Inertial Oscillation and Symmetric Motion Induced in an Inertio-Gravity Wave Critical Layer

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Abstract

An inertio-gravity wave critical layer is defined as the region in which the conventional WKB-type dispersion relation is mathematically invalid (Yamanaka and Tanaka, 1984b). This layer is bounded by a turning level at which wavefront parallels the basic isopycnic surface and by a level inside which geostrophic adjustment for the wave momentum can hardly take place. The valve-like critical 'level' absorption (Grimshaw, 1975) is explained by an anisotropic wavefront revolution due to the baroclinicity of the basic field. The induced zonal-mean field, correct to the second order of wave amplitude, resembles an inertial oscillation or a symmetric isopycnal motion. The net absorption rate of the critical 'layer', which coincides with that of the non-inertial waves (Booker and Bretherton, 1967), can be explained by a resonant energy conversion from the wave to the zonal-mean inertial oscillation. The geostrophic flow deceleration results from a kind of resonant redistribution of mean kinetic energy so as to maintain the zonal-mean isopycnal motion; this leads to a final equilibrium state in which Richardson number of the mean zonal flow in the critical layer is unity. Although the non-inertial approximation holds outside the layer with a basic Richardson number larger than unity, above-mentioned features are important to discuss the gravity-wave stress and quasi-horizontal diffusion in the stratosphere.

1. Introduction

Lindzen (1981), Holton (1982) and Matsuno (1982) have elucidated that stress and/or eddy viscosity both resulting from breaking gravity waves generate and maintain weak wind near the mesopause. A similar mechanism can work also in the middle stratosphere as pointed out recently by Tanaka and Yamanaka (1985). In these studies the gravity waves are assumed to be non-inertial, and they deposit momentum in the mean zonal flow in a wavebreaking layer below the critical level studied originally by Booker and Bretherton (1967). Namely the 'non-inertial gravity wave critical layer' can be interpreted as the region where the originally neutral

wave breaks into turbulence due to a 'local' instability of wave-superposed flow (see Yamanaka and Tanaka, 1984a, and references therein). Then the wave neutrality, as well as the basic-flow stability, outside the critical layer can be judged only by a 'local' Richardson number $\tilde{R}$ for the wave-superposed flow, *i.e.*, $\tilde{R} < 1/4$ to judgement of local (Kelvin-Helmholtz) instability. As an approximation, the criterion of local convective instability ($\tilde{J} < 0$) is conventional.

The neglect of the inertial effect seems quite reasonable in the upper mesosphere where the Ekman number is large (the viscous force dominates the Coriolis force). However, this situation may not always hold in the stratosphere; in fact long-period largescale gravity waves are frequently observed there (Sawyer, 1967; Thompson, 1978; Cadet and Teitelbaum, 1979; Barat, 1983; Yamanaka and

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Tanaka, 1984a, c; Maekawa *et al.*, 1984; Hirota and Niki, 1985). Some theoreticians have studied such 'inertio-gravity' waves in the stratosphere (Lindzen, 1970; Miyahara, 1976, 1981; Andrews, 1980; McIntyre, 1980; Kitchen and McIntyre, 1980; Tanaka, 1983; Dunkerton, 1984) by assuming sufficiently small vertical shears. In particular, Dunkerton studies the inertio-gravity wave propagation (and breaking) in rather realistic configuration of the stratosphere including the horizontal shear effects. However, the small-shear assumption requested in the conventional WKB method breaks down in the vicinity of the critical levels defined by

$$|\hat{\omega}| = f, \quad (1)$$

where $\hat{\omega}$ is the intrinsic frequency and f is the Coriolis factor (Jones, 1967; Grimshaw, 1975; Yamanaka and Tanaka, 1984b, hereafter referred to as YT). Although Dunkerton considers waves propagating outward from the critical levels (toward faster basic-flow regions) in case of the winter middle and upper stratosphere, waves upgoing through the lower stratosphere should encounter the critical levels as YT considered.

Although YT obtained an exact mathematical expression for the propagation of an inertio-gravity wave near the critical level (1) and its breaking due to the local convective instability, the following two questions were not answered:

(i) *Why can some inertio-gravity waves infiltrate through the critical level up to a turning level?* The turning levels are defined in YT (11) as follows:

$$|\hat{\omega}| = \gamma \\ \equiv \frac{f}{\sqrt{2J'}} \left[\sqrt{\left(J' - \frac{l^2}{k^2} - 3\right)^2 + 8J'} + \left(J' - \frac{l^2}{k^2} - 3\right) \right]^{1/2}, \quad (2)$$

where J' is the 3D Richardson number. Such an infiltration is complete when the eastward and northward components of wavenumber vector (k, l) have the same signs (the wavefront runs from northwest to southeast), and also occurs partially when they have opposite signs (the wavefront runs from northeast to

southwest).

(ii) *Why does the absorption rate*

$$\exp(-2\pi \cdot |l/k|) \quad (3)$$

of the inertio-gravity case (Grimshaw, 1975) differ so much from

$$\exp(-2\pi \cdot \sqrt{J' - 1/4}), \quad (4)$$

which is the rate in the pure gravity case (Booker and Bretherton, 1967)? It would seem conflicting that (4) holds at a level in finitely far from the critical level (1) (YT, Appendix B). As a speculative interpretation of this problem, YT mentioned that the wave transmitted downward across the critical level might also be partially re-reflected just under the critical level.

Consideration of above-mentioned points is the main purpose of this paper. YT have shown that both the infiltration and the absorption rate (3) are derived from the local dispersion relation which is valid around the critical level (1). In this sense we can define an *inertio-gravity wave critical layer* as

$$\gamma < |\hat{\omega}| < \sqrt{2} f. \quad (5)$$

[viz., YT(43)]. Therefore, an inertio-gravity wave-induced turbulence layer must be thinner than the critical layer (5), as illustrated in Fig. 1. It should be noted that short-wavelength gravity waves begin to break outside the inertio-gravity wave critical layer (5), as shown in YT §5.2 and also in Yamanaka and Tanaka (1984a, §§2.5-6, Figs. 4 and 5). This is because shorter waves break as non-inertial waves obeying a dispersion relation different from that in the critical layer (5).

As will be shown in §2.1, we must consider some absorption process other than the singular wavebreaking process in order to settle the problems (i) and (ii). Such a process is considered to appear inside the critical layer (5) and to vanish in the non-inertial cases. We notice an important property, *baroclinicity*, which appears in the rotating stratified atmosphere but disappears in the non-rotating system. This leads to a complexity in judging the wave neutrality, since there can be 'baroclinic instability' (cf. Eady, 1949; Stone, 1966; Tokioka, 1970). An heu-

Fig. 1 'Double' absorption of inertio-gravity wave critical layer (see text). Z_T means the thickness of turbulence (or wavebreaking) layer predicted by local convective (or Kelvin-Helmholtz) instability theories. 'G.W.' is an abbreviation of 'gravity waves'. The other notations of symbols are conventional and defined in the text or in Yamanaka and Tanaka (1984b). Note that f and γ are intrinsic frequencies at the critical and turning level, respectively.

ristic description of this instability is that a particle exchange inside the angle between horizontal and isopycnal is unstable (cf. Charney, 1973, Chapter VII). Note that this cannot be judged only by a basic-field parameter; in general a baroclinic geofluid is unstable for some disturbances of which the wavenumbers and growth rates depend upon the basic stratification and boundary conditions. Thus, there is no sufficient condition of the neutrality of an inertio-gravity wave and the basic field, despite that the critical-level concept requires it. Grimshaw (1975, p. 290, 1.34) also noticed this problem.

In §2.2 we find out that the wavefront inclination (the particle displacement direction correct to the first order of wave amplitude) varies with vertical propagation, which is

anisotropic because of the basic baroclinicity. It will be shown in §2.3 that a wave motion in the critical layer (5) is *destabilized* by the gravity (buoyancy) force. Although this case satisfies the aforesaid criterion of baroclinic instability, the wave remains neutral without any adjustment process. In §2.4 we show that another important property of the rotating stratified fluid, *geostrophic adjustment* (Rossby, 1937; cf. Charney, 1973, Chapter V), can hardly appear in the critical layer (5). It follows that some ageostrophic motions should last eternally if there is no dissipation.

Next we will study the second-order mean flow field accompanied (or induced) by the inertio-gravity wave, which is necessary to describe the budget of momentum and energy. In §3.1 the theorems on the momentum flux with inertio-gravity waves are reviewed. In §§3.2-3 we solve the second-order zonal-mean equations and obtain plane-wavelike solutions (of zero zonal wavenumbers), since the zonal-mean motion is not Doppler-modulated by the zonal basic flow. Hence the induced-mean motion itself has no critical nor turning levels anywhere, or else it is everywhere subject to critical or turning levels (inertial oscillation or symmetric isopycnal motion). The latter case leads to the ageostrophic motions in the critical layer (5) as expected before. It seems reasonable that the wave induces an inertial oscillation rather than a straight flow, because the wave momentum conserved in propagation in the non-rotating system should be rotated by an inertial period in the rotating system.

As shown in §3.4, the Lagrangian motion correct to the second order of wave amplitude is made up of the Eulerian motion and the Stokes drift. The singularities (1) in the Eulerian motion (Jones, 1967; Grimshaw, 1975; YT) appear also in the Lagrangian motion, although many works without solving the equations (Uryu, 1973; Andrews and McIntyre, 1976, 1978; Nakamura, 1979; McIntyre, 1980) did not mention it. The critical-level absorption (3) can be interpreted as the local instability like the non-inertial wave absorption by the factor (4); the difference is due to the anisotropic wavefront rotation found out in §2.2. However, since the absorption factor (4)

is obtained outside the critical layer (5) even in the inertio-gravity wave case, the problem (ii) is not settled in this stage.

Accordingly the connection between the inside and outside of the critical layer (5) is hopeless up to the second order of wave amplitude in the perturbation expansion. Thus, in §4.1, we will hypothesize a resonant interaction between the first-order wave field and the second-order zonal-mean field. The resonant part becomes predominant inside the critical layer (5) finally by an effect of the infinitesimal Rayleigh damping necessary to continue the wave part across the critical level (1). In §4.2 we show that the wave energy is transferred into the inertial oscillation near the critical level (1). When $J > 1$ (J : the basic (2D) Richardson number of the basic field), the basic kinetic energy is redistributed so as to generate the symmetric isopycnal motion near the turning level (2) inside the critical layer (5). When $J < 1$, the symmetric motion is unstable (Stone, 1966; Tokioka, 1970) in the whole domain considered. In §4.3 the final equilibrium state of redistribution is predicted (*cf.* Rossby, 1937) and the Richardson number for final zonal geostrophic flow is obtained as unity. On balance, the net absorption (4) of the critical layer (5) corresponds to the energy conversion from the wave to the inertial oscillation, and the mean geostrophic flow deceleration results from the energy redistribution of the basic flow itself.

In §5, we finally give the answers to the questions (i) and (ii) as a summary of the present paper. Introducing a critical layer (5), the absorption rate (4) of non-inertial case can be generalized for an inertio-gravity wave. It is important that the wave induces zonal-mean inertial oscillations rather than the zonal geostrophic flows. The mean kinetic energy is modified by zonal-mean symmetric isopycnal motions induced in the critical layer, which might contribute to quasi-horizontal diffusion process simulated by Kida's (1983) Lagrangian circulation model of the lower stratosphere.

2. Basic flow field and wave field

2.1 Configuration and formulation

The configuration of the atmosphere and notations of symbols are in principle the same as those treated in YT; otherwise they will be defined in each case. We distinguish equation and section numbers from that paper by the prefix YT.

The *basic* atmosphere YT (1a-d) is Boussinesq and baroclinic, and the inclination α of basic isopycnic surface is given by

$$\tan \alpha \equiv (\partial \bar{\sigma} / \partial y) / (\partial \bar{\sigma} / \partial z) = -f u_z / N^2, \quad (6)$$

where $\bar{\sigma} \equiv g \ln \bar{\rho}$ is the basic density field under the earth's gravity. (6) is a measure of the *baroclinicity*, which is the most important property of a vertically sheared rotating fluid system. (This is the Boussinesq version of the inclination of isentropic surfaces in the compressive fluid.) In the extratropical middle atmosphere ($f \sim 10^{-4} \text{s}^{-1}$, $N \sim 2 \cdot 10^{-2} \text{s}^{-1}$, $|u_z| \sim 2 \cdot 10^{-3} \text{s}^{-1}$) a typical value of α is 0.03° . It should be noted that the gravitational restoring force vanishes if the particle oscillation planes are parallel to the basic isopycnic surfaces (see §2.3). Also note that in the WKB (slowly varying waveguide) formalism in common use (Grimshaw, 1975; Kitchen and McIntyre, 1980; Dunkerton, 1984) these features cannot be found because the baroclinicity is omitted in the dispersion relation with neglect of the basic vertical shear.

The *wave* equation system YT (3a-e) is conventional in f -plane geofluid studies (*e.g.*, Eady, 1949; Stone, 1966; Jones, 1967; Andrews, 1980) and has neutral normal-mode solutions corresponding to inertio-gravity waves [YT(5), YT(10), YT(19a-f)] having the Jones' critical levels (1). This linear equation system can be derived in the following two ways (see Appendix A): the 'disturbance' for the zonal-mean flow, and the 'perturbation' subsystem of an asymptotic expansion of the primitive nonlinear system. As long as we consider the wave amplitude (say, a) to be infinitesimal ($a \rightarrow 0$), these two ways are equivalent; in the latter meaning the system YT (3a-e) is correct to $O(a)$. Then the mean field, in the mean-disturbance meaning, consists of

$O(1)$ part in equilibrium (the 'basic' field) and $O(a^2)$ part (the 'induced mean' field described later in §3.2) due to wave flux divergence terms (*cf.* Andrews and McIntyre, 1976, §4).

The conflicting results (3) and (4) for the absorption rate lead to the following alternative courses: if there is *only* the singular absorption at the Jones' critical level (1), we must consider that (3) is equal to (4), that is,

$$\frac{1}{4} < J < 1. \quad (7)$$

Otherwise we must consider some process unknown in the present stage. We cannot apply the former restriction (7) to the present study, because the non-inertial wave theory valid outside the critical layer (5) is not restricted by (7). Furthermore the wave neutrality is not guaranteed in case of (7), since there can be a symmetric instability (Stone, 1966; Tokioka, 1970; see also §3.3b). Therefore, we must seek for the unknown absorption process; we shall start from a reexamination of the wave solutions obtained by YT.

2.2 Wavefront revolution

When we obtain a set of locally well-defined wave parameters: $(k, l, m; \omega)$, the Boussinesq' continuity equation YT(3e) can be

generally rewritten as

$$\mathbf{k} \cdot \mathbf{u} = 0,$$

where $\mathbf{k} = (k, l, m)$ and $\mathbf{u} = (u, v, w)$. Thus the particle motion correct to $O(a)$ is along a wavefront. So long as the $\mathbf{u}$ -field is governed by a steady linear wave, the particle motion should be periodic; it has an elliptic orbit on the wavefront (Andrews, 1980, §5; also see Matsuno, 1980, §2 for planetary-wave case). Here we shall omit reproduction of these features in order to be straightforward. In a quasi-one-dimensional problem such as YT and this paper, ω , k and l are all constant, so that the inclination of wave front (or of particle orbit) varies only with modulation of the vertical wavenumber m .

The wavefront revolution according to wavepacket proceeding is expressed in the meridional plane projection as follows:

$$\mathcal{D}_t \varphi = -\frac{l}{m^2} W_g m_z = \frac{kl}{m^2} \bar{u}_z, \quad (8)$$

where $\varphi [= \arctan(l/m)]$ is the wavefront inclination in the meridional plane, and $\mathcal{D}_t$ denotes the time derivative with respect to the wavepacket. In derivation of (8) we use the Doppler frequency relation YT(7) and the group velocity definition YT(36). We can here set signs of the zonal wavenumber k and the

Fig. 2 Wavefront revolution due to a basic (negative) wind shear (a) for $l/k < 0$ (b) and for $l/k > 0$ (c). When a wave propagates upward from z_0 to z_1 , the wavefront indicated by solid lines becomes that indicated by dashed lines. Other notations of symbols are conventional (see text) and in common with those used in Yamanaka and Tanaka (1984b).

basic shear $\bar{u}_z$ without missing any generality. Thus the revolution direction of φ is dependent on the sign of meridional wavenumber l (see Fig. 2). This leads to the fact that wave behaviors should be different according to the sign of l if the wavefront approaches horizontal. The inclination α of the basic isopycnic surfaces (6) and the wavefront revolution (8) are both related to the sign of $\bar{u}_z$. Therefore, it k and l have opposite signs, an initially inclined wavefront becomes horizontal before it is parallel to the isopycnic surfaces. On the contrary, when k and l have the same signs, the wavefront becomes isopycnal earlier than horizontal.

2.3 Gravity destabilization

The inclination of the wavefronts in the critical layer (5) is obtained from the local dispersion relation YT(40) as

$$\tan \varphi = \left(\frac{f \bar{u}_z}{\bar{\omega}^2 - f^2} \pm \sqrt{\frac{N^2(\bar{\omega}^2 - \gamma^2)(\bar{\omega}^2 + \delta^2)}{\bar{\omega}^2(\bar{\omega}^2 - f^2)^2}} \right)^{-1} \quad \text{for } \gamma < |\bar{\omega}| < \sqrt{2}f, \quad (9a)$$

and from a finite value YT(44) of m at the critical level:

$$\tan \varphi = -\frac{2f}{\bar{u}_z} \left(J + \frac{2k^2}{k^2 + l^2} \right)^{-1} \quad \text{for } |\bar{\omega}| = f. \quad (9b)$$

If we use a large Richardson-number approximation such as

$$\gamma^2 = f^2 [1 - J^{-1} + O(J^{-2})],$$

we can obtain the following results from (6) and (9a-b):

$$\begin{aligned} \varphi &\simeq \alpha & \text{for } |\bar{\omega}| = \gamma, \\ \varphi &\simeq 0 \text{ or } 2\alpha & \text{for } |\bar{\omega}| = f. \end{aligned} \quad (10)$$

Note that (10) holds exactly for any J if $k=0$, but the proof is omitted here. Thus the turning level corresponds to the level at which the wavefronts are parallel to the inclined basic isopycnic surfaces (Fig. 3). Here we can see the physical meanings of the valve effect and the separation of the turning level from the critical level. Both of them are due to the basic baroclinicity (6) and disappear in the conventional WKB approximation ($\bar{u}_z \rightarrow 0$).

Fig. 3 Inclinations of oscillatory planes (indicated by thick arrows), which are along each wavefronts, for various ω in the case of a meridional wave ($k=0$; $\bar{u}_z < 0$). Meridional-plane projections of five cases participated by the critical-level frequency (f) and the turning-level frequency (γ) are shown. The basic isopycnic surface in each case is indicated by a dashed line. Note that the meridional wave has a plane wavefront which is perpendicular to the meridional plane and is not modulated by the basic zonal flow ($\bar{\omega} = \omega$).

We find out from (10) that the singular one of the two normal modes between the critical and turning levels ($\gamma < |\bar{\omega}| < f$) has a wavefront which is inclined between the basic isopycnic surface and the horizontal plane. Therefore, particle motions due to such a wave are not restored but destabilized by the gravity (buoyancy) force; they seem unstable when we are reminded of the instability criterion introduced heuristically in §1. Note that the wave is neutral as long as we assume that ω is real (a work done by the Coriolis force stabilizes the particle motion and cancels the gravity destabilization). A wave with positive (negative) l/k encounters such a gravity destabilization before (after) the critical-level absorption, that is, after (before) the turning-level reflection (see Fig. 2 of YT).

Similar but somewhat different gravity

Fig. 4 Particle orbit (an ellipse plotted by thick solid and dashed curves) of an upward propagating zonal wave ($l=0$). Thin dashed-dotted and dashed lines indicate intersects between the wavefront and the horizontal and vertical-zonal planes. Other symbols are almost conventional except for α denoting the inclination of the basic isopycnic surface. See text.

destabilizations are found also outside the critical level ($|\hat{\omega}| > f$). In two sectors (see Fig. 4) the orbit motions of particles are not restored but destabilized by the gravity, though the wave remains neutral. Interestingly the destabilization sectors of the upward propagating mode are on the opposite sides of the meridional line to those of downward propagating mode. Both orbits of the upward and downward propagating modes intersect at two 'marginal' points which correspond to the boundaries of the destabilization sectors. YT showed that the upward and downward moving modes at a far bottom must be mathematically continued to a linear combination of the upward and downward modes at around the critical level [*viz.*, YT(21a-b)].

We can easily confine that those destabilization sectors are enlarged in the vicinity of the critical level. At the outside boundary of the critical layer $|\hat{\omega}| = \sqrt{2}f$, the angle of such a sector is given by the following, using ϵ to be defined in (11):

$$\arctan \left[\frac{\sqrt{\left(\frac{2\pi}{\sqrt{k^2 + l^2}}\right)^2 + \left(\frac{2\pi}{m}\right)^2}}{\left|\frac{2\pi}{m}\right|} \cdot |\alpha| \right] \\ = \arctan \sqrt{\frac{1 + \epsilon}{J}} \sim J^{-1/2}.$$

For example, this angle becomes 6° for $J \sim 10^\circ$ and 45° for $J \sim 1$. Thus we can state that

the gravity destabilization becomes predominant after the (upgoing, say) wave passes the (bottom) boundary of the critical layer: $|\hat{\omega}| = \sqrt{2}f$. When we characterize the gravity (buoyancy) restoring force, we state that an *inertio-gravity wave loses the character of gravity wave gradually after infiltrating into the critical layer (5) and perfectly after (or before) the critical-level absorption.*

In this stage we have no information on what results from the gravity destabilization. Note again that the wave is neutral everywhere as long as ω is real, and that the conflicting results (3) and (4) for absorption rate appear when the basic state is stable with respect of the symmetric instability (7). As mentioned in §1, in case of the critical-level absorption of a non-inertial wave, we can interpret it as a result of the local convective (or Kelvin-Helmholtz) instability (wave-breaking) of wave-superposed flow (see Yamanaka and Tanaka, 1984a, §2). Then the wave is neutral as long as we include no adjustment process such as proposed by Lindzen (1981), and the absorption disappears if the basic state is unstable ($J' < 1/4$) as implied by (4). Even in the inertio-gravity wave case, the critical-level absorption (3) results from the local instability (YT-§5). Analogically we may consider that the critical-layer absorption (4) in case of the inertio-gravity wave is due to the gravity destabilization [in other words, the 'local baroclinic (symmetric) instability' of wave-superposed flow].

2.4 Geostrophic adjustability

Another important property of the rotating stratified fluid is *geostrophic adjustment* for a local instant addition of momentum (Rossby, 1937; see an extended discussion by Charney, 1973, Chapter V). This adjustment occurs when the ratio of the vertical scale to the horizontal scale of the momentum-applied space becomes sufficiently larger than

$$\tan \epsilon \equiv f/N. \quad (11)$$

A typical value of ϵ in the extratropical middle atmosphere is 0.3° . In a wave propagation problem, the wavefront inclination $\sqrt{k^2 + l^2}/|m|$ corresponds to the vertical/horizontal ratio of

the space to which a momentum is attached, since the wave momentum is defined with respect to one wave phase (*cf.* §3.1). Therefore, the wave momentum with a wave of which the front inclination is quite smaller than ε cannot contribute to geostrophic acceleration of the mean flow.

The wavefront inclination outside the critical layer (5) is approximately given by a dispersion relation $YT(39)'$, *i. e.*,

$$\frac{\sqrt{k^2 + l^2}}{|m|} \approx \frac{\sqrt{\hat{\omega}^2 - f^2}}{N} \quad \text{for } |\hat{\omega}| > \sqrt{2}f. \quad (12)$$

Approaching the critical layer, the wavefront revolves as described in (8). At the outside boundary of the critical layer we have

$$\frac{\sqrt{k^2 + l^2}}{|m|} \rightarrow \varepsilon \quad \text{for } |\hat{\omega}| \rightarrow \sqrt{2}f, \quad (13)$$

which does not essentially depend upon the sign and magnitude of l/k . From (6) and (11), using (8), (10) and (13), φ in the critical layer (5) is in general smaller than ε as long as $J > 1$. This implies that the inertio-gravity waves in the critical layer (5) can hardly accelerate (or decelerate) the mean geostrophic flow if they are absorbed there. Namely the wave momentum is returned outward from the critical layer by reflected or reemitted waves; otherwise it remains inside the critical layer as ageostrophic motions. However, the former is omitted because the wave absorption (4) measured at a far bottom is almost perfect as long as $J > 1$. Therefore, we can state that *in the inertio-gravity wave critical layer the geostrophic adjustment can hardly occur and the inertio-gravity wave momentum should be mainly transformed into some ageostrophic motions.*

The results (8), (10) and (13) lead to an explanation of the inertio-gravity wave critical layer (5) in view of the wavefront revolution, by which we can solve the problem (i) in §1. However, we cannot settle perblem (ii) except that the critical-level absorption (3) corresponds to the wavebreaking. Although we can speculate that the critical-layer absorption (4) may correspond to the gravity destabilization in the layer (5), it remains ambi-

guous how an inertio-gravity wave is absorbed in its critical layer, that is, how the wave momentum is added to the mean flow. In the next section, we shall advance against the flow field correct to the second order of the inertio-gravity wave amplitude.

3. Induced mean flow field

3.1 Wave momentum flux

In the presence of a steadily-forced inertio-gravity wave, the fluid gets horizontal momentum transferred from the wave source by the wave. The momentum addition from the wave to the fluid is governed by so-called 'pseudomomentum rule':

$$\overline{[(u + \bar{u}_z \zeta - f\eta)w]}_z = 0 \quad \text{for } |\hat{\omega}| \neq f, \quad (14)$$

where η and ζ are the Lagrangian meridional and vertical displacements due to the wave (Jones, 1967; Uryu, 1973; Andrews and McIntyre, 1976, 1978; Nakamura, 1979). The bar indicates the zonal mean; the vertical dependence of such a mean quantity can appear because the vertical wavenumber varies in the vertical direction. In case of non-inertial waves ($f \neq 0$) (14) gives the non-interaction theorem of Eliasson and Palm (1961), which states that the momentum flux divergence $(\bar{u}w)_z$ is zero [*cf.* (29b)]. However, the momentum flux with upward propagating inertio-gravity waves is in general diverged (*cf.* Tanaka, 1983) as in an equilibrium with the Stokes drift effect $f(\eta\bar{w})_z$.

In such a steady problem the mean mass flux in the meridional plane is along the basic isopycnic surface:

$$\frac{\overline{\sigma w}}{\overline{\sigma v}} = -\tan \alpha. \quad (15)$$

This implies that the direction of mass flux due to an inertio-gravity wave is independent of the wavefront inclination, and that it does not modify the basic baroclinic stratification. Note that the sign of a normal vector (6) is different from that of a flux vector (15). The mass flux, as well as the momentum flux, is diverged due to the basic baroclinicity:

$$(\overline{\sigma w})_z = -f \bar{u}_z (\eta\bar{w})_z = -\bar{u}_z (\overline{u w})_z. \quad (16)$$

These theorems were originally derived by Uryu (1973).

In addition, since l is constant in the present problem*, we can define a 'meridional pseudomomentum' as $(v + f\xi')$, where ξ' is not the Lagrangian zonal displacement but defined by $D_t\xi' = u$. Jones (1967) derived also a theorem similar to (14) such as

$$[(v + f\xi')w]_z = 0 \quad \text{for } |\hat{\omega}| \neq f, \quad (17)$$

and proved that the 'total pseudomomentum flux vector' should be parallel to the horizontal wavenumber vector:

$$\frac{(v + f\xi')w}{(u + \bar{u}_z \zeta - f\eta)w} = \frac{l}{k}. \quad (18)$$

It should be noted that (18) holds everywhere including the critical levels, so that the both components of pseudomomentum are changed in the same ratios if a wave absorption (by breaking, dissipation or any other processes) takes place. It is clear from (18) that the zonal and meridional pseudomomentum vanish when $k=0$ and $l=0$, respectively.

The wave fluxes can be related to the vertical velocity perturbation w and all the theorems (14)–(18) can be exactly confirmed by using those relations, although here we omit them (see Jones, 1967; Grimshaw, 1975). It should be noted that the exception of the critical levels shown in (14) and (17) can be obtained only from considerations using the exact solutions (*e.g.*, of w), because the basic equations YT(3a-e) do not explicitly express the existence of singularities corresponding to the Jones' (1967) critical levels. In this meaning, the critical level problems arise from the mathematical method to solve the equations, that is, the separation of independent variables.

3.2 Governing equation

Equations of the induced zonal mean field

* In this quasi-one-dimensional problem, the averages respect to zonal, meridional wavelengths and wave period are all equivalent to the ensemble mean. A pseudomomentum can be defined in the direction of which the one-wavelength mean is equivalent to the ensemble mean (Andrews and McIntyre, 1978, §2, McIntyre, 1980, §5).

(U, V, W, Σ, Φ) are now written as

$$U_t - fV + \bar{u}_z W = -(\overline{uw})_z, \quad (19a)$$

$$V_t + fU + \Phi_y = -(\overline{vw})_z, \quad (19b)$$

$$\Sigma + \Phi_z = 0, \quad (19c)$$

$$\Sigma_t + f\bar{u}_z V - N^2 W = -(\overline{\sigma w})_z, \quad (19d)$$

$$V_y + W_z = 0. \quad (19e)$$

These are almost the same as those used by Andrews (1980) except that the basic baroclinic flow is assumed here. The temporal, zonal and meridional derivatives of flux terms are omitted since we assume a pure plane wave in time and horizontal (ω , k and $l = \text{constant}$)*. The system (19a-e) is interpreted as $O(a^2)$ part of the zonal-mean field or as DC part of the 'second-order' asymptotic equations (see §2.1 and Appendix A).

Seeing Eq. (19e), we introduce a stream function Ψ for the induced mean meridional circulation as

$$V = -\Psi_z \quad \text{and} \quad W = \Psi_y. \quad (20a)$$

Then the following equation for Ψ can be derived from (19a-d):

$$(\partial_t^2 + f^2)\Psi_{zz} + 2f\bar{u}_z\Psi_{yz} + N^2\Psi_{yy} = -f(\overline{uw})_{zz}. \quad (20b)$$

The relation between the induced mean zonal velocity, the meridional circulation and the momentum flux is directly obtained from Eq. (19a) as

$$U_t = -(f\partial_z + \bar{u}_z\partial_y)\Psi - (\overline{uw})_z. \quad (20c)$$

Relations between (20a-c) and the equations used in foregoing studies (Andrews, 1980; Walterscheid, 1981; Holton, 1982; Matsuno, 1982; Tanaka, 1983; Kida, 1985; Miyahara, 1985; Tanaka and Yamanaka, 1985; Takahashi, private communication) are mentioned in Appendix B.

3.3 Classification of possible solutions

In this subsection we shall examine the solutions of the governing equation (20b) from

* It should be taken care that the induced mean field depends upon t , y and z whereas the wave fluxes depend only upon z .

an Eulerian viewpoint. Considering that the momentum flux depends only upon z , we introduce here a 'transformed Eulerian' stream function:

$$\phi \equiv \Psi + \frac{1}{f} \overline{uw}. \quad (21a)$$

Then we can rewrite Eq. (20b) in a homogeneous equation such as

$$(\partial_z^2 + f^2)\phi_{zz} + 2f\bar{u}_z\phi_{yz} + N^2\phi_{yy} = 0, \quad (21b)$$

which is convenient to discuss how the induced mean flow extends in space and in time, based on the differential equation theory (*cf.* Andrews, 1980; Miyahara, 1981).

In order to separate time-dependence, we take a complex Laplace transform of Eq. (21b)

Fig. 5 Classification of the zonal-mean field induced by an inertio-gravity waves for various values of the basic Richardson number (J) and the normal-mode 'frequency' squared (Ω^2). Here Ω is normalized by the Coriolis factor f . The hyperbolic domain of Eq. (24) is hatched; then the induced motion can extend from the place where the wave fluxes are diverged into the fluid. Areas indicated by (a)-(d) correspond to each classes discussed in the text. Note that these classifications are independent to whether the flux divergence is permanent (due to wavebreaking, viscous dissipation, *etc.*) or transient (typically with a slowly-varying wavepacket), and also to boundary conditions. See Stone (1966) for a boundary-value eigenvalue study and Andrews (1980) for a simplified forcing problem.

in terms of a 'frequency', say Ω :

$$(f^2 - \Omega^2)\phi_{zz} + 2f\bar{u}_z\phi_{yz} + N^2\phi_{yy} = 0, \quad (22)$$

where the same symbol ϕ is used to denote its transform, *i.e.*, the space-dependent part. Note that the zonal-mean flux terms such as $\overline{uw}$ consist only of the DC components, so that the structures of (second-order) solutions are not restricted directly by the frequency and wavenumbers of the (first-order) wave. From the form of Eq. (21b), Ω^2 is real, so that Ω is real or else pure imaginary. Equation (22) is hyperbolic, parabolic or elliptic according to

$$\Omega^2 \gtrless f^2 \left(1 - \frac{1}{J}\right).$$

We can classify normal modes of the wave-induced mean flows by the Richardson number J and the stability (sign of Ω^2), as summarized in Fig. 5. Note that $|\Omega_r|/f$, where Ω_r is the real part of Ω , corresponds to the Rossby number of the induced mean flow (*cf.* Appendix B). A boundary-value eigenvalue problem of Eq. (22) was solved by Stone (1966).

(a) Induced 'inertio-gravity waves'

Induced mean motions with real Ω satisfying $\Omega^2 > f^2(1 - 1/J)$ resemble internal inertio-gravity waves. From (21a-b), a formal solution is written as

$$\Psi = \phi_0 \exp[i(Ly + Mz - \Omega t)] - \frac{\overline{uw}}{f}, \quad (23a)$$

where ϕ_0 is a constant and L and M denote the meridional and vertical 'wavenumbers', respectively. The 'dispersion relation' is described as

$$\Omega^2 = f^2 + \frac{N^2 L^2 + 2f\bar{u}_z L M}{M^2}. \quad (23b)$$

This relation is the same as that of a plane inertio-gravity wave in the case of $k=0$. Suppose that a finite wavepacket passes through a level, and there will be left a second-order mean flow resembling an inertio-gravity wave at the level after the packet has passed. Even if the original inertio-gravity waves are localized like that, their effect can extend out of the packet through such an internal-wavelike induced mean flow

(i.e., a Cauchy's hyperbolic problem with open boundary).*

In general (23b) gives two values of L/M for a given Ω . If $|\Omega|=f$, labeled by (a)' in Fig. 5, one of them is equal to 2α and the other is

$$\frac{L}{M}=0 \quad (\text{for } |\Omega|=f). \quad (24a)$$

In the latter case we have $W=\Psi_y=0$ and $\Phi_y=0$, and the basic equations (19a-b) become

$$\begin{aligned} U_t - fV &= -(\overline{uw})_z, \\ V_t + fU &= -(\overline{vw})_z. \end{aligned} \quad (24b)$$

Thus, the wave-induced mean flow is an induced 'inertial oscillation'. We find that this case resembles a critical-level situation (of wave with $k=0$). However, breaking of the induced (second-order) 'wave' does not appear, since the local Richardson number is not so much decreased by such a mean modification.

For motions with $\Omega^2=f^2(1-1/J)$, labelled by (a)'' in Fig. 5, the wave-induced wave have a unique meridional structure:

$$\begin{aligned} \frac{L}{M} &= -\frac{f\bar{u}_z}{N^2} = \arctan \alpha \\ \text{for } |\Omega| &= f\sqrt{1-\frac{1}{J}}, \end{aligned} \quad (25a)$$

which parallels the basic isopycnic surfaces (6). We hereafter call such motions by 'symmetric isopycnical oscillation'. (In a compressible atmosphere, the term 'isopycnical' is replaced by 'isentropic', as suggested in §1). Then, from (20a, c), we have

$$\begin{aligned} U &= \pm i\sqrt{1-\frac{1}{J}} \left[V - \frac{1}{f}(\overline{uw})_z \right], \\ V &= \frac{N^2}{f\bar{u}_z} W + \frac{1}{f}(\overline{uw})_z. \end{aligned} \quad (25b)$$

We find that the induced mean flow in this case resembles an inertio-gravity wave (with

$k=0$) at the turning level. Note that the induced mean flow, as well as a wave with $k=0$, is a pure plane wave, so that there is nothing like a turning-level reflection. Although this case is, exactly speaking, a parabolic problem for Eq. (22), there is no problem when we consider it an extreme case of the hyperbolic domain ($|\Omega| \rightarrow f\sqrt{1-1/J}$).

According to Appendix F of YT, the two situations mentioned above can last long even after the original wave have passed, since the 'group velocity' coincides with the basic flow. When J becomes large, the isopycnic surfaces become almost horizontal and the two long-lasting modes of the induced mean flow may be very similar. The wavelike modes other than the inertial and isopycnical oscillations escape from the wavepacket region; these are considered to disappear as $t \rightarrow \infty$, although it is beyond the scope of this paper to seek the inverse Laplace transform of the solution of Eq. (22) (cf. Booker and Bretherton, 1967, §5).

(b) Induced symmetric instability

Figure 5 shows the existence of some unstable modes ($\Omega^2 < 0$) in the hyperbolic domain of Eq. (22). In fact, Eq. (22) with a set of slip rigid boundary conditions gives the same problem as a special case of Eady's (1949) baroclinic instability, that is, the *symmetric instability* (Stone, 1966; Tokioka, 1970). The growth rate Ω_i (imaginary part of Ω) of such a symmetric-unstable-wavelike induced mean flow satisfy

$$\Omega_i \leq f\sqrt{\frac{1}{J}-1}.$$

We can consider that, if a wavepacket passes through a level, the second-order mean field resembling a symmetric unstable wave will be induced after the packet has passed.*

The largest growing mode corresponds to the parabolic limit, labelled by (b)' in Fig. 5. As well as the isopycnical oscillation (25a), it is parallel to the basic isopycnic surface (6):

* McIntyre (1980, p. 165 1.13) noticed that there exist internal-wavelike modes in the mean field induced by an inertio-gravity wave. Recently, Miyahara (1985) and Takahashi (personal communication) discuss generation of planetary waves by somewhat different mechanisms including β -effect (see Appendix B).

* A possibility of instability of the flow incorporating gravity-wave effects was pointed out by Dunkerton (1982) in relation to the inertial stability of the equatorial mesosphere.

$$\frac{L}{M} = -\frac{f\bar{u}_z}{N^2} = \arctan \alpha$$

$$\text{for } \Omega_i = f\sqrt{\frac{1}{J}-1}, \quad (26a)$$

where a formal solution such as (23a) is again assumed. Then, from (20a, c), we have

$$U = -\sqrt{\frac{1}{J}-1} \left[V - \frac{1}{f} (\overline{uw})_z \right],$$

$$V = \frac{N^2}{f\bar{u}_z} W + \frac{1}{f} (\overline{uw})_z. \quad (26b)$$

We find again the same *meridional* structure as that of a turning-level inertio-gravity wave with $k=0$. However, U and V are almost anti-phase in this case, whereas they have a quarter-phase shift in case of the isopycnal oscillation (25b).

The baroclinic and Kelvin-Helmholtz unstable modes with nonzero zonal wavenumbers, which can be appeared in an instability problem of the basic flow, are degenerated in the induced *zonal-mean* field.* We can safely state that, if $J < 1$, an inertio-gravity waves induce a symmetric instability of the second-order mean field. (The energetics will be considered in § 4.2.) In this meaning, a *necessary* condition for the neutrality of the (first-order) inertio-gravity wave can be given by

$$J > 1 \quad (27)$$

(cf. § 1).**

(c) Induced steady flow

If the induced mean field is independent of time ($\partial_t = 0$, i.e., $\Omega = 0$), Eqs. (19a, d) are rewritten using (16) as

$$V = \frac{\bar{u}_z}{f} W + \frac{1}{f} (\overline{uw})_z$$

and

$$V = \frac{N^2}{f\bar{u}_z} W + \frac{1}{f} (\overline{uw})_z.$$

Therefore, we find that such a steady mean flow is induced only when

$$J = 1. \quad (28)$$

Note again that the relation (16) holding in case of an inertio-gravity wave is essentially necessary to derive (28).

The induced steady flow derived here is parallel to the basic isopycnal surfaces as well as the isopycnal oscillation (25b) and also the symmetric unstable flow (26b). Substituting (28) into (25b) or (26b) and using Eq. (19b), we have

$$U = 0 \quad \text{and} \quad \Phi_y = -(\overline{vw})_z.$$

Therefore, in this case, the induced mean flow does not have the zonal component and an induced meridional pressure gradient is balanced against the meridional-momentum flux divergence.

(d) Evanescent modes

In case of $\Omega^2 < f^2(1-1/J)$, (23b) gives two complex values of L/M ; the motions have exponential variations in space. First, when we adopt slip rigid boundary conditions, these modes do not appear as shown by Stone (1966). This implies that the condition (27) is also *sufficient* for the neutrality of an inertio-gravity wavepacket within a fluid volume bounded by slip rigid walls. Next, the evanescent mean motions appear in a simplified model [Appendix B, Eq. (A7a)] studied by Andrews (1980); we have a Dirichlet-Neumann's elliptic problem of closed boundary (see Fig. 3 of Andrews; also McIntyre, 1980, § 4). Namely, the external modes can extend only toward a steadily forcing portion of the boundary (the bottom, for example). Therefore, we may consider that the evanescent modes do not last long after the wavepacket has passed.

3.4 Particle trajectory

In the Eulerian description (21a-b), second-

* The baroclinic instability mentioned here falls under a narrower category omitting the symmetric instability (the broadest category is mentioned in § 1 and the narrowest is in a quasi-geostrophic framework). The '3D' Kelvin-Helmholtz instability of a zonal flow does not include any disturbances of zero zonal wavenumber. (cf. Stone, 1966; Tokioka, 1970; see also the next footnote).

** Another necessary condition for the neutrality of a wave can be obtained view of the '3D' Kelvin-Helmholtz instability as $J' \equiv J(1+l^2/k^2) > 1/4$. This stability condition is satisfied by any mode when $J > 1/4$, and by any J for $k=0$.

order zonal-mean flow is induced by the wave momentum flux divergence $(\overline{uw})_z$, and the following three modes can last long:

- 1) the inertial oscillation, (24a-b);
- 2) the symmetric isopycnal oscillation, (25a-b); and
- 3) the symmetric unstable motion, (26a-b).

Air particles move horizontal in case of 1), whereas they move along the basic isopycnal surfaces in 2) or 3). As shown in Fig. 5, 1) exists for any $J > 0$, and 2) and 3) appear for $J > 1$ and $0 < J < 1$, respectively. For $J = 1$, there exist 1) and also a steady neutral isopycnal motion intermediate between 2) and 3). When $0 < J < 1$, 3) is most predominant. These features are quite different from the non-acceleration features in case of non-inertial waves (see § 3.1). They appear significantly inside the critical layer (5), but vanish approximately outside it since a wave must be almost non-inertial there (see §§ 1, 2.1).

In this subsection we shall consider these features of the second-order zonal-mean field from a Lagrangian viewpoint. At first, the mean meridional and vertical displacements (Y, Z) correct to the second order of wave amplitude are given by

$$Y_t = V + (\overline{\zeta v})_z, \quad Z_t = W + (\overline{\zeta w})_z, \quad (29a)$$

where the Stokes drift has an 'antisymmetric' character:

$$\overline{\zeta v} = -\overline{\eta w}, \quad \overline{\eta v} = \overline{\zeta w} = 0 \quad (29b)$$

(cf. Andrews, 1980, § 5; Matsuno, 1980, § 2).

Differentiating or integrating (29a), with using (29b), (20a) and (23a), we finally obtain

$$\begin{aligned} Y_{tt} &= i\Omega\phi_z, & Z_{tt} &= -i\Omega\phi_y, \\ Y &= -\frac{\phi_z}{i\Omega} + \frac{1}{f} \int^t \overline{[(u - f\eta)w]}_z dt, \\ Z &= -\frac{\phi_y}{i\Omega}. \end{aligned}$$

Hence, we have the following equations governing the second-order particle trajectory in the meridional plane:

$$\begin{aligned} Y_{tt} + \Omega^2 Y &= \frac{\Omega^2}{f} \int^t \overline{[(u + \bar{u}_z \zeta - f\eta)w]}_z dt, \\ Z_{tt} + \Omega^2 Z &= 0. \end{aligned} \quad (30a)$$

Furthermore, we can derive the following theorems from (19a-b) and (29a-b):

$$(U + \bar{u}_z Z - fY)_t = -\overline{[(u + \bar{u}_z \zeta - f\eta)w]}_z. \quad (30b)$$

$$(V + fX')_t + \Phi_y = -\overline{[(v + f\xi')w]}_z, \quad (30c)$$

where we use $X'_t = U + (\overline{\zeta u})_z$.

From (14) and (30a-b) we can state that the zonal pseudomomentum transported by an inertio-gravity wave does not contribute to generation of the mean zonal flow in Lagrangian meaning (cf. Andrews and McIntyre, 1978; Nakamura, 1979). Thus we can define the pseudomomentum as an invariant for a propagating wave. A fluid particle receives by a Lagrangian-mean zonal momentum $(U + \bar{u}_z Z)$ with the wave, but at the same time it is drifted by a Lagrangian-mean meridional motion which cancels the received zonal momentum through the Coriolis force $(-fY)$.* Similarly, Eqs. (17) and (30c) imply that the meridional pseudomomentum does not join in the Lagrangian-mean meridional momentum equation. Although many studies did not state this point, these features of the meridional pseudomomentum are important, in particular on the critical-level absorption as shown below.

As so far shown by Jones (1967) and Grimshaw (1975), such a 'Lagrangian non-interaction' situation breaks at the critical level (1), where a gap described by the factor (3) appears in the vertical pseudomomentum flux. This important fact has not been mentioned in so-called 'generalized Eliassen-Palm theorem', e.g., Eq. (4-63) of Nakamura (1979), which is only a reduction of Eq. (19a) without any information of the wave solutions (cf. § 3.1). We find from (30a-c) that there are two ways to accelerate the mean zonal flow at the critical level. One is that the zonal pseudomomentum is directly consumed to generate the meridional Lagrangian motion, as seen in (30a), and the zonal motion is accelerated so as to satisfy (30b). The other is that the meridional pseudomomentum modifies the mean geostrophic balance as described in (30c). (18) implies that the former is caused

* In Eulerian meaning, the mean zonal flow U is balanced with the Stokes' drift effect $(\bar{u}_z Z - fY)$.

by a wave with small $|l/k|$, whereas the latter is by a wave with large $|l/k|$.

Therefore, the Grimshaw's absorption factor (3) implies that if $|l/k|$ is large then the absorbed wave momentum contributes to modification of the mean geostrophic balance at the critical level (1). In spite of (7), both (3) and (4) become zero if $|l/k| \rightarrow \infty$. Here we remember that an infinitesimal Rayleigh damping YT(24) is necessary to describe the critical-level absorption as considered by Booker and Bretherton (1967) for a non-inertial case. The theorems (14) and (17) become invalid if the wave receives a permanent modification such as wavebreaking, viscous dissipation or diabatic cooling (see Appendix B), and the mean field is also modified as described in (30b-c). In fact, a wavebreaking due to the local convective instability appears in the vicinity of the critical level (1), and the turbulence layer thickness is thicker as $|l/k|$ is larger [see YT(55)].

However, when $|l/k|$ is small, the critical-level absorption process can hardly appear and the mean meridional displacements induced by the meridional pseudomomentum (30b) is predominant. Nevertheless, the Booker-Bretherton's absorption rate (4) must be realized outside the critical layer (5). The connection between the inside and outside of the critical layer seems hopeless as long as we persist in a linear study using an asymptotic expansion of the governing equations as shown in Appendix A, at least to the second order of wave amplitude. As summarized in the beginning of this subsection, the induced Eulerian-mean flow should be ageostrophic if it lasts long, in either case with or without wavebreaking. In the next section we shall propose a hypothesis concerning the interaction between the wave and the induced mean motion.

4. Resonant interaction between wave and induced mean flow

4.1 Approximation to the nonlinear process

A particle is oscillated in the meridional plane by the total restoring force ($\hat{\omega}^2\eta$, $\hat{\omega}^2\zeta$) in the (linear) first-order approximation. The wavefront cross-sections shown in Fig. 2 can

be regarded as the particle trajectories in the meridional plane in this approximation. As shown in Eq. (30a), the particle is also forced by the second-order zonal-mean restoring force (Ω^2Y , Ω^2Z). If the wave has a sufficiently small amplitude of $O(a)$, the trajectory correct to $O(a^2)$ is simply expressed by a linear combination of the first- and second-order field ($\eta+Y$, $\zeta+Z$) (see, *e.g.*, Matsuno, 1980, §2; However, as discussed in §3.4, it is physically insufficient to describe what happens in the inertio-gravity wave critical layer (5), so that we need a nonlinear study. In a weak nonlinear stage (so-called Landau's hypothesis) only the interaction between $O(a)$ and $O(a^2)$ should be considered, since $O(a)$ - $O(a)$ interaction leads to harmonics generation such as wavebreaking turbulence.

The nonlinear effects expected to appear in the propagation of a small-amplitude wave was preliminarily discussed by Andrews and McIntyre (1976, §10) in relation to the wave dissipation mechanisms. However, the local instabilities as examined by YT may be put aside until §4.3, because they are effectively replaced by the Grimshaw's (1975) critical-level absorption (3), as mentioned in §3.4. The interactions with pre-existing disturbances are also omitted in this study. Here we shall consider an interaction between the first-order wave and the second-order mean motion. Although separation of the first- and second-order fields is mathematical rather than physical, we can formulate them as if they were originated independently, as shown in §3.3.

In general, again as mentioned in Andrews and McIntyre (1976), suitably weak nonlinear effects can be incorporated approximately as external forces onto the linear disturbance equations. Here we shall hypothesize the following first-order oscillation with a nonlinear correction such as

$$D_t^2\eta + \hat{\omega}^2\eta = F(t), \quad D_t^2\zeta + \hat{\omega}^2\zeta = G(t), \quad (31a)$$

where D_t denotes the Lagrangian time derivative and

$$\begin{aligned} F &\equiv -Y_{tt} + \phi_{tz} = +M\Omega\tilde{\phi}e^{-i\Omega t}, \\ G &\equiv -Z_{tt} - \phi_{tz} = -L\Omega\tilde{\phi}e^{-i\Omega t}, \end{aligned} \quad (31b)$$

based on Eqs. (23a) and (21a). We write here $\tilde{\phi} \equiv \phi_0 \exp(iLy + iMz)$, and expect that D_t , η and ζ are correct to the second-order of wave amplitude.

It should be noted that (31a) has two normal modes, $\pm\hat{\omega}$, but one of them is realistic for each problem, because $\hat{\omega}$ is determined from the basic field configuration $\bar{u}(z)$ and a set of given parameters (ω ; k). The solutions of (31a-b) are

$$\begin{aligned}\eta &= \eta_0 e^{-i\hat{\omega}t} + \frac{M\Omega\tilde{\phi}}{\hat{\omega}^2 - \Omega^2} \cdot e^{-i\Omega t}, \\ \zeta &= \zeta_0 e^{-i\hat{\omega}t} - \frac{L\Omega\tilde{\phi}}{\hat{\omega}^2 - \Omega^2} \cdot e^{-i\Omega t},\end{aligned}\quad (32)$$

where η_0 the ζ_0 are functions only of space and $\hat{\omega}t \approx \omega t - kx$. We are reminded that the infinitesimal Reyleigh-damping factor $\omega_i (>0)$, e.g., an infinitely small imaginary part of ω , is necessary for branch connections of the linear parts [the first terms of RHS's of (32)] at the critical levels [see § 3.4 and YT(24)]. It is clear from (32) that the large-time asymptotics of an inviscid solution ($\omega_i=0$ and then $t \rightarrow \infty$) is *different* from the inviscid limit of a large-time solution ($t \rightarrow \infty$ and then $\omega_i \rightarrow 0$). We consider the latter case in view of the critical-level connection; thus *the first-order motions vanish and the second-order motions become conspicuous as time passes*.

The long-lasting second-order motions have been summarized in § 3.4. If $J > 1$, 1) and 2) become predominant to the first-order wave near the critical and turning levels, respectively:

$$\eta \rightarrow \frac{Mf\tilde{\phi}}{\hat{\omega}^2 - f^2} \cdot e^{-if t}, \quad \zeta \rightarrow 0 \quad \text{for } |\hat{\omega}| \rightarrow f, \quad (33a)$$

$$\left. \begin{aligned}\eta &\rightarrow \frac{Mf\sqrt{1-1/J}\tilde{\phi}}{\hat{\omega}^2 - \gamma^2} \cdot e^{-if\sqrt{1-1/J}t} \\ \zeta &\rightarrow \frac{Mf^2\bar{u}_z\sqrt{1-1/J}\tilde{\phi}}{N^2(\hat{\omega}^2 - \gamma^2)} \cdot e^{-if\sqrt{1-1/J}t}\end{aligned}\right\} \quad \text{for } |\hat{\omega}| \rightarrow \gamma. \quad (33b)$$

On one hand, (33a) implies that the wave absorption rate (3) derived from the critical-level continuation YT(23a-b) of the first-order (linear) part is not essential. The branch continuation of (33a) at the critical level (1),

e.g., $\hat{\omega} \rightarrow f \pm 0$, is equivalent to those of the resonance problems appeared in many fields in physics; the phase shift between the two branches is π for $\omega_i \rightarrow 0$ and the amplitude is *not* attenuated. Therefore, as an explanation from an energetical viewpoint will be given in the next subsection, there is left an inertial oscillation around the critical level (1) after the singular wave absorption (3). On the other hand, a similar consideration on (33b) will lead to the fact that there is left an isopycnical oscillation around the turning level (2) after the wave reflection.

The induced mean field itself is a plane wavelike structure with a constant amplitude, but its resonant interaction with the first-order wave leads to an amplification of 1) or 2) in the critical layer (5). From (33a), (24b) and (29a-b), we have

$$\left| \frac{Mf\tilde{\phi}e^{-if t}}{\hat{\omega}^2 - f^2} \right| / |Y| = \frac{f^2}{\hat{\omega}^2 - f^2} \rightarrow 1 \quad \text{for } |\hat{\omega}| \rightarrow \sqrt{2}f. \quad (34)$$

Thus, at the boundary of the critical layer (5), the interaction is almost negligible and the motion correct to the second order of wave amplitude is expressed by a linear combination of the basic, wave and induced-mean fields. The fraction used in (34) becomes infinite with approaching the critical level (1), so that the interaction process is essentially important, as mentioned above. Besides, when the wave is far away from the critical layer, 1) and 2) are both negligibly small:

$$\eta \rightarrow \eta_0 e^{-i\hat{\omega}t}, \quad \zeta \rightarrow \zeta_0 e^{-i\hat{\omega}t} \quad \text{for } |\hat{\omega}| \rightarrow \infty. \quad (35)$$

Therefore, the absorption rate (4) defined outside the critical layer (5) remains valid, and *the net absorption in the critical layer can be replaced by the hypothesis of wave-mean resonance*.

If we set $J < 1$ and take a set of slip rigid boundary conditions for the second-order field, the inertio-gravity wave may smear out as 3) grows:

$$\eta \approx \frac{iMf\sqrt{\frac{1}{J}-1}\tilde{\phi}}{\hat{\omega}^2 + f^2\left(\frac{1}{J}-1\right)} \cdot \exp\left[f\left(\frac{1}{J}-1\right)^{1/2} \cdot t\right]$$

$$\zeta \approx \frac{iMf^2\bar{u}_z\sqrt{\frac{1}{J}-1}\tilde{\phi}}{N^2\left[\hat{\omega}^2 + f^2\left(\frac{1}{J}-1\right)\right]} \cdot \exp\left[f\left(\frac{1}{J}-1\right)^{1/2} \cdot t\right]$$
(36)

Note again that $\hat{\omega}$ has a small imaginary part ω_i . The growth rate is constant but the amplitude increases as $|\hat{\omega}| \rightarrow 0$. Furthermore, 1) can be also induced in the critical layer (5) even in case of $J < 1$. The unstable mode (36) leads to redistribution of the zonal-mean energy, that is, the basic field should be permanently modified.

4.2 Energy conversion

The energy density E with a wavepacket passing by the group velocity W_g leads to an energy flux EW_g , which is a function of z (see § 3.1 and references therein). W_g in the critical layer (5) has been derived in Appendix F of YT. In general, E correct to the second order of wave amplitude consists of the correlations of first-order quantities (the 'wave part': E_w) and the correlations of zeroth- and second-order quantities (the 'mean part': E_m). The wave part is written as $E_w = (\bar{u}^2 + \bar{v}^2 + \bar{w}^2 + \sigma^2/N^2)/2$, which is *finite* for $|\hat{\omega}| \neq \gamma$ as clarified from $|w|$ derived in YT(32). The mean part is a *linear* function of z because $\bar{u}$ and $\bar{\sigma}$ are assumed as linear functions of z and the amplitude of the second-order quantities are independent of z . However, when we incorporate the resonant interaction processes between the first- and second-order fields, the mean part magnifies in the critical layer as shown in (34) and (36). We shall consider the energy budget of the wave and zonal-mean field in the interaction process.

When $|\hat{\omega}| \rightarrow f$, YT(32), (33a) and YT(A28) lead to

$$E_w \rightarrow \text{const}, \quad E_m \propto |\hat{\omega}^2 - f^2|^{-1}$$

$$\text{and } W_g \propto |\hat{\omega}^2 - f^2|.$$

Thus the wave and mean parts of the energy flux become

$$E_w W_g \rightarrow 0 \quad \text{and} \quad E_m W_g \rightarrow \text{const}$$

$$\text{for } |\hat{\omega}| \rightarrow f. \quad (37)$$

This implies that the wave energy is absorbed into the mean field near the critical level. Such a feature appears commonly in the critical-level problem (see Booker and Bretherton, 1967, § 6), but in the inertio-gravity wave case the resonant interaction process is essentially important.

When $J > 1$ and $|\hat{\omega}| \rightarrow \gamma$, YT(32), (33b) and YT(A28) imply

$$E_w \propto |\hat{\omega}^2 - \gamma^2|^{-1/2}, \quad E_m \propto |\hat{\omega}^2 - \gamma^2|^{-1}$$

$$\text{and } W_g \propto |\hat{\omega}^2 - \gamma^2|^{1/2}.$$

Thus we confirm that the wave is perfectly reflected at the turning level:

$$E_w W_g \rightarrow \text{const} \quad \text{and} \quad E_m W_g \rightarrow \infty$$

$$\text{for } |\hat{\omega}| \rightarrow \gamma. \quad (38)$$

Then the zonal-mean symmetric isopycnal motion must draw its energy from the basic field. In the present problem, unlike Andrews (1980) and McIntyre (1980), particles can move along the basic baroclinic isopycnal surface (6) without any displacement of the centroid, as shown in (15) and (29a). Hence the mean potential energy is unchanged everywhere to the second order of wave amplitude, and the mean kinetic energy (initially of the basic zonal geostrophic flow) should be redistributed so as to generate and maintain the symmetric isopycnal motion. It follows that the geostrophic component of the zonal-mean flow field is *reduced in any case* in the critical layer (5). Since the critical level (1) locates always on the wave-source side of the steering level ($\hat{\omega} = 0$), this geostrophic-flow reduction works just like the mean-flow acceleration of non-inertial waves except for very fast phase speed $C(C > f/k$ and $C < -f/k$ for $\bar{u}_z > 0$ and $\bar{u}_z < 0$, respectively; see Fig. 6). Note that the *total* amount of the kinetic energy is *not reduced* because both the basic flow and the isopycnal oscillation are zonal-mean. The energy conversion is finite and instant because the isopycnal symmetric motion is a neutral oscillation as long as $J > 1$.

Fig. 6 Mean geostrophic flow deceleration induced by an inertio-gravity wave (thick arrow) and a non-inertial wave (thin arrow) for various cases of the basic zonal flow $\bar{u}$ and the wave zonal phase velocity $C (\equiv \omega/k; k=\text{const.})$. When the wave phase speed is large and contrary to the lower-layer basic flow ($\omega > f$ for $\bar{u}_z > 0$; $\omega < -f$ for $\bar{u}_z < 0$), the inertio-gravity wave acceleration directions are opposite to those of non-inertial waves.

When $J < 1$, the wave energy is absorbed as shown in (37). However, the zonal-mean symmetric isopycnal motion is unstable everywhere (even outside the critical layer) in this case, so that the zonal-mean kinetic energy is redistributed all the time. Although such a feature resembles what appears in symmetric unstable modes of the first-order boundary-value eigenvalue problems treated by Stone (1966) and Tokioka (1970), the total amount of the zonal-mean kinetic energy is not reduced in the present case whereas it is permanently exchanged into the eddy energy in the usual symmetric-instability problems. In case of $J < 1$, the geostrophic flow is reduced every altitudes in the waveguide and the symmetric isopycnal motion grows there.

We have found that the zonal-mean symmetric isopycnal motion is generated and the mean kinetic energy is redistributed in any case. It is ambiguous how large the kinetic energy is converted into the symmetric isopycnal motion, but it may be somewhat difficult to solve exactly an initial-value problem of the inertio-gravity wave-mean flow interaction. However, as will be shown in the next subsection, we can predict the final equilibrium state which should result from the interaction.

4.3 Final equilibrium state

We shall express a zonal-mean quantity in Eqs. (A2a-e) at the final *equilibrium* state of the wave-mean flow interaction by putting $\langle \rangle$, and assume a geostrophic relation there:

$$\langle \bar{v} \rangle_t = 0, \quad \langle \bar{v} \rangle = -\frac{1}{f} \langle \bar{p} \rangle_y. \quad (39a)$$

Considering that $\bar{u}_y = 0$ at the initial basic state, we write the potential vorticity conservation as

$$\langle \bar{v} \rangle_y = 0. \quad (39b)$$

Using (39a-b), we derive the following equations from (A2a-e) for the final equilibrium state:

$$-f \langle \bar{v} \rangle + \langle \bar{v} \rangle_z \langle \bar{w} \rangle = 0, \quad (40a)$$

$$\langle \bar{v} \rangle_y \langle \bar{v} \rangle + \langle \bar{v} \rangle_z \langle \bar{w} \rangle = 0, \quad (40b)$$

$$\langle \bar{s} \rangle + \langle \bar{p} \rangle_z = 0, \quad (40c)$$

$$\langle \bar{s} \rangle_y \langle \bar{v} \rangle + \langle \bar{s} \rangle_z \langle \bar{w} \rangle = 0, \quad (40d)$$

$$\langle \bar{v} \rangle_y + \langle \bar{w} \rangle_z = 0. \quad (40e)$$

Note that a quantity with $\langle \rangle$ is not a total quantity at the final state, so that a neutral oscillation around it can survive even in the final state but any induced unstable motions have subsided there (*cf.* Rossby, 1937, p. 244).

The final geostrophic and hydrostatic equi-

libria, (39a) and (40c), lead to the final thermal-wind equilibrium:

$$\langle \bar{S} \rangle_y = f \langle \bar{U} \rangle_z. \quad (41)$$

From (40d) and (41) we have

$$\langle \bar{W} \rangle = - \frac{f \langle \bar{U} \rangle_z}{\langle \bar{S} \rangle_z} \cdot \langle \bar{V} \rangle.$$

Comparing this with (40a), the Richardson number for the final geostrophic flow should satisfy

$$\langle J \rangle \equiv - \frac{\langle \bar{S} \rangle_z}{\langle \bar{U} \rangle_z^2} = 1. \quad (42)$$

Even if the trivial case, $\langle \bar{V} \rangle = \langle \bar{W} \rangle = 0$, is realized, (42) holds as long as $\bar{V}$ and/or $\bar{W}$ can take non-zero values during the initial and final state. Therefore, the final zonal-mean field corresponds to a marginal state of the symmetric instability [cf. (28)].* Note that the final equilibrium field in general has a spatial variation in the meridional plane, although we leave Eqs. (40a-e) unsolved.

When the initial state satisfies $J > 1$, an inertio-gravity wave induces the zonal-mean inertial and isopycnal oscillations lasting long inside the critical layer (5). The latter motion is neutral (not amplified in time), but it redistributes the zonal-mean flow so as to satisfy (39a-b), (41) and (42). If $J < 1$, the wave induces the zonal-mean inertial oscillation and the symmetric unstable motion; the latter grows drawing its energy from the basic kinetic energy until the same final equilibrium state mentioned above. Note again that $\langle \bar{U} \rangle$ seen in (42) and other equations is only the geostrophic equilibrium component of $\bar{U}$ after a very long time has passed.

It has been shown in §3.3 (b) that there are no Kelvin-Helmholtz-type unstable modes in the induced mean (second-order) field as long as the initial basic field is valid and unchanged. However, the Kelvin-Helmholtz stability of the final equilibrium state is not guaranteed. From (42) and the criterion of

Kelvin-Helmholtz stability for the final equilibrium flow ($\langle \bar{U} \rangle$, $\langle \bar{V} \rangle$),

$$\frac{1}{4} < \frac{-\langle \bar{S} \rangle_z}{\langle \bar{U} \rangle_z^2 + \langle \bar{V} \rangle_z^2} \leq 1,$$

that is,

$$0 \leq \langle \bar{V} \rangle_z^2 < 3 \langle \bar{U} \rangle_z^2. \quad (43)$$

Using again (42), we can interpret (43) as the Richardson number for the final meridional flow being larger than 1/3.

As discussed in §§2.3 and 3.4, wave breaks in the vicinity of the critical level (1) which corresponds to the Grimshaw's absorption (3). Lindzen (1981) proposed an adjustment, so-called 'saturation', for the non-inertial wave-breaking, and YT §5 showed that the breaking of an inertio-gravity wave can take place only in the vicinity of the critical level (1) inside the critical layer (5). However, the critical level absorption of inertio-gravity waves cannot be explained by such a simple wave-breaking mechanism as the non-inertial wave case. Even if a wavebreaking adjustment appears actually, the resulting flow field must satisfy (39a-b) and (41)-(43) at the final equilibrium state (cf. Appendix B). In the rotating atmosphere, if turbulence lasts long and participates in the final equilibrium state, the effective thickness of long-lasting turbulence layer is also limited just like an 'Ekman-layer' problem. Thus, the effect of eddy viscosity (or diffusivity) resulting from the inertio-gravity wavebreaking is considered to be suppressed. It is an interesting and important problem in the future how the non-inertial wavebreaking parameterization is updated on the basis of the results of this paper.

In summary, the redistribution of the zonal-mean geostrophic flow itself toward $\langle J \rangle = 1$ is essential in the mean-flow ($J > 1$ at the basic state) acceleration process of the inertio-gravity wave critical layer (5). This results from the symmetric isopycnal oscillation (25a-b) triggered by the wave near the turning level (2), but the 'absorbed' wave energy is entirely converted into the inertial oscillation (24a-b) around the critical level (1). The wave momentum, which is conserved along a non-rotating waveguide (in a inertial reference

* In YT and in this paper we neglect any external forces/heatings, which maintain the annual oscillation of the middle-atmospheric circulation (see, e.g., Matsuno, 1980, §3). Thus the final state in such a realistic problem should be reconsidered.

frame), is no longer conserved in wave propagation in the rotating atmosphere; this cannot be neglected inside the critical layer where the wave-momentum revolution approaches the earth's rotation. Hence the wave momentum is transferred into the zonal-mean inertial oscillation (rest if we observed from the inertial frame) rather than the geostrophic flow (rotating with the earth).

As mentioned in §1, our motivation to the inertio-gravity wave study is brought up by the observational evidence in the stratosphere. Although further discussions are left to subsequent papers, the inertial-oscillation generation mechanism found out in this paper may explain to produce the predominancy of quasi-inertial oscillations (see references in the second paragraph of §1) and quasi-horizontal mixing (*cf.* Kida, 1983) in the stratosphere. The wavebreaking turbulence and its vertical diffusivity are not so severe except that waves break before reaching their critical layers as modeled by Lindzen (1981) and Holton (1981) for the mesosphere.

5. Conclusions

The YT theory on Jones' (1967) critical level absorption of inertio-gravity waves is advanced. Now we can answer the two questions in §1 as follows:

(i) The valve effect (Grimshaw, 1975) and the existence of turning levels are due to the anisotropic wavefront rotations in the basic baroclinic field. It is essential that an inertio-gravity wave critical layer ($\gamma < |\omega| < \sqrt{2}f$) works totally as an absorption layer, rather than the concept that a wave infiltrates through the critical level up to the turning level.

(ii) The critical-level absorption by a factor dependent upon the meridional-zonal wavenumber ratio (Grimshaw, *op. cit.*) corresponds to wavebreaking in the vicinity of the critical level. However, this is recessive in the presence of the critical-layer absorption by a factor dependent upon the basic Richardson number (Booker and Bretherton, 1967). The latter corresponds to gravity destabilization of the wave and resultant inertial-oscillation induction inside the layer, which can be replaced by a resonant interaction between the wave

and the wave-induced zonal-mean field.

The final equilibrium state of an inertio-gravity wave critical layer is a mixed structure of the zonal-mean inertial oscillation, the zonal-mean symmetric isopycnal oscillation and the geostrophic zonal flow. The latter two motions result from redistribution of the basic kinetic energy, and the Richardson number for the final zonal flow becomes unity. Although the net absorption rate of the inertio-gravity wave critical layer is the same as that in case of non-inertial waves as mentioned in the answer (ii), the internal structures and the induced mean motions are quite different. In the mesosphere the inertio-gravity wave critical layer may be buried inside a thicker wavebreaking layer, that is, the turbulence layer of a non-inertial gravity wave. However, in considering the microstructure formation and quasi-horizontal diffusion in the stratosphere, the inertio-gravity wave critical layer should not be neglected. In a subsequent paper, the author is going to discuss the stratospheric microstructure in comparison with a series of balloon observations (*cf.* Yamanaka and Tanaka, 1984a, c; Yamanaka *et al.*, 1985).

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Appendix A

Basic equations and expansions

The basic equations for an adiabatic, inviscid and Boussinesq geofluid are written as follows (*cf.* Andrews and McIntyre, 1976,

§ 2-4):

$$\mathcal{U}_t + \mathcal{U}\mathcal{U}_y + \mathcal{C}\mathcal{V}\mathcal{U}_y + \mathcal{W}\mathcal{U}_z - f\mathcal{C}\mathcal{V} + \mathcal{P}_x = 0, \quad (\text{A1a})$$

$$\mathcal{C}\mathcal{V}_t + \mathcal{U}\mathcal{C}\mathcal{V}_x + \mathcal{C}\mathcal{V}\mathcal{C}\mathcal{V}_y + \mathcal{W}\mathcal{C}\mathcal{V}_z + f\mathcal{U} + \mathcal{P}_y = 0, \quad (\text{A1b})$$

$$\mathcal{S} + \mathcal{P}_z = 0, \quad (\text{A1c})$$

$$\mathcal{S}_t + \mathcal{U}\mathcal{S}_x + \mathcal{C}\mathcal{V}\mathcal{S}_y + \mathcal{W}\mathcal{S}_z = 0, \quad (\text{A1d})$$

$$\mathcal{U}_x + \mathcal{C}\mathcal{V}_y + \mathcal{W}_z = 0, \quad (\text{A1e})$$

where $(\mathcal{U}, \mathcal{C}\mathcal{V}, \mathcal{W}, \mathcal{S}, \mathcal{P})$ denotes the total quantities.

The zonal-mean subsystem of (A1a-e) is as follows:

$$\begin{aligned} \mathcal{U}_t + \mathcal{C}\mathcal{V}\mathcal{U}_y + \mathcal{W}\mathcal{U}_z - f\mathcal{C}\mathcal{V} \\ = -(\mathcal{U}'\mathcal{C}\mathcal{V}')_y - (\mathcal{U}'\mathcal{W}')_z, \end{aligned} \quad (\text{A2a})$$

$$\begin{aligned} \mathcal{C}\mathcal{V}_t + \mathcal{C}\mathcal{V}\mathcal{C}\mathcal{V}_y + \mathcal{W}\mathcal{C}\mathcal{V}_z + f\mathcal{U} + \mathcal{P}_y \\ = -(\mathcal{C}\mathcal{V}'\mathcal{C}\mathcal{V}')_y - (\mathcal{C}\mathcal{V}'\mathcal{W}')_z, \end{aligned} \quad (\text{A2b})$$

$$\mathcal{S} + \mathcal{P}_z = 0, \quad (\text{A2c})$$

$$\mathcal{S}_t + \mathcal{C}\mathcal{V}\mathcal{S}_y + \mathcal{W}\mathcal{S}_z = -(\mathcal{S}'\mathcal{W}')_y - (\mathcal{S}'\mathcal{C}\mathcal{V}')_z, \quad (\text{A2d})$$

$$\mathcal{C}\mathcal{V}_y + \mathcal{W}_z = 0, \quad (\text{A2e})$$

If we assume that the disturbance has a non-dimensional magnitude a , the double-underlined terms are $O(a^2)$, the asterisk terms are $O(a^4)$ and the other terms are $O(a^2)$.

The disturbance subsystem of (A1a-e) is as follows:

$$\begin{aligned} \mathcal{U}'_t + \mathcal{U}\mathcal{U}'_x + \mathcal{C}\mathcal{V}\mathcal{U}'_y + \mathcal{W}\mathcal{U}'_z + \mathcal{C}\mathcal{V}'\mathcal{U}_y \\ + \mathcal{W}'\mathcal{U}_z - f\mathcal{C}\mathcal{V}' + \mathcal{P}'_x \\ = -(\mathcal{U}'\mathcal{U}')_x - (\mathcal{U}'\mathcal{C}\mathcal{V}' - \mathcal{U}'\mathcal{C}\mathcal{V}')_y \\ - (\mathcal{U}'\mathcal{W}' - \mathcal{U}'\mathcal{W}')_z, \end{aligned} \quad (\text{A3a})$$

$$\begin{aligned} \mathcal{C}\mathcal{V}'_t + \mathcal{U}\mathcal{C}\mathcal{V}'_x + \mathcal{C}\mathcal{V}\mathcal{C}\mathcal{V}'_y + \mathcal{W}\mathcal{C}\mathcal{V}'_z + \mathcal{C}\mathcal{V}'\mathcal{C}\mathcal{V}_y \\ + \mathcal{W}'\mathcal{C}\mathcal{V}_z + f\mathcal{U}' + \mathcal{P}'_y \\ = -(\mathcal{C}\mathcal{V}'\mathcal{U}')_x - (\mathcal{C}\mathcal{V}'\mathcal{C}\mathcal{V}' - \mathcal{C}\mathcal{V}'\mathcal{C}\mathcal{V}')_y \\ - (\mathcal{C}\mathcal{V}'\mathcal{W}' - \mathcal{C}\mathcal{V}'\mathcal{W}')_z, \end{aligned} \quad (\text{A3b})$$

$$\mathcal{S}' + \mathcal{P}'_z = 0, \quad (\text{A3c})$$

$$\begin{aligned} \mathcal{S}'_t + \mathcal{U}\mathcal{S}'_x + \mathcal{C}\mathcal{V}\mathcal{S}'_y + \mathcal{W}\mathcal{S}'_z + \mathcal{C}\mathcal{V}'\mathcal{S}_y + \mathcal{W}'\mathcal{S}_z \\ = -(\mathcal{S}'\mathcal{U}')_x - (\mathcal{S}'\mathcal{C}\mathcal{V}' - \mathcal{S}'\mathcal{C}\mathcal{V}')_y \\ - (\mathcal{S}'\mathcal{W}' - \mathcal{S}'\mathcal{W}')_z, \end{aligned} \quad (\text{A3d})$$

$$\mathcal{U}'_x + \mathcal{C}\mathcal{V}'_y + \mathcal{W}'_z = 0, \quad (\text{A3e})$$

where the underlined terms are $O(a)$ and the others are smaller.

In the present study, we use the following asymptotic expansions:

$$\mathcal{U} \equiv \bar{u} + U + O(a^2), \quad \mathcal{U}' = u + O(a^3), \quad \text{etc.}, \quad (\text{A4})$$

where $\bar{u} = O(a^0)$ is of the basic field, $u = O(a)$ is of the wave field and $U = O(a^2)$ is the induced mean field.

Appendix B

Comments on the induced mean equations

Andrews (1980) solved a transient problem of Eq. (20b) using a motionless initial state ($\bar{u}=0$), a small-Rossby-number approximation ($|\Omega| \ll f$) and a channel model $[(\bar{u}w)_y \neq 0]$; then (20b-c) can be rewritten as follows:

$$(f^2\partial_z^2 + N^2\partial_y^2)\Psi = -f(\bar{u}w)_{zz}, \quad (\text{A5a})$$

$$U_t = -f\Psi_z - (\bar{u}w)_z. \quad (\text{A5b})$$

In his problem $(\bar{u}w)_z$ appears due to transience at the top of a semi-infinite upgoing inertia-gravity wavepacket. His results from (A7a-b) were fully discussed by McIntyre (1980, § 3). Note that the hyperbolic-type solutions (see Fig. 5) are degenerated in (A5a) by the small-Rossby-number approximation.

If we eliminate Ψ from (A5a-b), we have

$$(f^2\partial_z^2 + N^2\partial_y^2)U_t = -N^2\partial_y^2(\bar{u}w)_z, \quad (\text{A6})$$

which is a Boussinesq, non-dissipative version of the governing equations of Holton (1982) and of Matsuno (1982). In Holton's model, $(\bar{u}w)_z$ results from non-inertial gravity wave attenuation with breaking (Lindzen, 1981); an eddy viscosity due to the wavebreaking turbulence is also incorporated. Tanaka and Yamanaka (1985) simulate the middle-stratospheric weak wind by applying Holton's model. In Matsuno's model, $(\bar{u}w)_z$ is due to the wave dissipation during vertical wavepacket propagation in a viscous atmosphere (*viz.*, Plumb

and McEwan, 1978). Recently, Miyahara (1985) points out a possibility of planetary-wave generation by the Matsuno's mechanism, using a 2D barotropic quasi-geostrophic β -plane model and a zonally localized wave-packet. In those foregoing studies the baroclinicity of the basic field and the inertial effect on the waves were neglected or suppressed.

When we assume that Y and Z are both sufficiently small, Eqs. (19a) and (29a-b) yield an equation similar to (A5b):

$$U_t = -[(\bar{u} - f\eta)\bar{w}]_z. \quad (\text{A7})$$

Tanaka (1983a) incorporated effects of vertical density variation and eddy viscosity into Eq. (A7). He calculated the pseudomomentum flux divergence corresponding to RHS of Eq. (A7) for inertigravity waves and for non-inertial waves ($f=0$) by using the conventional WKB approximation similar to (12) and a vertical profile of $\bar{u}$. His results suggest that there may be a serious overestimation of U_t by neglect of f in particular longer-wavelength gravity waves. However, in order to possess validity of the WKB solutions, he treated only waves free from critical levels throughout the calculated basic flow range, that is, those having a phase velocity in the opposite direction to $\bar{u}$.

In addition, there are some studies treating more primitive models by numerical calculations. Walterscheid (1981) used linear wave-mean equations similar to YT(3a-e) and (19a-e) but assumed a time-dependent basic flow resembling semidiurnal tide near the mesopause. He showed a semidiurnal oscillation induced by a pair of oppositely directed waves forced at a bottom. Recently, Kida (1985) simulates successfully the mesopause circulation in a 3D GCM with external random forcings at the bottom corresponding to the lower stratosphere. Takahashi (private communication) also simulates a planetary-wave generation using a 3D primitive model with a bottom forcing. In these studies the inertial effect is implicitly incorporated, but we can no longer expect to find any critical layers of monochromatic waves in such a nonlinear switch-on problem (cf. Dickinson, 1969). It

seems that the mesopause weak wind generation may not be so dependent on the forced phase velocity, if it is small, which is consistent to the results shown in §4.2 and in Fig. 6.

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慣性重力波臨界層で生成される慣性振動と軸対称運動

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慣性重力波臨界高度の特性は、非慣性波のそれと (i) 「弁」的透過・(ii) 吸収率において異なる (Grimshaw, 1975; Yamanaka and Tanaka, 1984b)。本論文ではこの場合の「臨界層」を WKB 近似無効領域で定義し、層内の (i) (ii) と層外の特性 (非慣性波と同様) とを矛盾なく物理的に説明する。

まず、基本場が回転成層系特有の異方性 (傾圧性) を持つことから、波動場についても波面 (≡ 粒子振動面) の傾きに注目して考察する。臨界層の一方の境界面である転移高度では波面 ≡ 等密度面となり浮力復元力が 0 となる。他方の境界面では波面の傾きは Coriolis 因子と Väisälä-Brunt 振動数との比で与えられる。両面間の臨界層内においては浮力が粒子振動の復元力としては働きにくく、また波の発散する運動量は地衡風調節されにくいことがわかる。臨界高度では波面が水平のモードと等密度面より大きい傾きを持つモードが存在し得るが、それらの南北波数は互いに逆符号である。前者は局所的成層不安定となり碎波吸収を受け、後者は透過し転移高度反射の後で前者のモードに移る。以上から (i) および臨界層内の (ii) については、基本場の持つ傾圧性が波の鉛直構造に強い水平構造依存性をもたらした結果であると解釈できる。

次に、波について 2 次の帯状平均場には「慣性 (水平) 振動」的モードおよび「軸対称 (等密度面) 運動」モードが卓越することがわかるが、Lagrange 的非加速定理により場が安定かつ定常である限り臨界高度以外では生成され得ない。そこで一つの仮説として、臨界高度近傍で無限小粘性により波の時間的減衰が起こり波と共鳴の状態にある帯状平均慣性振動が残ると考える。同様に転移高度近傍では軸対称運動が残るとする。この仮説に基いてエネルギー的に調べると、波は慣性振動生成に使われ、軸対称運動は基本場帯状地衡流が減速して作られることがわかる。基本場 Richardson 数 > 1 である限り、以上の過程は臨界層内において帯状平均流の Richardson 数が 1 となるまで続き、層外では非慣性波臨界高度の場合 (Booker and Bretherton, 1967) と同じ吸収率が得られることになる。

以上から慣性重力波の吸収率 (ii) は、本論文で定義した臨界層内において碎波とは別に慣性振動・軸対称運動生成を考えれば、非慣性波の場合と同一となる。つまり慣性波特有の伝播 (i) や臨界高度碎波は、平均流減速効果に関しては余り意味を持たないと言える。しかしながら現実成層圏の鉛直渦拡散を定量的に説明する際に慣性化した重力波の碎波は考慮すべきであり、また非地衡風成分の生成は準水平拡散効果の正体を解明する上で無視し得ないものとなるであろう。

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