}$.

When the linear resonant condition is perfectly satisfied, i.e. $\delta = 0$, and without dissipation $P_1 = 0$, A is given by

$$A = ih_0 P_2 T. \quad (4.32)$$

The amplitude grows in proportion to time, which is a general property of linear resonance (cf. Tung and Lindzen, 1979).

If a small departure from the resonant condition ($\delta \neq 0$) and/or the damping ($P_1 \neq 0$) are included, A is written as

$$A = \frac{ih_0 P_2}{P_1 + iP_3 \delta} + C_0 e^{-(P_1 + iP_3 \delta)T}, \quad (4.33)$$

where C_0 is determined by the initial condition. Then, as $T \rightarrow \infty$, A tends to the forced solution which is the same as (3.13) based on the linear theory;

$$\begin{aligned} A_s = & iU_r h_0 \left/ \left[i\beta \delta \frac{Br}{U_r \sin^2(\varphi_0)} \right. \right. \\ & \left. \left. + \left(c \frac{\beta Br}{k U_r \sin^2(\varphi_0)} + E^{1/2} \frac{k^2 + l^2}{k} \right) \right] \right., \end{aligned} \quad (4.34)$$

where P_1, P_2 and P_3 have been substituted.

(d) Non-linear effect

The non-linear effect appears through the induced mean zonal flow. As the wave grows by the resonance, the zonal flow is induced. Then, the resonant condition $U_0=U_r$ is broken down. This means that as is anticipated by (4.33) the maximum response of wave is attained not at $\delta=0$ but at some small, non-zero value of δ .

As is seen from eqs. (4.27.3) to (4.27.5), the zonal mean field can be induced by two effects, i.e., wave-self interaction and wave-forcing term (topography and/or thermal forcing) interaction. If the wave is forced by mountain with no dissipation, no mean zonal flows are induced (cf. Charney and Drazin, 1961). This result can be readily obtained from eqs. (4.27.3) to (4.27.5); we have first from (4.27.1) that $\sin(\Theta)=0$, and hence

$$\frac{\partial D_1}{\partial T} = P_8 R \frac{\partial R}{\partial T} = 0, \quad (4.35.1)$$

$$\frac{\partial D_3}{\partial T} = P_{13} R \frac{\partial R}{\partial T} = 0. \quad (4.35.2)$$

Thus, it follows that the time change of φ_{00} is zero if $\partial R/\partial T=0$.

Further, it is noted that dissipation can cause the zonal mean field through the changes in phase and amplitude of wave (cf. Dickinson, 1969; Andrews and McIntyre, 1976). In order to see the effects of non-linear coupling of wave and external forcings, we should fully solve the set of eq. (4.27).

5. Equilibrium solutions and their stabilities

The set of equations (4.27) has some equilibrium solutions, which can be obtained by setting $\partial/\partial T=0$. Such solutions are determined by the balance of many terms whose roles have been separately discussed in the pre-

vious section. In this section, we shall discuss the equilibrium states and their stabilities.

5.1 Method

Setting $\partial/\partial T=0$ in eqs. (4.27) and eliminating Θ , D_1 , D_2 and D_3 , we can obtain the following equation for R ;

$$A_1 X^2 + A_2 X^2 + A_3 X + A_4 = 0, \quad (5.1)$$

where $X=R^2$, and coefficients A_1 to A_4 are summarized in Appendix B. Since eq. (5.1) is of third order, then it has, generally, three solutions. We can examine the stabilities of these three solutions as follows (e.g., Sattinger, 1973).

The solution is divided into the equilibrium solution and the time dependent perturbation as follows;

$$\begin{pmatrix} R \\ \Theta \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} R_s \\ \Theta_s \\ D_{1s} \\ D_{2s} \\ D_{3s} \end{pmatrix} + \begin{pmatrix} R'(T) \\ \Theta'(T) \\ D'_1(T) \\ D'_2(T) \\ D'_3(T) \end{pmatrix}, \quad (5.2)$$

where suffix s indicates the equilibrium. Near the equilibrium, the primed quantities are small, and hence we can linearize the set of equations as follows,

$$\frac{d}{dT} \begin{pmatrix} R' \\ \Theta' \\ D'_1 \\ D'_2 \\ D'_3 \end{pmatrix} = A(R_s, \Theta_s, D_{1s}, D_{2s}, D_{3s}) \begin{pmatrix} R' \\ \Theta' \\ D'_1 \\ D'_2 \\ D'_3 \end{pmatrix} \quad (5.3)$$

where A is the so-called Jacobi matrix defined by

$$\begin{pmatrix} -P_1 & P_2 h_0 \cos(\Theta_s) & 0 & 0 & 0 \\ -P_4 h_0 \cos(\Theta_s) & -P_4 h_0 \sin(\Theta_s) & P_5 & J_0 P_6 & P_7 \\ R_s^2 & R_s & & & \\ P_8(2cR_s - 2P_1 R_s + P_2 h_0 \sin(\Theta_s)) & P_2 P_8 h_0 R_s \cos(\Theta_s) & -c & 0 & 0 \\ P_9 \cos(\alpha - \Theta_s) & P_9 R_s \sin(\alpha - \Theta_s) & 0 & -c & 0 \\ P_{13}(2cR_s - 2P_1 R_s + P_2 h_0 \sin(\Theta_s)) & P_2 P_{13} h_0 R_s \cos(\Theta_s) & & & \\ + P_{14} J_0 \cos(\alpha - \Theta_s) & + P_{14} J_0 R_s \sin(\alpha - \Theta_s) & P_{11} & J_0 P_{12} & -P_{10} \\ + P_{15} h_0 \sin(\Theta_s) & + P_{15} h_0 R_s \cos(\Theta_s) & & & \end{pmatrix} \quad (5.4)$$

and it is noted that A includes the equilibrium solutions alone as given coefficients. Therefore, if we assume that the perturbations are of $e^{\sigma t}$ (σ is complex in general; $\sigma = \sigma_r + i\sigma_i$), then σ is the eigenvalue of matrix A , and we can examine the stability of the system near equilibrium solution; for positive σ_r the wave is unstable, while it is stable for σ_r negative.

5.2 Results

We shall show some numerical results. In the calculations, we use U_r^* (dimensional linear resonant speed of zonal flow, for example, $U_r^* = 14.9$ m/sec for $k=4$ ($\lambda^* = 7850$ km at $\varphi = 45^\circ$)) as the scaling factor, and hence its non-dimensional counterpart U_r is equal to 1. The numerical values of various parameters are presented in Appendix A.

(a) Dependence of the response on damping ($E^{1/2}$ and c) and topographic forcing (h)

We set $k=4$ (cf. Murakami, 1967) and $J_0=0$. In Fig. 1(a, b), shown are the dependences of the response curves on $E^{1/2}$ with $h_0=5$ and $c=1$ (c is fixed at this value throughout this work). The abscissa is the detuning parameter, $\delta = (U - U_r)/\varepsilon^2$, while the ordinates are the wave amplitudes (Fig. 1(a)) and the phases

Fig. 1(a) Amplitude responses of wave $k=4$ for topographic forcing with $h_0=5$ alone ($J_0=0$, $c=1$); symbols *, Δ and $\square$ correspond to equilibrium solutions for $E^{1/2}=0.5$, 1.0 and 2.0 respectively. The abscissa denotes the detuning parameter $\delta = (U - U_r)/\varepsilon^2$, i.e., the difference of the basic zonal flow from the linear resonant condition, while the ordinate is the amplitude of the wave. It is seen that there exist three equilibrium states, for example, in case of $E^{1/2}=0.5$ ($-6 < \delta < -12$).

Fig. 1(b) Phase- δ relation, corresponding to Fig. 1(a). The ordinate is the phase of the wave.

(Fig. 1(b)), respectively. It is seen that as $E^{1/2}$ decreases, the maximum amplitude becomes larger and its position shifts toward δ negative. This inclination of the amplitude response curve is characteristic of non-linear resonance problem (cf. Nayfeh and Mook, 1979). However, as will be shown later, it depends on k whether the amplitude response curve inclines toward δ negative or positive. It is further noted that the maximum amplitude is attained at $\Theta = \pi/2$, as anticipated from the linear theory.

It is emphasized that for small $E^{1/2}$ (e.g. $E^{1/2}=0.5$) there are three equilibrium solutions in a certain range of δ (e.g. $-12 \leq \delta \leq -6$), similarly to the results by CD and others. The stability calculations mentioned above give that the states on the upper and the lower branches are stable, while those on the middle branch are unstable. These stability properties are commonly seen in non-linear resonance problems (cf. Nayfeh and Mook, 1979). In Fig. 2, we show the induced mean zonal flow at the center of the channel (cyclic), corresponding to three equilibrium states at $\delta = -8$. As the wave amplitude decreases, the induced mean zonal flow speed becomes small. Reminding that the mean zonal flow in the presence of wave is the superposition of the basic zonal flow and the induced mean zonal flow, it follows that large amplitude waves in a weak westerly flow and small amplitude in a strong westerly flow are stable.

Fig. 2 The mean zonal flows induced by the waves for $E^{1/2}=0.5$ and $\delta=-8$ shown in Fig. 1(a); (a) is for large amplitude wave, (b) for moderate amplitude wave and (c) for small amplitude wave.

As mentioned by CD , the former could correspond to the so-called low-index flow, while the latter to the high-index flow. It is, however, emphasized that, as will be shown later, such identifications could be somewhat misleading, particularly if we would include thermal forcings.

Next, we shall examine how the equilibrium solutions depend on the magnitude of topographic forcing, h_0 . We set $E^{1/2}=0.5$ and $c=1$ (also $J_0=0$). In Fig. 3 shown are the dependences of amplitude on h_0 for several different δ 's, while Fig. 3(a) is the projection of the solution surface onto $\delta-h_0$ plane. It is seen that the existence of multiple equilibria depends not only on δ but also h_0 , and it can be confirmed again that the middle branch (Fig. 3(b)) is unstable. Further, it is seen from Figs. 1(a) and 3(a) that (i) the maximum wave amplitude is proportional to h_0 and to $(E^{1/2})^{-1}$ and (ii) its position on δ axis shifts in proportion to $(E^{1/2})^{-2}$ and to h_0^2 . These characteristics can be justified by eq. (4.27) as follows. Since, in equilibrium, the amplitude becomes maximum at $\Theta=\pi/2$ for some δ (e.g., $\delta=-12$ for $E^{1/2}=0.5$), we fix Θ at $\pi/2$. Then, we can readily see that the resulting set of equilibrium equations is invariant with respect to the following transformation;

$$(h_0, J_0, E^{1/2}, c, \delta, R, D_1, D_2, D_3)$$

$$\rightarrow (a\hat{h}_0, a\hat{J}_0, b^{-1}\hat{E}^{1/2}, b^{-1}\hat{c}, a^2b^2\hat{\delta}, ab\hat{R}, a^2b^2\hat{D}_1, ab^2\hat{D}_2, a^2b^2\hat{D}_3) \quad (5.5)$$

where a and b are arbitrary constants, and also we have used that P_1, P_{10}, P_{11} , and P_{12} are transformed into $b^{-1}P_1, b^{-1}P_{10}, b^{-1}P_{11}$ and $b^{-1}P_{12}$, respectively. Thus, if h_0 and $E^{1/2}$ are multiplied by a and b^{-1} , respectively, then the amplitude R is multiplied by ab , and the position of the maximum amplitude on δ axis shifts as a^2b^2 .

Fig. 3(a) The projection of the solution surface for $k=4$ and $E^{1/2}=0.5$ onto $(h_0-\delta)$ plane, when $J_0=0$ and $c=1$. Broken line indicates the positions of maximum amplitude for each δ .

Fig. 3(b) The cross sections of the solution surface at some constant δ 's.

(b) Dependence of the response curve on wave number

As has been shown in Fig. 1(a), the amplitude response curve for $k=4$ inclines toward δ negative. This means that the maximum response of $k=4$ occurs in a westerly flow weaker than that in the case of linear resonance ($\delta=0$). However, this result changes with k ; if k exceeds some critical value k_c ($=5.44$ in the present situation), the response curve inclines toward δ positive, as is shown in Fig. 5(a). Correspondingly, the phase response also changes with k (see Fig. 5(b)). It is noted that at $k=k_c$ we have only single and stable equilibrium state for every δ . The dependence of the response curve on k can be attributed to non-linear feed-back effects of the induced mean zonal flow as follows (see Appendix C for more detailed discussions).

The feed-back effects of the induced mean zonal flow on the wave appear as the sum of two terms in the potential vorticity equation; one is the zonal advection of wave potential vorticity $\bar{u}\partial/\partial x \nabla^2 \varphi'$, the other is the meridional advection of zonal mean potential vorticity due to the wave $v'\partial/\partial y \nabla^2 \bar{\varphi}$. The magnitude of these terms depend on k . Neglecting y dependences of the solution, we can obtain that $\bar{u}\partial/\partial x \nabla^2 \varphi' \propto k^3$ and $v'\partial/\partial y \nabla^2 \bar{\varphi} \propto k$. In the present situation, these two terms tend to cancel each other; in other words, the phase of the wave tends to shift eastward by the effect of the meridional transport of the zonal mean vorticity by the wave, while

westward owing to the advection. This situation is schematically shown in Fig. 4(a). At $k=5.44$ ($\equiv k_c$), these two effects just cancel out, i.e., the feed-back effect vanishes. These dependences of amplitude response on wave number are somewhat analogous to the behaviors of non-linear spring described by a Duffing equation with third order non-linearity, in which the inclination of amplitude response curve changes, depending on the sign of the coefficient of the non-linear term (cf. Nayfeh and Mook, 1979).

It is interesting to note here that the amplitude response curve in CD's model inclines toward δ positive for all wave numbers. The difference between the results of CD and the present paper may be due to that of the lateral boundary condition; under the present cyclic condition, $\bar{\varphi} \propto \sin(2ly)$ while $\bar{\varphi} \propto \cos(2ly)$ in CD's highly truncated system. This structural difference in the wave-induced mean flow is crucial for the inclination of the response curve. As is seen from Fig. 4(b), although the effects of $\bar{u}\partial/\partial x \nabla^2 \varphi'$ and $v'\partial/\partial y \nabla^2 \bar{\varphi}$ in CD model tend to cancel each other as in our model, we can show that the magnitude of the former is larger than the latter for all k (see Appendix C). As a result, the wave tends to propagate westward. This, in turn, implies that in order for the resonance to occur, a mean zonal flow more easterly than that of the linear case is required; the amplitude response curve inclines toward δ positive.

Further, Fig. 5(c) is the projection of the

Fig. 4 The schematic profiles of the induced zonal flow $\bar{u}$, its vorticity $-\bar{u}_y$, and the streamlines of perturbation φ' . (a) is for present model and (b) for CD model.

Fig. 5(a) Amplitude responses for some k 's. *, $\triangle$ and $\square$ correspond to $k=4$, 5.44 and 7, respectively.

Fig. 5(b) Same as Fig. 5(a) but for phases.

Fig. 5(c) The projection of the solution surface for $E^{1/2}=0.5$ onto the $(k-\delta)$ plane, when $h_0=5$, $J_0=0$ and $c=1$.

solution surface onto $k-\delta$ plane. It is seen that for $|\delta| \leq 2.5$, there exist single equilibrium states for all k , and that short waves have multiple equilibria (two stable and one unstable) for large δ (far from the linear resonance condition).

(c) Response for thermal forcing

If we add the thermal forcing J_0 to the case of Fig. 1(a, b), we have Fig. 6(a, b), for example, in which $J_0=10$ and $\alpha=180^\circ$ are assumed. In the case with $E^{1/2}=0.5$, the maximum response is attained near $\delta=-12$ and $\Theta=90^\circ$. It is interesting that the amplitude response curve forms a loop for $-12 \leq \delta \leq -6$, i.e., for a certain δ (e.g. $\delta=-6$ for $E^{1/2}=0.5$), there are two states with equal amplitudes but with different phases ($\Theta=135^\circ$ and 40°). Note that the largest amplitude states corresponding to the middle branch in

Fig. 6(a) Same as Fig. 1(a) but for $h_0=5$, $J_0=10$, $d=0.1$, $\alpha=180^\circ$, and $c=1$ with $E^{1/2}=0.5$ (*), 1.0 ($\triangle$) and 2.0 ($\square$).

Fig. 6(b) Same as Fig. 6(a) but for phases.

the phase response (Fig. 6(b)) are unstable. Such a 'loop phenomenon' seems not to be a peculiarity of our model; if we adopt the lateral boundary conditions and truncated Fourier system both of which are similar to those of *CD*, the amplitude response curve tends to form a loop with increasing the magnitude of thermal forcing (see Appendix D), and also in this case the largest amplitude states are unstable. It is noted that there is no loop for all k without thermal forcing in our case, while in *CD*'s model we have a loop for h_0 larger than some value even if thermal forcing is zero (Appendix D). The loop due to thermal forcing seems to be overlooked so far. In other words, it should be emphasized that the largest amplitude states are not always stable.

The equilibrium states depend also on α , the phase difference between the topographical forcing and the thermal forcing. Fig. 7(a) is the projection of the solution surface onto $\delta-\alpha$ plane. It is seen that the region of δ to give the multiple equilibria (three equilibrium solutions) is narrow near $\alpha=90^\circ$ and wide near $\alpha=270^\circ$. In Fig. 7(b, c), the cross-sections of the solution surface at some δ 's are shown; Fig. 7(b) are for amplitude, while Fig. 7(c) for phase. It is seen that at $\delta=-8$ the largest amplitude states for $-140^\circ \leq \alpha \leq 60^\circ$ are stable, while those for $120^\circ \leq \alpha \leq 220^\circ$ are unstable. At $\delta=-14$, the largest amplitude states which are unstable are confined to

Fig. 7(a) The projection of solution surface onto ($\delta-\alpha$) plane. The other parameters are the same as in Fig. 6.

(b-1)

(b-2)

(b-3)

Fig. 7(b) Cross sections of the solution surface (amplitude) along some constant δ 's. * and $\square$ indicate stable and unstable solutions, respectively. Amplitude response for α changing slowly as $\alpha=11.25T-60$ with time is also shown in Fig. 7(b-3).

$210^\circ \leq \alpha \leq 240^\circ$. It is emphasized again that the stabilities of multiple equilibria depend seriously on thermal forcing. Thus, it is not so easy to find from these simplified models the correspondence of the equilibrium solutions to the actual atmospheric flow patterns such as the so-called high index flow, low index flow and blockings etc..

Fig. 7(c) Same as Fig. 7(b) but for phases.

6. Time dependent solutions

In this section, we shall examine the time-dependent behaviors of quasi-resonant Rossby waves by performing time-integration of the set of eq. (4.27). We shall deal with the following two problems; one is to solve the equations for a given set of initial conditions in $(R, \theta, D_1, D_2, D_3)$ space, the other is to examine the response to the small change in external parameters such as δ, α and so on near the bifurcation points between stable and unstable states.

(i) Initial value problem

Fig. 8 shows the trajectories projected onto

Fig. 8 Trajectories projected onto amplitude-phase plane for the case of Fig. 1 ($E^{1/2}=0.5$, $\delta=-8$). $dD_n/dT=0$ ($n=1, 2, 3$) at $T=0$ is assumed. $\square$ indicates an equilibrium unstable solution.

amplitude-phase ($R-\theta$) plane, along which states move toward the stable equilibrium points from some specified initial states. In this figure the external parameters are fixed at $h_0=5$, $E^{1/2}=0.5$, $c=1$, $k=4$, $J_0=0$ and $\delta=-8$ (see Fig. 1). For all trajectories, the initial phase is set at 135° , which is on the middle branch in phase response curve (Fig. 1(b)), and further we assume $dD_n/dT=0$ at $T=0$. It is clearly seen that if the initial amplitude is smaller (larger) than 1.0, then the trajectories converge to a stable equilibrium point with small (large) amplitude. It is emphasized that all the trajectories converge to either of two stable states so far as we have calculated for many other initial conditions. This suggests that 5-dimensional phase space $((R, \theta, D_1, D_2, D_3)$ space) is separated into two subspaces, i.e., the attractor domains of large amplitude state and of small amplitude state. In other words, any trajectory cannot go across the 'separation wall' on which middle unstable equilibrium solutions lie, though not proved rigorously in mathematical sense. Thus, if an initial state belongs to, for example, the large amplitude domain, then the trajectory converges finally to the large amplitude state.

(ii) Response to slow change of δ

Here, we shall examine how Rossby waves respond to the slow change of the basic zonal

Fig. 9(a) Amplitude responses for δ changing slowly as $\delta=5T-20$ and $\delta=-5T+20$ for the case of Fig. 1 ($E^{1/2}=0.5$).

flow. Fig. 9 shows the time variation of wave amplitudes when δ changes gradually as $\delta=5T-20$ under the same conditions as those in Fig. 1(a). As the initial condition ($T=0$), we take the equilibrium solution at $\delta=-20$. It is seen that as δ comes close to the bifurcation point (between the lower and the middle branches), the amplitude grows suddenly, i.e., there occurs a state transition from small amplitude to large one. After this transition, the wave amplitude tends to settle down to an equilibrium solution oscillating slowly. The middle branch states are never attained, as already anticipated by the stability analysis. Fig. 9 also shows a result obtained by a reverse movement of δ given by $\delta=20-5T$. In this case, too, there occurs a sudden change of the amplitude at the bifurcation point, and the middle branch states cannot be attained. Further, such a rapid growth of wave as in Fig. 9 might be a cause of anomalous amplification leading to the sudden warming (cf. Matsuno, 1971). The zonal mean fields induced by the wave can be calculated from eq. (2.6), eq. (2.7) and eq. (2.3) as follows;

$$\bar{u} = -\varphi_{00,y} e^{z/2}, \quad (6.1)$$

$$\bar{v} = -e^z \int \frac{\partial}{\partial z} (\bar{w} e^{-z}) dy, \quad (6.2)$$

$$\begin{aligned} \bar{w} = & -\frac{Ro}{Br} \sin(\varphi_0) \left[\left(\frac{\partial}{\partial T} + c \right) (e^{z/2} \varphi_{00})_z \right. \\ & \left. + 4k e^z \varphi_{10} \frac{\partial}{\partial z} \varphi_{12,y}^{(2)} - 4k e^z \varphi_{12}^{(2)} \frac{\partial}{\partial z} \varphi_{10,y} \right], \end{aligned} \quad (6.3)$$

Fig. 10(a) Time change of the wave amplitude for $2(\delta=-10) \leq T \leq 6(\delta=10)$ in the case of Fig. 9(a) ($\delta=5T-20$).

Fig. 10(b) Time-height section of mean zonal flow $\bar{u}$ induced by wave at $y=1/2$, corresponding to Fig. 10(a). The upper figure shows the time change of $\bar{u}$ at $z=1.2$ indicated by $\triangleleft$ in the lower figure.

Further, it is readily seen that in equilibria, we have

$$Roc\bar{u} = \sin(\varphi_0)\bar{v}. \quad (6.4)$$

In Fig. 10(b), shown are the time-height sections of $\bar{u}$ which correspond to the change of wave amplitude in Fig. 10(a). It is seen that the zonal mean fields change almost simultaneously throughout the entire layer. This is because the wave is barotropic. It is noted that $\bar{u}$ is easterly; the mean meridional circulation is induced as a result of the wave-topography interaction and meridional heat flux convergence, and Coriolis force acts to cause easterly flows.

(iii) Response to slow change of α

Fig. 7(b-3) shows the response for the change of α which is given by $\alpha=11.25T-60$. We see also a sudden change from the large amplitude to the small amplitude state near the bifurcation point.

7. Concluding remarks

We have discussed non-linear behaviors of resonant Rossby waves by the perturbation expansion method. The main results are summarized as follows.

In the case of topographic forcing alone, as has already been pointed out by Charney and DeVore (1979) and others, there are multiple equilibria (three equilibrium states) for some external parameter region; two stable states with either large amplitude or with small amplitude, and one unstable state with moderate amplitude. However, owing to our lateral cyclic boundary condition, the characteristics of the amplitude response curves change, depending on wave number under the same condition; for longer waves ($k < 5.44$) the maximum amplitude responses appear at westerly zonal flows smaller than the linear resonance value, while shorter waves ($k > 5.44$) have their maximum amplitude at larger westerly flows. This result can be attributed to the wave-number dependence of non-linear feed back effect of the induced mean zonal flow. It is noted that in Charney and DeVore's model, the maximum response appears in larger westerly flow than that at the linear resonance for all wave numbers. It is further noted that the present results would be somewhat analogous with those of a non-linear spring governed by a Duffing equation with third order non-linearity (Nayfeh and Mook, 1979, e.g.).

One more point which should be emphasized is that if thermal forcing is incorporated the amplitude response curve forms a 'loop' for some range of δ (deviation of the basic zonal flow from the linear resonant value), and hence this means that the largest amplitude states are unstable. Such a loop phenomenon is not a peculiarity of our model, because we have confirmed that even without thermal forcing in Chaney and DeVore's model such a loop can appear if the height of topography exceeds some value, and that the loop becomes wider with increasing thermal forcing. In our model the loop does not appear if no thermal forcings are imposed. The loop phenomenon seems to be overlooked so far. It would not be, however, meaningless to examine, in the light of the present results, whether the actual atmospheric parameters give the multiple equilibria. For example, in the case of topographic forcing alone (see Fig. 1(a)), if we adopt $h^* = 250$ m for $k = 4$

Table 1 Correspondence to dimensional values for the wave $k = 4$.

U_r^*	14.9 (m/s)
h^*	$5.20 \times 10^3 \varepsilon^2 h_0$ (m)
$\frac{K}{H} / ^*$	$6.28 \times 10^{-6} \varepsilon^3 J_0$ (m/s ³)
ψ^*	$1.48 \times 10^4 \varepsilon e^{z/2} \text{Re}(\varphi_{10})$ (m ² /s ²)
$\bar{U}^*$	$14.9(1 + \varepsilon^2(\delta + \bar{u}))$ (m/s)
$\bar{u}^*$	$14.9 \varepsilon^2 \bar{u}$ (m/s)
$\bar{w}$	$0.021 \varepsilon^2 \bar{w}$ (m/s)
t^*	$3.88 \varepsilon^{-2} T$ (day)
$\lambda^*_{(\text{wave length})}$	$3.14 \times 10^7 k^{-1}$ (m)

from the harmonic analysis of earth's topography (e.g., Charney, Shukla and Mo, 1981), then we obtain that $\varepsilon = 0.21$ from $h^* = 5.2 \times 10^3 \varepsilon^3 h_0$ m with $h_0 = 5$. Thus, it follows that $E^{1/2} = 0.024$ and 0.048 for $\tilde{E}^{1/2} = 0.5$ and 1.0 , respectively. The former value of Ekman number is about one half of that usually assumed ($E^{1/2} = 0.053$ see Appendix A), while the latter is about 80 per cent. With the latter Ekman number, the damping time $1/P_1$ is about 7.4 days (see section 4.3(b)).

As is shown in Fig. 1(a), the multiple equilibria occur for $-5 \leq \delta \leq -4$ for $\tilde{E}^{1/2} \approx 1.0$ with the maximum amplitude about 0.7. This range of δ is not inconsistent (though critical) with the expansion by ε , and readily translated into $11.6 \text{ m/s} \leq U_0 \leq 12.3 \text{ m/s}$. In addition, the meridional wind component corresponding to the maximum amplitude wave state becomes about 17.5 m/s.

If the above estimation would be plausible, the results mentioned in this paper which are focused on the case of $E^{1/2} = 0.5$ should be taken as somewhat exaggerated without loss of the physical essence. However, we should remind that the usual estimation of Ekman number which is crucial for resonance is considerably ambiguous. If the earth's atmosphere is less viscous than usually assumed, the non-linear resonance could be expected, though we should emphasize again that the present model is too simplified to be applicable directly to the actual atmosphere.

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Appendix A

The following symbols are used unless otherwise mentioned.

(1) Representative values used for the transformation to the nondimensional forms.

L representative horizontal length, assumed to be one-period-width of the cyclic channel ($=5 \times 10^6$ m)

H scale-height ($=RT_0/g=7 \times 10^3$ m)

U representative speed of the basic zonal flow (linear resonant zonal flow $U_r^*=14.9$ m/s for wave number $k=4$)

g acceleration of gravity ($=9.8$ m/s²)

φ_0 reference latitude of β -plane ($=45^\circ$)

Ω angular velocity of the earth ($=0.7 \times 10^{-4}$ s⁻¹)

f Coriolis parameter ($=2\Omega \sin(\varphi_0)=0.99 \times 10^{-4}$ s⁻¹)

$\hat{a}^*$ radius of the earth ($=6.4 \times 10^6$ m)

c_p specific heat of dry air at constant pressure (1004 JK⁻¹ kg⁻¹)

R gas constant for dry air ($=287$ JK⁻¹ kg⁻¹)

κ ratio of gas constant to specific heat at constant pressure ($=R/c_p=2/7$)

T_0 a basic state temperature ($=T_0(z)$)

N^2 buoyancy frequency squared ($=R/H(dT_0/dz+\kappa T_0/H)=4 \times 10^{-4}$ s⁻²)

(2) Nondimensional parameters and variables

Ro Rossby number ($=U/2\Omega L=2.1 \times 10^{-2}$ for wave number $k=4$)

β Rossby parameter ($=Ro^{-1} \cos(\varphi_0)L/\hat{a}^*$)

Br Burger number ($=(NH/2\Omega L)^2=0.04$)

S static stability parameter ($=(NH)^2/gH=0.286$)

$E^{1/2}$ Ekman number ($=(\nu/(2H^2f)) \sin(2\alpha_0)$

$Br/Ro \sin(\varphi_0)=0.0538$, with $\nu=8$ m² s⁻¹ and $\alpha_0=22.5^\circ$, where ν and α_0 being eddy viscosity and the angle between isobars and surface wind direction, respectively)

c coefficient of Newtonian cooling and Rayleigh friction ($=1.0$)

t time

x, y eastward, northward Cartesian coordinates, respectively

z a measure of height ($z^*=-H \ln(p/p_0)$, where p and p_0 being pressure and reference pressure, respectively)

u, v x and y components of velocity, respectively

$\mathbf{u}$ horizontal velocity vector (u, v)

w a measure of vertical velocity

$\mathbf{k}$ vertical unit vector

∇_H horizontal gradient vector

k wave number in the longitudinal direction

l wave number in the meridional direction ($=\pi$)

Ψ geopotential

T temperature

h height of mountains

J heating function

Appendix B

(1) Coefficients of evolution equations (4.27)

Numerical values are written for $k=4$, $l=\pi$, $E^{1/2}=0.5$ and $c=1$.

$$P_1 = - \frac{c \frac{\beta Br}{U_r \sin^2(\phi_0)} + E^{1/2}(k^2 + l^2)}{S + \frac{\beta Br}{U_r \sin^2(\phi_0)}} = 6.36$$

$$P_2 = - \frac{U_r}{S + \frac{\beta Br}{k U_r \sin^2(\phi_0)}} = 1.70$$

$$P_3 = \frac{\beta P_2 Br}{U_r^2 \sin^2(\phi_0)} = 3.51$$

$$P_4 = P_2 = 1.70$$

$$P_5 = - \frac{l P_2 (4l^2 - \frac{\beta}{U_r}) Br}{U_r \sin^2(\phi_0)} = 5.80$$

$$P_6 = \frac{\frac{\sin^2(\phi_0)}{Br} \frac{l}{d} - \frac{l}{1+1/d} \left\{ \frac{\sin^2(\phi_0)}{Br} \left(\frac{1}{d} + \frac{1}{d^2} \right) - 4l^2 + \frac{\beta}{U_r} \right\}}{\frac{\beta}{kU_r} + \frac{\sin^2(\phi_0)S}{kBr}} = 0.527$$

$$P_7 = \frac{4l^3 - \frac{\beta l}{U_r}}{\left\{ \frac{\beta}{kU_r} + \frac{\sin^2(\phi_0)S}{kBr} \right\} \left\{ \frac{1}{2} + \left(\frac{1}{4} + 4l^2 \frac{Br}{\sin^2(\phi_0)} \right)^{1/2} \right\}} = 2.47$$

$$P_8 = \frac{-\beta}{2lU_r^2} = -4.12$$

$$P_9 = \frac{-2l \left(1 + \frac{1}{d} \right) \frac{\sin^2(\phi_0)}{Br}}{U_r \left\{ \frac{\sin^2(\phi_0)}{Br} \left(\frac{1}{d^2} + \frac{1}{d} \right) - 4l^2 \right\}} = -0.647$$

$$P_{10} = \frac{\left\{ \frac{1}{2} - \left(\frac{1}{4} + 4l^2 \frac{Br}{\sin^2(\phi_0)} \right)^{1/2} \right\} c - 4l^2 E^{1/2}}{\frac{1}{2} - \left(\frac{1}{4} + 4l^2 \frac{Br}{\sin^2(\phi_0)} \right)^{1/2} - S} = 12.9$$

$$P_{11} = \frac{4l^2 E^{1/2} - Sc}{\frac{1}{2} - \left(\frac{1}{4} + 4l^2 \frac{Br}{\sin^2(\phi_0)} \right)^{1/2} - S} = -11.9$$

$$P_{12} = P_{11} = -11.9$$

$$P_{12} = \frac{SP_8 - \frac{2l\beta}{U_r^2} \frac{Br}{\sin^2(\phi_0)}}{\frac{1}{2} - \left(\frac{1}{4} + 4l^2 \frac{Br}{\sin^2(\phi_0)} \right)^{1/2} - S} = 8.68$$

$$P_{14} = \frac{\left(S + \frac{1}{d} \right) P_9 + \frac{2l}{U_r}}{\frac{1}{2} - \left(\frac{1}{4} + 4l^2 \frac{Br}{\sin^2(\phi_0)} \right)^{1/2} - S} = 0.228$$

$$P_{15} = \frac{2kl}{\frac{1}{2} - \left(\frac{1}{4} + 4l^2 \frac{Br}{\sin^2(\phi_0)} \right)^{1/2} - S} = -15.4$$

(2) Coefficients of equation (5.1) for equilibrium

$$A_1 = P_{16}^2 P_{17}^2 + P_{19}^2$$

$$A_2 = 2\delta P_{16}^2 P_{17} + 2P_{18} P_{19} - P_{16}^2 P_{19}^2$$

$$A_3 = P_{18}^2 + \delta^2 P_{16}^2 - 2P_{16}^2 P_{18} P_{19}$$

$$A_4 = -P_{16}^2 P_{18}^2$$

where

$$P_{16} = \frac{P_2 h_0}{P_1}$$

$$P_{17} = -\frac{J_0 \sin(\alpha)}{P_3 P_{16}} \left\{ \frac{P_6 P_9}{c} + \frac{P_7 P_9 P_{12}}{c P_{10}} + \frac{P_7 P_{14}}{P_{10}} \right\} - \frac{1}{P_3} \left\{ \frac{P_7 P_{15} h_0}{P_{10} P_{16}} + P_8 P_8 + \frac{P_7 P_8 P_{11}}{P_{10}} + \frac{c P_7 P_{13}}{P_{10}} \right\}$$

$$P_{18} = \frac{P_4 h_0}{P_3}$$

$$P_{19} = \frac{J_0 \cos(\alpha)}{P_3} \left\{ \frac{P_6 P_9}{c} + \frac{P_7 P_9 P_{12}}{c P_{10}} + \frac{P_7 P_{14}}{P_{10}} \right\}$$

Appendix C

On the inclination of response curve

The result shown in section 5.2(b) that the position of the maximum response on δ axis changes largely depending on wave number can be attributed to the wave number dependence of non-linear feed-back effect of the induced zonal flow on the wave. This can be shown simply as follows.

Eq. (2.5) can be simplified by omitting thermal forcing term and by replacing d/dt by $(U_r + \delta + \bar{u})\partial/\partial x$ as follows;

$$\begin{aligned} A &= \frac{-(U_r + \delta + pl)h_0}{(1/2 - a)(U_r + \delta + pl) + E^{1/2}(k^2 + l^2)i/k} \\ &\sim \frac{U_r h_0}{\frac{Br}{\sin^2(\varphi_0)} \left[-4pl^3 + \frac{\beta}{U_r}(Pl + \delta) \right] - E^{1/2}(k^2 + l^2)i/k} \\ a &= \sqrt{\frac{1}{4} + \frac{Br}{\sin^2(\varphi_0)} \left(k^2 + l^2 - \frac{\beta + 4pl^3}{U_r + \delta + pl} \right)} \\ &\sim \frac{1}{2} + \frac{Br}{\sin^2(\varphi_0)} \left[-\frac{4}{U_r} pl^3 + \frac{\beta}{U_r^2} (pl + \delta) \right] \end{aligned} \quad (C-3)$$

where the last approximation is based on small δ and p . Thus, for a given set of $E^{1/2}$, k and l , the maximum amplitude is attained at $\delta = p(4l^3 U_r / \beta - l)$ where $\beta = (k^2 + l^2)U_r$. If $\delta = 0$, i.e., $k = \sqrt{3}l = 5.44$, then the non-linear effect vanishes. That is, the amplitude response curve has its maximum at $\delta = 0$. Further, if $k < \sqrt{3}l (= 5.44)$, then the position of the maximum response is located in the negative side of δ , while in the positive side for $k > \sqrt{3}l (= 5.44)$. This result agrees remarkably well with the numerical calculation in section 5.2(b).

On the other hand, in Charney and DeVore's case, $\bar{\varphi}$ is approximated as $\bar{\varphi} = -q \cos(l y) e^{-z/2}$ with a small positive q . Then, by a similar manipulation to the above, it can be shown that the maximum amplitude is attained at $\delta = 8qk^2/3(k^2 + l^2) > 0$. Consequently, the amplitude response curve inclined toward the

$$\begin{aligned} (U_r + \delta + \bar{u}) \frac{\partial}{\partial x} \left[\nabla^2 + \frac{\sin^2(\varphi_0)}{Br} \left(\frac{\partial^2}{\partial z^2} - \frac{1}{4} \right) \right] \varphi' \\ + \beta \frac{\partial}{\partial x} \varphi' + v' \frac{\partial}{\partial y} \nabla^2 \bar{\varphi} = 0 \end{aligned} \quad (C-1)$$

where c has been tentatively omitted, and the boundary condition at the bottom becomes as

$$\begin{aligned} (U_r + \delta + \bar{u}) \frac{\partial}{\partial x} \left(\varphi'_z + \frac{1}{2} \varphi' \right) \\ = -(U_r + \delta + \bar{u}) \frac{\partial h}{\partial x} - E^{1/2} \nabla^2 \varphi' \end{aligned} \quad \text{at } z=0. \quad (C-2)$$

Here, we assume $\varphi' = A e^{-az} \sin(l y) e^{ikx}$ as wave solution and $\bar{\varphi} = p \sin(2ly) e^{-z/2}$ ($\bar{u} = -e^{z/2} \partial \bar{\varphi} / \partial y$) as the induced zonal mean field, where p is a small negative parameter. Then, substituting these solutions, multiplying by $\sin(l y)$ and integrating the result for $0 \leq y \leq 1$, we have

positive side of δ for all k .

Appendix D

Charney and DeVore model

In order to compare our results with Charney and DeVore's, we have applied their lateral boundary condition and highly truncated Fourier system to our model and calculated the resulting equations.

Retaining the notational correspondence to CD's we have assumed as follows;

$$\begin{aligned} \varphi = \varphi_A \sqrt{2} \cos(y/L) + \varphi_K 2 \cos(nx/L) \sin(y/L) \\ + \varphi_L 2 \sin(nx/L) \sin(y/L), \end{aligned} \quad (D-1)$$

$$h = h_0/H \cos(nx/L) \sin(y/L), \quad (D-2)$$

where L is the width of the channel, H is the depth of the fluid layer and n is mode number in the zonal direction. Then, we can obtain that

Fig. D1(a) Amplitude and phase responses for $h_0/H=0.01$ and $\varphi_j^*=0.0$. * and □ indicate stable and unstable states, respectively.

Fig. D1(b) Same as Fig. D1(a) but for $h_0/H=0.02$.

Fig. D1(c) Same as Fig. D1(a) but for $h_0/H=0.05$.

$$\dot{\phi}_A = -k(\phi_A - \phi_A^*) + h_{01}\phi_L, \quad (D-3)$$

$$\dot{\phi}_K = -k\phi_K - (\alpha_{n1}\phi_A - \beta_{n1})\phi_L + \phi_J^* \cos(\alpha), \quad (D-4)$$

$$\dot{\phi}_L = -k\phi_L + (\alpha_{n1}\phi_A - \beta_{n1})\phi_K - h_{n1}\phi_A - \phi_J^* \sin(\alpha), \quad (D-5)$$

where

$$\begin{cases} h_{01} = \frac{4\sqrt{2}n}{3\pi} \frac{h_0}{H} \\ h_{n1} = \frac{n}{n^2+1} \frac{4\sqrt{2}}{3\pi} \frac{h_0}{H} \\ \alpha_{n1} = \frac{8\sqrt{2}}{3\pi} \frac{n^3}{n^2+1} \\ \beta_{n1} = \frac{n}{n^2+1} \frac{L}{a} \cot(\varphi_0) \end{cases}$$

where $\phi_J^* \cos(\alpha)$ corresponds to thermal forcing with factor $J \propto \phi_J^* \cos(n\pi/L + \alpha)$, where α being the phase difference of the thermal forcing from the topography h . ϕ_A^* is the forcing of mean zonal flow. The role of this term corresponds to our δ .

Equilibrium solutions can be obtained from the following equation

$$\begin{aligned} Q_1\phi_A^3 + (Q_2 + Q_3\phi_A^*)\phi_A^2 + (Q_4 + Q_5\phi_J^* \cos(\alpha) \\ + Q_6\phi_A^*)\phi_A + Q_7\phi_J^* \cos(\alpha) + Q_8\phi_A^* \\ + Q_9\phi_J^* \sin(\alpha) = 0, \end{aligned} \quad (D-6)$$

where Q_n are listed in the last part of this appendix.

We have examined the stabilities of the equilibrium solutions for several values of parameters. A part of the results is presented in Fig. D1. The numerical values of parameters are as follows; $k=10^{-2}$, $L/a=1/4$, $\varphi_0=45^\circ$, $H=10^4$ and $n=2$. It is seen in case of $\varphi_J^*=0$ that (i) the resonance curve inclines toward φ_A^* (i.e., δ) positive side as already mentioned in Appendix C, (ii) the response curve tends to form a loop with increasing the height of mountain h_0 , and (iii) such a loop becomes wider with increasing thermal forcing φ_J^* as is shown in Fig. D2 (phase responses are not shown, because they are similar to Fig. D1). It is further emphasized that the largest amplitude states on the loop is unstable, and that for $h_0=0.05$ and $\varphi_J^*=$

Fig. D2(a) Same as Fig. D1(a) but for $h_0/H=0.05$, $\varphi_J^*=0.0001$ and $\alpha=0.0$.

Fig. D2(b) Same as Fig. D2(a) but for $\varphi_J^*=0.0005$ and $\alpha=0.0$.

0.0005, almost all equilibrium states on the loop are unstable, i.e., small amplitude states alone are stable (Fig. D2(b)). Fig. D3 shows the cross-sections of the solution surface (amplitude) at various values of φ_A^* . It is seen also from these figures that the largest amplitude state among three equilibrium states is stable only for limited regions of α .

List of coefficients of amplitude equation (D-6)

D3(a)

D3(b)

Fig. D3 Cross-sections of the solution surface (amplitude) along some ϕ_A^* 's for $h_0/H=0.05$ and $\phi_j^*=0.0005$.

$$Q_1 = \frac{k\alpha_{n1}^2}{h_{01}}$$

$$Q_2 = -\frac{2k\alpha_{n1}\beta_{n1}}{h_{01}}$$

$$Q_3 = -Q_1$$

$$Q_4 = k\left(\frac{k^2}{h_{01}} + h_{n1}\right) + \frac{k\beta_{n1}^2}{h_{01}}$$

$$Q_5 = -\alpha_{n1}$$

$$Q_6 = -Q_2$$

$$Q_7 = \beta_{n1}$$

$$Q_8 = -\left(\frac{k^3}{h_{01}} + \frac{k\beta_{n1}^2}{h_{01}}\right)$$

$$Q_9 = k$$

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準共鳴ロスビー波について、第 I 部 外部波

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子午面方向に周期的な境界条件のもとで、摂動展開法を用いて、ロスビー波の非線型共鳴について調べた。モデルは、波と地形（又は加熱）そして波自身の相互作用によって作られる平均帯状流のフィードバック効果を含んでいる。強制として正弦型の山と加熱を、散逸としてエクマン減衰、レーリー摩擦そしてニュートン冷却を想定した。地形強制のみの場合、長い（短い）波の振幅応答曲線は、パラメーター δ （基礎平均流の線型共鳴条件からのずれ）の負（正）の方向に傾き、粘性が小さければ、 δ のある範囲に、いわゆる多重平衡解を持つ（Charney and DeVore 1979, 参照）。 δ 軸上での最大応答の位置は、強制の大きさの 2 乗、散逸の -2 乗に比例して移動する。ある臨界波数では非線型効果の相殺の結果、解応答曲線は、線型応答に似て解の多重性が消える。平衡解の安定性を摂動法で調べた結果、Charney and DeVore (ibid) がすでに指摘しているように、多重平衡解のうち上部及び下部分岐上の解は安定であるが、中間分岐上の状態は不安定である事がわかった。又、多重解領域の外はすべて安定である。

しかし、応答は熱強制が加わると、その大きさや地形に対する位相差によって複雑に変わる。例えば、 δ に関して振幅応答曲線を描けばループ状になり、その場合大振幅解は不安定である（位相でみると中間分岐にある）。このループは加熱を増せば大きくなる。このようなループは、Charney and DeVore モデルでも山を高くすればあらわれるが、これまで見落されてきたようである。最後に、発展方程式の数値積分によって、解の時間的振舞を調べた。