A Spectral Limited-area Model with Time-dependent Lateral Boundary Conditions and Its Application to a Multi-level Primitive Equation Model

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Abstract

A spectral limited-area primitive-equation model with a time-dependent lateral boundary condition is described. The model is nested in one way to a low resolution model. A spectral representation of the horizontal fields of prognostic variables is performed by introducing a modified double Fourier series. This series is composed of the usual orthogonal double Fourier series, that is suitable for prediction in the limited-area with a free-slip wall boundary condition, and a few additional bases by which the boundary condition which does not belong to the wall condition will be satisfied. Fluxes of mass, momentum, energy, etc. across the boundary can be specified using the additional bases. The proposed spectral method prescribes the amplitude of the additional bases, using the historical data from a coarse-mesh model forecast. The model predicts the amplitude of only the orthogonal bases that satisfy the wall boundary condition. A boundary relaxation technique is incorporated in order to reduce the amplitude of a spurious solution near the boundary due to ill-posed lateral boundary condition.

A one-dimensional, linear advection equation is integrated by the present spectral method and by a grid-point method. The comparison of these two methods reveals that the spectral method give far better results than those of the grid point method. This is mainly due to the improved accuracy in estimating horizontal derivatives in the spectral method *i.e.*, the computational dispersion and the systematic phase error that are unavoidable in the grid-point method are completely excluded in the spectral method.

The spectral formulation of a limited-area model is applied to the Japan Meteorological Agency (JMA) operational grid-point 12-level fine-mesh limited-area model (12L-FLM; March, 1983 –) and the forecasts of the two 12L-FLMs are compared. The spectral 12L-FLM almost perfectly duplicates the forecast of the grid-point 12L-FLM for relatively large-scale disturbances, within the resolution limit of the latter. However, the forecast of the spectral 12L-FLM is superior to the grid-point model for relatively small-scale disturbances. The compared case includes a well-organized intense typhoon in the initial field. The horizontal scale of the typhoon is almost marginal to be resolved by the grid-point 12L-FLM. The predicted typhoon by the spectral model is more realistically intense and symmetric than that by the grid-point model.

The amount of computation required by the spectral 12L-FLM, whose transform grid is the same as that of the grid-point model, is 20~30 per cent larger than the amount required by the grid-point 12L-FLM. We conclude from this that the computational economy of the spectral 12L-FLM is practically superior to the grid-point 12L-FLM, since the effective resolution of the spectral model is about 1.5 times better than the grid-point model. We plan to apply the spectral method to the next operational limited-area model in JMA.

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1. Introduction

Grid-point methods or finite-difference schemes are generally used to construct the horizontal structure for representing dynamical part of many limited-area models. Especially, finite-difference schemes are almost exclusively used in the models with time-dependent lateral boundary conditions. The exceptions may be the use of finite element methods (*e.g.*, Staniforth, 1979) and the use of semi-Lagrangian techniques in advection terms (*e.g.*, Bates, 1984).

While, the spectral method has already been widely accepted in global and hemispherical models in many NWP centers (*e.g.*, Kanamitsu *et al.*, 1983) as well as in many research centers. Spectral limited-area models with periodic or wall-boundary condition are developed for research works such as two dimensional turbulence (*e.g.*, Bennet and Haidvogel, 1983), energy transfer due to nonlinear wave-wave interactions (*e.g.*, Tanaka 1984), and bubble convection (*e.g.*, Daley and Merillees, 1971). The major reasons for the use of the spectral method are: (a) its accuracy due to complete absence of the truncation error and aliasing error associated with the finite-difference method, (b) its simplicity that a grid-point model cannot compete, and (c) its computational economy obtained by the transform method for integrating spectral form of equations (Orszag, 1974).

Although the spectral method has much advantage, it has hardly been introduced to a limited area model with time-dependent lateral boundary condition. The difficulty of introducing a spectral method is related to the formulation of the lateral boundary conditions. For example, usual orthogonal basic function series such as the spherical harmonics or the double Fourier series do not satisfy a given time-dependent lateral boundary condition. Therefore, it requires some thought in developing limited-area spectral method for time-dependent lateral boundary conditions.

The purpose of this paper is (a) to present one possible way of constructing a spectral limited-area model with time-dependent lateral boundary condition, (b) to show its practicality in applying to three-dimensional limited-area primitive-equation models, and (c) to show

various advantages when it is applied to Japan Meteorological Agency (JMA) operational grid-point 12-level fine-mesh limited-area model (12L-FLM).

The basic idea of introducing the spectral method to a limited-area model is described in section 4, including:

- the treatment of time-dependent lateral boundary conditions,
- the spectral representation of the field,
- the spectral prognostic equations corresponding to the basic equations in the physical space.

Then we describe the multi-level, spectral limited-area model, together with a brief description of the corresponding grid-point model in section 5. Various numerical results of the spectral method are discussed in sections 4.7 and 6, for the one-dimensional, linear advection model and the multi-level primitive-equation model, respectively.

2. Basic equations for the spectral limited area model

The basic equations of the spectral limited area model are written in the σ -coordinate, using the advection form, on a conformal projection map. The equations are written by introducing a map factor m . Therefore, the model can be applied to any map projection that uses one map factor m , *i.e.* the polar stereographic, the Mercator and the Lambert conformal conical projections.

The following set of prognostic variables are used:

$$u^* = \frac{u}{m}, \quad v^* = \frac{v}{m}, \quad T, \quad q, \quad \pi^* = \ln p_s \quad (2.1)$$

where p_s is the pressure at the model's surface, u and v are x and y component of wind, T is temperature, and q is specific humidity. The basic equations are

$$\begin{aligned} \frac{\partial u^*}{\partial t} = & -m^2 \left\{ u^* \frac{\partial u^*}{\partial x} + v^* \frac{\partial u^*}{\partial y} \right\} \\ & - \frac{1}{2} u^* \left\{ u^* \frac{\partial m^2}{\partial x} + v^* \frac{\partial m^2}{\partial y} \right\} \\ & - \sigma \frac{\partial u^*}{\partial \sigma} - \left\{ \frac{\partial \phi}{\partial x} + RT \frac{\partial \pi^*}{\partial x} \right\} \\ & + v^* \left\{ f + \frac{1}{2} \left(u^* \frac{\partial m^2}{\partial y} - v^* \frac{\partial m^2}{\partial x} \right) \right\} \quad (2.2) \end{aligned}$$

$$\begin{aligned}
\frac{\partial v^*}{\partial t} = & -m^2 \left\{ u^* \frac{\partial v^*}{\partial x} + v^* \frac{\partial v^*}{\partial y} \right\} \\
& - \frac{1}{2} v^* \left\{ u^* \frac{\partial m^2}{\partial x} + v^* \frac{\partial m^2}{\partial y} \right\} \\
& - \dot{\sigma} \frac{\partial v^*}{\partial \sigma} - \left\{ \frac{\partial \phi}{\partial y} + RT \frac{\partial \pi^*}{\partial y} \right\} \\
& - u^* \left\{ f + \frac{1}{2} \left(u^* \frac{\partial m^2}{\partial y} - v^* \frac{\partial m^2}{\partial x} \right) \right\} \quad (2.3)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial T}{\partial t} = & -m^2 \left\{ u^* \frac{\partial T}{\partial x} + v^* \frac{\partial T}{\partial y} \right\} \\
& - \sigma^* \dot{\sigma} \frac{\partial}{\partial \sigma} (T \sigma^{-\kappa}) \\
& + \kappa T \left\{ \frac{\partial \pi}{\partial t} + m^2 \left(u^* \frac{\partial \pi^*}{\partial x} + v^* \frac{\partial \pi^*}{\partial y} \right) \right\} \quad (2.4)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \pi^*}{\partial t} = & - \int_0^1 m^2 \left\{ \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right\} d\sigma \\
& - \int_0^1 m^2 \left\{ u^* \frac{\partial \pi^*}{\partial x} + v^* \frac{\partial \pi^*}{\partial y} \right\} d\sigma \quad (2.5)
\end{aligned}$$

$$\begin{aligned}
\dot{\sigma}_{\sigma+\Delta\sigma} = & \dot{\sigma}_{\sigma} - m^2 \int_{\sigma}^{\sigma+\Delta\sigma} \left\{ \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right\} d\sigma \\
& + u^* \frac{\partial \pi^*}{\partial x} + v^* \frac{\partial \pi^*}{\partial y} \Big|_{\sigma}^{\sigma+\Delta\sigma} - \Delta\sigma \frac{\partial \pi^*}{\partial t} \quad (2.6)
\end{aligned}$$

$$\frac{\partial \phi}{\partial \sigma^*} = -C_p T \sigma^{-\kappa}; \quad \frac{\partial \phi}{\partial \sigma} = -\frac{RT}{\sigma} \quad (2.7)$$

$$\frac{\partial q}{\partial t} = -m^2 \left\{ u^* \frac{\partial q}{\partial x} + v^* \frac{\partial q}{\partial y} \right\} - \dot{\sigma} \frac{\partial q}{\partial \sigma} \quad (2.8)$$

It is noted that in section 6 we compare the forecast performance of the proposed spectral limited-area model with a grid-point one *i.e.*, 12L-FLM (JMA current operational fine-mesh limited-area Asia model: see Electronic Computation Center JMA (1983)). The equations for the operational 12L-FLM are written in a flux form using the following set of prognostic variables:

$$u, v, \theta, q, p, \quad (2.9)$$

where θ is a potential temperature.

The differences between the forecast equations used in the 12L-FLM and the present spectral limited-area model is not essential. The

selections given by (2.1)~(2.8) are simply for reducing the total amount of numerical calculation by omitting physical processes.

3. Spectral limited area model with a wall-boundary condition

In this section we describe a spectral limited-area model with a free-slip wall-boundary condition. The model with the wall-boundary condition is not our final goal. This model will be modified in the next section in order to satisfy a time-dependent lateral boundary condition.

3.1 Spectral representation for a free-slip wall-boundary condition

The following spectral representation of the variables obviously satisfies a free slip wall boundary conditions, or strictly speaking, the mirror boundary condition:

$$\begin{aligned}
U_{Okl} = & \varepsilon_k \varepsilon_l \int_0^\pi \int_0^\pi u^* \sin k\hat{x} \cos l\hat{y} d\hat{x} d\hat{y} \\
V_{Okl} = & \varepsilon_k \varepsilon_l \int_0^\pi \int_0^\pi v^* \cos k\hat{x} \sin l\hat{y} d\hat{x} d\hat{y} \\
S_{Okl} = & \varepsilon_k \varepsilon_l \int_0^\pi \int_0^\pi T^* \cos k\hat{x} \cos l\hat{y} d\hat{x} d\hat{y} \quad (3.1)
\end{aligned}$$

$$\Pi_{Okl} = \varepsilon_k \varepsilon_l \int_0^\pi \int_0^\pi \pi^* \cos k\hat{x} \cos l\hat{y} d\hat{x} d\hat{y}$$

where $\hat{x}$ and $\hat{y}$ are horizontal coordinates normalized by the distance of the forecast domain L_x and L_y , *i.e.*,

$$\hat{x} = \frac{\pi x}{L_x}, \quad \hat{y} = \frac{\pi y}{L_y} \quad (3.2)$$

k and l are the wavenumber (integer) in x and y direction, respectively, and ε is a normalizing factor defined by

$$\varepsilon_n = \begin{cases} 1/\pi & \text{for } n=0 \\ 2/\pi & \text{for } n=1, 2, 3, \dots \end{cases}$$

By use of the normalized distance the forecast area becomes $0 \leq \hat{x}, \hat{y} \leq \pi$. The choice of sine or cosine series for different physical variables is based on the consideration of appropriate wall-boundary conditions.

In general, there are four different sets of double-Fourier bases, in the form of $\{\cos k\hat{x} \cos l\hat{y}\}$, $\{\cos k\hat{x} \sin l\hat{y}\}$, $\{\sin k\hat{x} \cos l\hat{y}\}$ and $\{\sin k\hat{x} \sin l\hat{y}\}$ respectively. The bases within a set are orthogonal to each other in the forecast re-

gion. Each set satisfies different lateral boundary conditions. All the bases of every set are eigenfunctions of the Laplacian operator in the rectangular forecast area on a conformal projection plane. This fact gives us a great advantage in solving the semi-implicit time-integration scheme (see Appendix C).

In order to simplify notations, we introduce the matrix representation for variables in the physical space and their spectral representations by $\underline{a}$ and $\underline{A}$, respectively. Then (3.1) is simply written by introducing a Fourier transform operator in the two dimensional space, $\mathcal{F}_0$, as

$$\underline{A}_0 = \mathcal{F}_0 \underline{a} \quad (3.3)$$

We also introduce the inverse operator to transform from wave to physical space, which is defined by

$$\underline{a}_0 = \mathcal{F}_0^{-1} \underline{A}_0 \quad (3.4)$$

where $\underline{a}_0$ and $\underline{a}$ are not necessarily the same: the former always satisfies the wall-boundary condition while the latter need not satisfy the condition. The operator $\mathcal{F}_0$ appears in the r.h.s. of (3.3) removes boundary values that do not satisfy the wall-boundary condition in the initial field.

There may exist fluxes of mass, momentum and energy across the boundary in the field given by $\underline{a}$. On the other hand, the field $\underline{a}_0$ completely excludes the flux across the boundary, since the wind component normal to the boundary, the gradient of T and π^* normal to the boundary given by $\underline{a}_0$ are always zero.

3.2 Spectral prognostic equations in case of the wall boundary condition

We define differential operators D_x and D_y expressed by the relations

$$\mathcal{F}_0^{-1} D_x \underline{A}_0 = \frac{\partial \underline{a}_0}{\partial x}, \quad \mathcal{F}_0^{-1} D_y \underline{A}_0 = \frac{\partial \underline{a}_0}{\partial y} \quad (3.5)$$

Application of this operator to U , for example, produces:

$$D_x U_{0kl} = \frac{\pi k}{L_x} U_{0kl}, \quad D_y U_{0kl} = -\frac{\pi l}{L_y} U_{0kl}$$

The prognostic equations in the physical space can be written in the form

$$\frac{\partial}{\partial t} \underline{a} = \underline{a}_t \left(\underline{a}, \frac{\partial}{\partial x} \underline{a}, \frac{\partial}{\partial y} \underline{a}, \frac{\partial}{\partial \sigma} \underline{a} \right) \quad (3.6)$$

where $\underline{a}_t$ corresponds to the r.h.s. of (2.2)~(2.5) and (2.8). By applying $\mathcal{F}_0$ operator to (3.6), we obtain

$$\frac{\partial}{\partial t} \underline{A}_0 = \mathcal{F}_0 \underline{a}_t (\mathcal{F}_0^{-1} \underline{A}_0, \mathcal{F}_0^{-1} D_x \underline{A}_0, \mathcal{F}_0^{-1} D_y \underline{A}_0, \delta_\sigma \mathcal{F}_0^{-1} \underline{A}_0) \quad (3.7)$$

This is the spectral prognostic equations of the spectral limited-area model with the wall-boundary conditions, where δ_σ represents the vertical differential operator in the physical space. This equation is solved by a transform technique in the horizontal, and by finite-difference schemes in the vertical.

A spectral model with the wall-boundary condition is one possible method to carry out the time integration, when there is no substantial interaction across the lateral boundary between the inside and the outside of the forecast region. Spectral hemispherical models with spherical harmonics assume the wall-boundary conditions along the equatorial plane, and, in most cases, this assumption seems sufficient for short range forecast in the extratropical region. In Appendix A, we compare two northern hemispherical barotropic primitive models: one with the usual spherical harmonics and the other with the double-Fourier series by (3.1). We will show in Appendix A that the two model produce substantially the same forecasts over the extratropical region.

4. Spectral method for a time-dependent lateral boundary condition

4.1 Spectral representation for a time dependent lateral boundary condition – Definition of the modified Fourier series expansion –

General idea of introducing the modified Fourier series

The basic equations (2.2)~(2.8) are simultaneous, first-order partial differential equations. They include the prognostic variable themselves as well as their first-order partial derivatives. The logic of solving an initial/boundary-value problem suggests that, if there is no real boundary e.g., a rigid wall boundary, we must specify not only the values but also the gradients of all prognostic variables as the initial condition. It is obvious that the usual global model, which

uses spectral representation by the spherical harmonics, satisfies this condition.

The lateral boundary of the limited area model with a time-dependent boundary condition is not a real boundary but an artificial one: the forecast area is only a partial region of the earth's atmosphere which has no lateral boundary at all. The spectral representation by the wall-boundary bases (3.1) does not allow winds across the boundary or non-zero gradients of T and π^* normal to the boundary. The wall-boundary bases are not suitable to a model with time-dependent lateral boundary condition, since the fields of the prognostic variables are either zero at the boundary or zero-gradient normal to the boundary.

We solve this problem by introducing some additional bases to each set of wall-boundary bases, *i.e.*, we define four sets of additional bases for the four sets of wall-boundary bases, respectively. The new set consists of the wall-boundary bases and the additional bases. The additional bases supplement the restricted boundary condition of the wall-boundary bases. It is noted that the wall-boundary bases and the corresponding additional bases are not orthogonal to each other since each set of the wall-boundary bases forms a complete set of orthogonal functions for the corresponding particular boundary condition. Hereafter, we call the new sets "the modified Fourier series" which will be described in this section.

One-dimensional case

We have the Fourier sine series and the Fourier cosine series for one-dimensional orthogonal bases. Neither of these does not simultaneously satisfy the non-zero value and the non-zero gradient conditions at the boundary. For the Fourier sine series, it is necessary to specify non-zero boundary values. We introduce the following sinusoidal function $f_N(\hat{x})$ which consists of two additional bases:

$$f_N(\hat{x}) = F_0 + F_{-1} \cos \hat{x} \quad (4.1)$$

Here the coefficients F_0 and F_{-1} are determined in order that $f_N = f$ at $\hat{x} = 0$ and $\hat{x} = \pi$, where $f(\hat{x})$ is an arbitrary field to be expanded. We introduce the notation F_N which represents the spectrum of the additional bases, *i.e.*, $F_N = (F_0, F_{-1})^T$.

Then, the spectral representation of $f(\hat{x})$ denoted hereafter by F , as

$$F = F_N + F_0 = F_N + \mathcal{F}_0(f(\hat{x}) - f_N(\hat{x})) \quad (4.2)$$

where the operator $\mathcal{F}_0$ corresponds to the Fourier sine transform in the x direction in this case. We call F defined by (4.2) as the spectrum of the modified Fourier sine series. The suffix 0 and N, imply that the variables and the operators with 0 and N belong to the Orthogonal (wall boundary) bases and the Non-orthogonal (additional) bases, respectively.

By the same way, we introduce a modified Fourier cosine series in which additional bases specify non-zero $\partial/\partial x$ at the boundary (this corresponds to define non-zero gradient normal to the boundary). We introduce a function which consists of two successive sine bases:

$$g_N(\hat{x}) = G_{-1} \sin \hat{x} + G_{-2} \sin 2\hat{x} \quad (4.3)$$

Here the coefficients G_{-1} and G_{-2} are determined by requiring that $\partial g_N / \partial \hat{x} = \partial g / \partial \hat{x}$ at $\hat{x} = 0$ and $\hat{x} = \pi$. The spectrum of an arbitrary function $g(\hat{x})$ by the modified Fourier cosine series is given by

$$G = G_N + G_0 = G_N + \mathcal{F}_0[g(\hat{x}) - g_N(\hat{x})] \quad (4.4)$$

where the operator $\mathcal{F}_0$ implies the Fourier cosine transform in x direction.

We introduce here the modified Fourier sine/cosine transform operator $\mathcal{F}$. Then, we simply write (4.2) and (4.4) as

$$F = F_N + F_0 = \mathcal{F}[f(\hat{x})]$$

By the definition of the modified Fourier sine/cosine series, we may call the present spectral representation "a sinusoidal-subtracted Fourier sine/cosine series expansion". This is contrasted with the polynomial-subtracted Fourier sine/cosine series expansion which was discussed by Gottlieb and Orszag (1977).

From a mathematical point of view, the additional bases introduced above reduce the Gibbs phenomenon at the boundary. If $f(\hat{x})$ is infinitely differentiable for $0 \leq \hat{x} \leq \pi$ and $f^{(2P+1)}(0) = f^{(2P+1)}(\pi) = 0$ for $p = 1, 2, 3, \dots$ then the Fourier cosine coefficient F_k approaches zero rapidly as $k \rightarrow \infty$, *i.e.*, the Fourier cosine expansion of $f(\hat{x})$ rapidly converges. However, if $f^{(2P+1)}(0) \neq 0$ and $f^{(2P+1)}(\pi) \neq 0$, then the Gibbs phenom-

enon appears at the boundary. In our modified Fourier cosine series, we subtract $f_N(\hat{x}) = f_{-1} \sin \hat{x} + f_{-2} \sin 2\hat{x}$ from $f(x)$. This partly reduces the amplitude of Gibbs phenomenon at the boundary. The same discussion can be made for the Fourier sine series expansion.

Although there is a wide selection of functional forms of the additional bases, it is desirable to select them such that they reduce the expansion error of $f(\hat{x}) - f_N(\hat{x})$ as much as possible. From practical point of view, we have the following choices:

- (a) low-order polynomials
- (b) small-wavenumber sinusoidal functions

In the present method we introduce (b). There is no theoretical reason in this selection. The practical reasons considered here are; (a) all mathematical operations and numerical calculations can be performed in wavenumber spaces, (b) the Fast Fourier Transform algorithm can be fully utilized, (c) it simplifies the model structure and offers computational economy.

Two dimensional case

In order to simplify the discussion, we introduce the notation $\cos^*$ and $\sin^*$ to show one dimensional modified Fourier bases by the following definition:

$$\begin{aligned}\cos^* k\hat{z} &= \cos k\hat{z} & (k=0, 1, 2, \dots) \\ \cos^* k\hat{z} &= \sin(-k\hat{z}) & (k=-1, -2) \\ \sin^* k\hat{z} &= \sin k\hat{z} & (k=1, 2, 3, \dots) \\ \sin^* k\hat{z} &= \cos(-k\hat{z}) & (k=0, -1)\end{aligned}\quad (4.5)$$

We have four different sets of double Fourier series in $0 \leq \hat{x}, \hat{y} \leq \pi$ by different boundary conditions in x and y direction. A given two dimensional field $f(\hat{x}, \hat{y})$ can be expanded by the four sets as follows:

$$\begin{aligned}\Sigma \Sigma F_{kl}^{CC} \cos k\hat{x} \cos l\hat{y} \\ \Sigma \Sigma F_{kl}^{CS} \cos k\hat{x} \sin l\hat{y} \\ \Sigma \Sigma F_{kl}^{SC} \sin k\hat{x} \cos l\hat{y} \\ \Sigma \Sigma F_{kl}^{SS} \sin k\hat{x} \sin l\hat{y}\end{aligned}\quad (4.6)$$

The fields given by the double Fourier spectral form (4.6) do not satisfy non-zero boundary value and non-zero gradient at the boundary simultaneously. Therefore it is impossible to represent a time dependent lateral boundary

condition by the spectral representation by (4.6).

We define the four sets of modified double Fourier spectral bases by substituting $\sin/\cos$ series of (4.6) by $\sin^*/\cos^*$ series in both the x and y directions. Using the modified bases we have four sets of two dimensional modified double Fourier spectral expansion of a given field $f(\hat{x}, \hat{y})$ as follows:

$$\begin{aligned}\Sigma \Sigma F_{kl}^{CC*} \cos^* k\hat{x} \cos^* l\hat{y} \\ \Sigma \Sigma F_{kl}^{CS*} \cos^* k\hat{x} \sin^* l\hat{y} \\ \Sigma \Sigma F_{kl}^{SC*} \sin^* k\hat{x} \cos^* l\hat{y} \\ \Sigma \Sigma F_{kl}^{SS*} \sin^* k\hat{x} \sin^* l\hat{y}\end{aligned}\quad (4.7)$$

where summations for k and l are taken followed by the definition of $\cos^*$ and $\sin^*$ given by (4.5).

Similar to the one dimensional case, the modified double Fourier bases of each set can be divided into two parts *i.e.*, the orthogonal bases and the additional bases. It is obvious that those bases which include at least one of the additional bases in one dimensional cases (*e.g.*, $\cos^*(-\hat{x})$, $\cos^*(-2\hat{x})$ or $\sin^*(-\hat{y})$) belong to the additional bases. The bases within the additional ones are not orthogonal to those within the orthogonal bases. Similar to the one dimensional case the boundary condition which can not be satisfied by the orthogonal bases are represented by the additional bases.

In order to get the unique spectral expansion of the given field $f(\hat{x}, \hat{y})$, we consider the following three boundary conditions:

- (a) boundary value
- (b) gradient normal to the boundary ($\partial f / \partial x$ or $\partial f / \partial y$)
- (c) $\partial^2 / \partial x \partial y$ at the four corner points.

We determine the amplitude of the additional bases using these boundary conditions except those which can be specified using the orthogonal bases.

Consider about the discrete case *i.e.*, $f(\hat{x}, \hat{y})$ ($0 \leq \hat{x}, \hat{y} \leq \pi$) is given by the grid-point data f_{ij} with the distances Δx and Δy where $i=1 \sim I$, $j=1 \sim J$ are the grid point indexes, $\Delta x = \pi/(I-1)$ and $\Delta y = \pi/(J-1)$. The discrete Fourier series expansion of f_{ij} by the orthogonal double Fourier bases can be uniquely defined using the bases with $k \leq I-1$ and $l \leq J-1$. We also truncate the modified Fourier

series expansion at $k=I-1$ and $l=J-1$. It is noted that this truncation is used only in the spectral representation of a given field by the modified Fourier series. As is described later we introduce the other wavenumber truncation (see section 4.3). Then the number of the additional bases of each set becomes as follows:

$$C-C^*: 2(I+J)+4$$

$$C-S^*: 2(I+J)$$

$$S-C^*: 2(I+J)$$

$$S-S^*: 2(I+J)-4$$

where $C-C^*$ represents $\{\cos^* k\hat{x} \cos^* l\hat{y}\}$ bases, and $C-S^*$, $S-C^*$ and $S-S^*$ represents the other three bases by the similar way.

We define the amplitude of the above additional bases using the following boundary conditions which can not be given by the orthogonal bases:

$C-C^*$: condition (b) and condition (c)

$C-S^*$: condition (a) at $j=1$ and J , and condition (b) at $i=1$ and I

$S-C^*$: condition (a) at $i=1$ and I , and condition (b) at $j=1$ and J

$S-S^*$: condition (a)

It is obvious that the number of the above boundary conditions and the number of the additional bases are the same, therefore we have the unique spectrum of the additional bases.

We denote the spectrum of the additional bases as F_N and the field given by the F_N as f_N , then the spectrum of the orthogonal bases F_0 can be obtained using the ordinary double Fourier transform as

$$F_0 = \mathcal{F}_0(f - f_N)$$

where the operator $\mathcal{F}_0$ denotes the double Fourier series expansion. Finally we have the modified double Fourier spectrum F as

$$F = F_0 + F_N$$

Hereafter we denote these modified double Fourier transform of $f(x, y)$ using the operator $\mathcal{F}$ as

$$F = F_0 + F_N = \mathcal{F}f(\hat{x}, \hat{y}) \quad (4.8)$$

We define the inverse operator of $\mathcal{F}$ by $\mathcal{F}^{-1}$. The inverse operator for $C-C^*$ series can be written as

$$\mathcal{F}^{-1}F^{C-C^*} = \sum_k \sum_l F_{kl}^{C-C^*} \cos^* k\hat{x} \cos^* l\hat{y} \quad (4.9)$$

Using the operator $\mathcal{F}$ we have the spectral representation of the prognostic variables of the basic equation (2.2)~(2.8) which is written formally as

$$\underline{A} = \underline{A}_0 + \underline{A}_N = \mathcal{F}\underline{a}(\hat{x}, \hat{y}) \quad (4.10)$$

where

$$\underline{a} = (u^*, v^*, T, \pi^*, q)^T \quad (4.11)$$

$$\underline{A} = (U, V, S, \Pi, Q)^T$$

The orthogonal bases of each prognostic variable is selected in order to satisfy the wall-boundary condition, i.e.,

$$C-C^*: T, \pi^*$$

$$C-S^*: v^*$$

$$S-C^*: u^*$$

$$S-S^*: q$$

$$(4.12)$$

It is noted that the wall-boundary condition suggests that q should be expanded using $C-C^*$ series. However, in the adiabatic part of the model q is only passively advected quantity and no feed back works to the dynamics of the model atmosphere. Therefore we may apply any set of the four series for q . The selection of the bases for q in (4.12) is introduced only by economical advantage for reducing computation in the model.

4.2 The spectral prognostic equation for a time-dependent lateral boundary condition

Lateral boundary condition

In order to derive the spectral prognostic equation of the limited-area model using the modified double Fourier series, we introduce the following lateral boundary condition by prescribing the spectrum of the additional bases $\underline{A}_N$. In other words, we prescribe the following quantities at the boundary

— boundary value: u^* at $\hat{x}=0$ and $\hat{x}=\pi$, v^* at $\hat{y}=0$ and $\hat{y}=\pi$, and q

— gradient normal to the boundary: T, π^*

Strictly speaking we specify $\partial^2/\partial x \partial y$ at the four corner points for T and π^* . When we specify these quantities to be always zero, the spectral limited area model becomes the model with free slip wall boundary condition (it specifies zero

boundary for q). This implies that non-zero fluxes, *i.e.*, mass, momentum, heat, water vapor, energy, etc., across the lateral boundary can only be specified by non-zero spectrum of the additional bases $\underline{A}_N$. The non-zero $\underline{A}_N$ will support a time dependent lateral boundary in our spectral limited area model. The other quantities at the boundary such as u^* at $\hat{y}=0$ and $\hat{y}=\pi$ are all predicted in the model.

However, the above boundary condition does not satisfy the so-called well-posedness condition: either under-specification or over-specification may occur. Spurious solution due to over/under-specification may appear in the model forecast. In order to reduce the amplitude of such spurious solution we introduce a lateral boundary relaxation technique which will be described later.

Before starting the time-integration, we prepare $\underline{A}_N$ using the historical data from an appropriate coarse-mesh model forecast. Hereafter, the coarse-mesh model forecasts are distinguished from those of the limited area model by $\sim$ mark.

The modified double-Fourier spectral representation of the prognostic variable $\underline{a}(t)$, under the present boundary condition, is written by

$$\underline{A}(t) = \widetilde{\underline{A}}_N(t) + \mathcal{F}_0(\underline{a}(t) + \mathcal{F}^{-1}\widetilde{\underline{A}}_N(t)) \quad (4.13)$$

where $\widetilde{\underline{A}}_N(t)$ is the prescribed value at the boundaries, which are known through the forecast period. Considering the definition of the modified Fourier series expression, *i.e.*, $\underline{A} = \underline{A}_N + \underline{A}_0$, we have

$$\underline{A}_0(t) = \mathcal{F}_0(\underline{a}(t) - \mathcal{F}^{-1}\widetilde{\underline{A}}_N(t)) \quad (4.14)$$

Thus $\underline{A}_0(t)$, the orthogonal part of the spectral representation of $\underline{a}(t)$ can be determined by (4.14). It is noted that $\underline{a}(t)$ in (4.14) and $\underline{a}^* = \mathcal{F}^{-1}(\underline{A}_N + \underline{A}_0)$ are not necessarily the same.

Spectral prognostic equation

We can easily derive the spectral prognostic equation from (4.14). Differentiating the eqs. (4.14) with respect to t , we have

$$\frac{\partial}{\partial t} \underline{A}_0 = \mathcal{F}_0 \left(\frac{\partial}{\partial t} \underline{a} - \mathcal{F}^{-1} \frac{\partial}{\partial t} \widetilde{\underline{A}}_N \right) \quad (4.15)$$

where $\partial \underline{a}(t)/\partial t$ represents time tendencies of all prognostic variables in the physical space, and it

is defined by the set of basic equations (2.2)~(2.8). As was described in the previous section, the basic equations can also be written in the following form:

$$\frac{\partial}{\partial t} \underline{a} = \underline{a}_t(\mathcal{F}^{-1}\underline{A}, \mathcal{F}^{-1}D_x\underline{A}, \mathcal{F}^{-1}D_y\underline{A}, \delta_\sigma \mathcal{F}^{-1}\underline{A}) \quad (4.16)$$

where D_x and D_y are differential operators with respect to x and y for the modified double Fourier spectrum and $\underline{A} = \underline{A}_N + \underline{A}_0$. Substituting (4.16) into (4.15), we have

$$\frac{\partial}{\partial t} \underline{A}_0 = \mathcal{F}_0 \left[\underline{a}_t(\mathcal{F}^{-1}\underline{A}, \mathcal{F}^{-1}D_x\underline{A}, \mathcal{F}^{-1}D_y\underline{A}, \delta_\sigma \mathcal{F}^{-1}\underline{A}) - \mathcal{F}^{-1} \frac{\partial}{\partial t} \widetilde{\underline{A}}_N \right] \quad (4.17)$$

This is the spectral prognostic equations of the limited-area model which satisfies the lateral boundary condition prescribed by $\widetilde{\underline{A}}_N$. It is obvious that (4.17) reduces to the spectral prognostic equation with wall-boundary condition (3.7) when $\widetilde{\underline{A}}_N = 0$.

Using the prescribed $\widetilde{\underline{A}}_N(t)$ and the spectral prognostic equation (4.17), we can integrate the equation in time and obtain the time evolution of $\underline{A}_0(t)$. The fields of the prognostic variables in the physical space can be obtained from the following conversion.

$$\underline{a}(t) = \mathcal{F}^{-1}[\underline{A}_0(t) + \widetilde{\underline{A}}_N(t)] \quad (4.18)$$

4.3 Transforms and aliasing

Similar to conventional spectral methods, we introduce the transform method to evaluate the r.h.s. of the spectral prognostic equation (4.17) using grid-point values of $\underline{a}$, $\partial \underline{a}/\partial x$ and $\partial \underline{a}/\partial y$. We define grid points with a constant grid distance $\Delta x = \Delta y = d$ on a conformal projection map. The total number of grid points in x and y directions are $I = L_x/d + 1$ and $J = L_y/d + 1$, respectively. It is noted that physical processes are also evaluated using this grid in the multi-level spectral limited area model described in the next section.

Aliasing may occur in evaluating nonlinear terms in the double Fourier transform operation $\mathcal{F}_0(\underline{a}_t)$ when the number of grid points I and J are not sufficient. Let us discuss the Fourier transform along the x axis. There are $2(I-1)$ number of grid points within $0 \leq \hat{x} \leq 2\pi$.

Let the spectral representation of the square of the map factor, m^2 , be truncated at K_m , and those of the prognostic variables and the coriolis parameter be truncated at wave number K . The quadratic nonlinear terms, e.g., $m^2 u^* \partial u^* / \partial x$ will yield waves with maximum wave number of $2K + K_m$. In order to eliminate aliasing on those quadratic nonlinear terms when $\mathcal{F}_0(\underline{q}_t)$ operator is applied, the following inequality relation must hold.

$$2(I-1) > (2K + K_m) + K = 3K + K_m \quad (4.19)$$

The same condition is also obtained in the y direction,

$$2(J-1) > 3L + L_m \quad (4.20)$$

where L_m is the maximum wavenumber for the spectral representation of m^2 in y direction. Finally we have

$$\begin{aligned} I &> (3K + K_m)/2 + 1 \\ J &> (3L + L_m)/2 + 1 \end{aligned} \quad (4.21)$$

There exist, however, some cubic nonlinear terms in the basic equations (2.2)~(2.8) for example;

$$\kappa T \left[m^2 \left(u^* \frac{\partial \pi^*}{\partial x} + v^* \frac{\partial \pi^*}{\partial y} \right) \right], \sigma \frac{\partial u^*}{\partial \sigma}, \text{ etc.} \quad (4.22)$$

Although the condition (4.21) will not provide aliasing-free transforms for those cubic terms, we specify the number of grid points I and J by (4.21). This assumption is commonly made in the global spectral models, and the effect of aliasing of the cubic terms is shown to be insignificant (Bourke *et al.*, 1977).

4.4 Wavenumber truncation

The spectral expression of the variable $\underline{a}(\hat{x}, \hat{y})$ by the modified double Fourier series can be written using complex notation as

$$\underline{a} = \sum \underline{A}_{kl} \exp(ik\hat{x} + il\hat{y}) \quad (4.23)$$

where $i^2 = -1$, k and l are wavenumbers (integer) normalized by the distance of the forecast domain L_x and L_y , and k^* and l^* are the unnormalized wave number which are written as

$$\begin{aligned} k^* &= \pi k / L_x \\ l^* &= \pi l / L_y \end{aligned} \quad (4.24)$$

We truncate waves with a total wavenumber in the unnormalized space in such a way that $(k^{*2} +$

$l^{*2})^{1/2}$ is equal to a given maximum wavenumber along x axis $K^* = \pi K / L_x$. This truncation implies that we predict the amplitude of the Fourier bases (orthogonal bases) which satisfy

$$k^2 + (L_x / L_y)^2 l^2 \leq K^2 \quad (4.25)$$

This is "elliptic truncation" in the normalized wavenumber space. We refer (4.25) as the elliptic truncation at wavenumber K , and write it as E46 truncation when $K=46$. It should be noted that (4.25) does not mean "elliptic truncation" or anisotropic truncation, rather it implies an isotropic (homogeneous) truncation" in the projected space (not in the actual space). It is noted that we also truncate the amplitude of the additional bases by their total wave number in the model.

4.5 Boundary relaxation

The proposed lateral boundary condition provides basically "reflective condition" since it is built up on a model with free-slip lateral boundary condition. Moreover, we prescribe either the boundary value or the gradient normal to the boundary of the prognostic variable without considering the well-posedness condition. In order to reduce the amplitude of spurious solutions due to ill-posed boundary condition, we introduce a boundary relaxation technique described as follows:

$$\frac{\partial \underline{A}_0}{\partial t} = \dots + \mathcal{F}_0[\alpha \nabla^2 \mathcal{F}^{-1}(\underline{A} - \tilde{\underline{A}}) - \beta \mathcal{F}^{-1}(\underline{A} - \tilde{\underline{A}})] \quad (4.26)$$

where $\nabla^2 \mathcal{F}^{-1} \equiv \mathcal{F}^{-1}(D_x D_x + D_y D_y)$, and α, β are functions of i_B which is length from the nearest boundary and $\nabla^2 \equiv \partial^2 / \partial x^2 + \partial^2 / \partial y^2$. In the multi-level spectral limited area model being discussed later, we define α and β by

$$\begin{aligned} \left[\frac{\alpha(i_B)}{\beta(i_B)} \right] &= \begin{cases} \frac{1}{2} \left[\frac{\alpha^*}{\beta^*} \right] \left(1 + \cos \frac{2\pi i_B}{I_B} \right) & (0 \leq i_B \leq I_B) \\ 0 & (I_B < i_B) \end{cases} \end{aligned} \quad (4.27)$$

where I_B is the width of the relaxation area measured by the number of the grid lengths. We may refer α^* and β^* as the relaxation parameters of red and white spectrum, respectively. This tendency is added to the spectral prognostic equations.

4.6 Comparison with a grid-point method

It is interesting to compare the present method with grid-point method. Let us try to interpret the spectral method as one of the grid-point methods. If we imagine that the time integration starts not from the spectral representation, from $\underline{A}(t)$, but from the corresponding grid point data $\underline{a}^*(t)$. In this case the time integration procedure will be expressed by the following four stages:

- (1) We convert the grid point value $\underline{a}^*(t)$ to the spectral form $\underline{A}(t)$, using the prescribed lateral boundary condition given by $\underline{A}_N(t)$. Then, we apply the wave number truncation on $\underline{A}_0(t)$.
- (2) We calculate the grid point value of $\underline{a}$, $\partial \underline{a} / \partial x$, $\partial \underline{a} / \partial y$ by applying $\mathcal{F}^{-1}$ operator on $\underline{A}$, $D_x \underline{A}$ and $D_y \underline{A}$. Here, $\underline{a}^*$ and $\underline{a}$ are not necessarily the same.
- (3) We calculate time tendencies of the prognostic variables, $\partial \underline{a} / \partial t$, using $\underline{a}$, $\partial \underline{a} / \partial x$ and $\partial \underline{a} / \partial y$. Here, the present spectral method evaluates $\partial \underline{a} / \partial x$ and $\partial \underline{a} / \partial y$ analytically by fitting analytical functions that satisfy the given boundary conditions. This analytical evaluation is the fundamental difference from the usual grid-point model which calculates them by finite-difference methods.
- (4) We calculate the next time-step values of $\underline{a}^*$. Then, return back to the stage (1) until we complete the forecast.

It is obvious that these prediction procedures are exactly the same as those derived from the spectral prognostic equations (4.17) operated by $\mathcal{F}_0^{-1}$ on both sides. Comparing the spectral method with the usual grid-point method, we can point out the following differences:

- The way of estimating horizontal derivatives
- Mathematical formulation of the lateral boundary condition

This comparison reveals that there is no substantial differences between the grid point and the spectral models from the physical point of view. Differences may appear from the mathematical representation of the basic equations using a finite number of freedom; the total number of grid points for the grid-point model, and the total number of bases (waves) in the

spectral model.

The main purpose of developing the present spectral model is to improve the accuracy of estimating $\partial \underline{a} / \partial x$ and $\partial \underline{a} / \partial y$ by introducing the modified double Fourier series representation of $\underline{a}$. There are various possible ways of selecting analytical functions in order to satisfy the above requirement, *e.g.*, orthogonal polynomials and basic spline function. however, when we consider the computational economy, our selection of the modified double Fourier series seems quite desirable.

4.7 One-dimensional example

Basic equation and model description

Since a test of the scheme using a linearized model gives us clear insight into the performance of the scheme in many cases, the present spectral method is applied to a one-dimensional advection equation written as

$$\frac{\partial f}{\partial t} + U \frac{\partial f}{\partial x} = 0 \quad (4.28)$$

where U is a constant. There are two selections of the spectral representation of $f(x, t)$: one is those by the modified sine series and the other by the modified cosine series.

$$\sum F_k^S \sin^* k \hat{x} = F_0^S + F_{-1}^S \cos k \hat{x} + \sum F_k^S \sin k \hat{x} \quad (4.29)$$

$$\sum F_k^C \cos^* k \hat{x} = F_{-1}^C + F_{-2}^C \sin k \hat{x} + \sum F_k^C \cos k \hat{x} \quad (4.30)$$

In the present experiment, the forecast area ($0 \leq \hat{x} \leq \pi$) is covered by 73 grid points with $d=90\text{km}$ grid length. The size of the forecast area L_x is thus 6,480km. We introduce the wave-number truncation at $K=46$. This truncation, as well as the number of grid points in x direction, is just the same as is used in the multi-level spectral limited area model described in the next section. U is set to 25 m/sec (advection speed of one grid distance per one hour). Time integration is performed by the leapfrog scheme (except for the first time step) with time increment of 360 seconds. Time truncation error is expected to be negligibly small because $U \Delta t / \Delta x = 1/10$.

The lateral boundary relaxation scheme de-

finied by (4.26) is applied using the following values:

$$\begin{aligned} I_B &= 7 \\ &\quad (\text{width of the relaxation area}) \\ \alpha^* &= [3\Delta t (\pi K/L_x)^2]^{-1} \\ &\quad (\text{relax. const. with red spectrum}) \\ \beta^* &= 1/4 \text{ hour}^{-1} \\ &\quad (\text{relax. const. with white spectrum}) \end{aligned} \quad (4.31)$$

The leapfrog scheme is used with the Asselin's (1972) time filter with $\nu = 0.001$.

Experiments of zero lateral boundary condition

Zero boundary value of f and $\partial f/\partial x$ at the boundary are assumed in this experiments. This means that zero flux of f across the boundary is prescribed by the coarse mesh model forecast. The zero boundary condition is given by prescribing zero value of the amplitude of the additional bases

Initial field of $f(x)$ is expressed by a mathematical pattern, a solitary wave expressed by one cycle cosine with the width of L , are placed at $x=24d$ (about $1/3$ distance of the forecast area from the upstream boundary):

$$f(x, 0) = \begin{cases} \frac{1}{2} \left\{ 1 + \cos \frac{2\pi(x-x_0)}{L} \right\}, & |x-x_0| \leq L/2 \\ 0, & |x-x_0| > L/2 \end{cases} \quad (4.32)$$

A grid model with the fourth-order differencing scheme is constructed for a reference. The lateral boundary of the grid model is placed at $x=-50d$ and $x=(I+50)d$ and the boundary relaxation given by (4.31) is also used *i.e.*, the forecast area is expanded to both sides of the boundary with $50d$ length therefore influence of the lateral boundary might be ignored in the forecast of the region $0 \leq x \leq 72d$. The forecasts of the spectral model are compared with the analytical solution as well as with the grid model.

Fig. 1 shows the forecasts by the spectral sine model (left panel), the spectral cosine model (middle panel) and the grid-point model (right panel). The forecasts of two different solitary waves are shown in the top and the bottom figures: the width of the solitary waves are $6d$ and $10d$. In the figure the forecast and the

corresponding analytical solution are shown by the thick and thin solid lines, respectively, for every 15 hour interval from initial to 60 hour forecast. At 60 hour from the initial the wave is completely out of the downstream boundary in the analytical solution.

For the solitary wave with $6d$ width, the forecasts of both the spectral sine and cosine models are very close to the corresponding analytical solutions. The forecasts of the $10d$ -width wave almost coincide with the analytical solutions. The spectral model forecasts are slightly modified in the relaxation area when the wave enter into the area. However, the error is confined in the relaxation area, and does not penetrate into the outside. At $t=60$ hour the solitary waves predicted by the spectral models have almost completely disappeared, and we observe only small-amplitude wave pattern left behind in the $6d$ solitary wave forecasts.

In contrast to the forecasts of the spectral models, however, large-amplitude, upstream noise due to computational dispersion as well as systematic phase error is observed in the $6d$ solitary wave forecast of the grid point model. The initial steepness of the solitary wave cannot be maintained in the $6d$ wave forecast. The computational upstream dispersion still exists in the $10d$ wave forecast although its amplitude reduces considerably. When the 2nd-order accuracy horizontal scheme is applied to the grid model (we do not show the results of it), the numerical solutions of it become worse than the forecast of the model with the 4th-order scheme shown in Fig. 1.

Experiments of a non-zero lateral boundary conditions

The same spectral model for the one dimensional limited area is tested for the case in which a sinusoidal wave with wave length of L passes through the forecast area with a constant phase speed U . This experiment is performed in order to see the response of the spectral method to the lateral boundary forcing prescribed by $\tilde{F}_N$. We compare the forecasts of the two spectral models (sine and cosine models) with the corresponding analytical solution, and not with those of the grid model. The initial and the boundary condition of the experiment is taken

Fig. 1 The comparison of the numerical and analytical solutions of the linear advection model with a constant phase speed U . Solitary wave with the width of L is placed in the initial condition. The numerical solutions (thick solid line) of the spectral limited area model with the modified sine and cosine series and the grid point model with 4th order scheme are shown in the left, center and right columns, respectively for $L=6d$ and $L=10d$ cases. The analytical solutions are superposed by the thin solid line. Each panel shows the forecasts at every 15 hour interval from top (initial) to bottom (60 hours). Zero lateral boundary condition is assumed in this experiment.

from the following analytical solution of the linear advection equation (4.28).

$$f(x, t) = f_0 \cos \left[\frac{2\pi}{L} (x - Ut) + \phi_0 \right] \quad (4.33)$$

The results are shown in Fig. 2. The sine and cosine version model forecasts are shown in the left and right columns, and the numerical and the analytical solutions are shown by the thick and thin solid lines, respectively. The forecasts of the wave lengths, $L=2,000\text{km}$ ($\sim 22.2d$) and $10,000\text{km}$ ($\sim 111d$), are shown in the top and the bottom figures, respectively.

No error is detected for both the $2,000\text{km}$ and $10,000\text{km}$ wave cases in Fig. 2, since the thin and the thick curves are completely overlapped each other. We see from Fig. 2 that the sinusoidal waves pass through the upstream and downstream boundary without any substantial modification over the whole forecast area in our spectral limited-area model.

Remarks on the one-dimensional experiments

The major results obtained in these one-dimensional experiments are that the use of the modified Fourier series and the boundary relaxation scheme work very well for the limited-area

model with time-dependent boundary conditions. Other advantages of the scheme are

- very small systematic phase error, and
- negligibly small computational dispersion,

which are mainly due to the analytical estimation of the derivatives. We may conclude from these experiments that the spectral method is superior to the grid-point method.

5. Application to multi-level PE model

The proposed spectral formulation of a limited-area model is applied to the JMA operational grid-point 12-level fine-mesh limited-area model (12L-FLM). The spectral version of the 12L-FLM is outlined in this section.

5.1 Spectral prediction equations

The set of prediction equations in the physical space are shown in (2.2)–(2.8). We introduce here the column matrix with K elements by the notation $\underline{X}$, where X and K represent the prognostic/diagnostic variables and the number of the vertical levels of the model, respectively. Using this matrix notation and the notations defined in the previous sections, we specify the spectral representation of the prognostic variables as

Fig. 2 Same as Fig. 1 except for the numerical solutions of the spectral limited area model when sinusoidal wave with wavelength of L passing through the forecast area with a constant phase speed U .

$$\begin{pmatrix} \underline{U} \\ \underline{V} \\ \underline{S} \\ \underline{Q} \\ \underline{\Pi} \end{pmatrix} = \begin{pmatrix} \tilde{\underline{U}}_N \\ \tilde{\underline{V}}_N \\ \tilde{\underline{S}}_N \\ \tilde{\underline{Q}}_N \\ \tilde{\underline{\Pi}}_N \end{pmatrix} + \begin{pmatrix} \underline{U}_0 \\ \underline{V}_0 \\ \underline{S}_0 \\ \underline{Q}_0 \\ \underline{\Pi}_0 \end{pmatrix} = \tilde{\underline{A}}_N + \underline{A}_0 \equiv \underline{A} \quad (5.1)$$

where $\tilde{\underline{A}}_N$ is the prescribed lateral boundary condition using forecast of a coarse mech model. Prognostic variables in the physical space are defined by the following conversion

$$(1, \partial/\partial x, \partial/\partial y) \begin{pmatrix} \underline{u}^* \\ \underline{v}^* \\ \underline{T} \\ \underline{Q} \\ \underline{\pi}^* \end{pmatrix} = \mathcal{F}^{-1}(1, D_x, D_y) \begin{pmatrix} \tilde{\underline{U}}_N + \underline{U}_0 \\ \tilde{\underline{V}}_N + \underline{V}_0 \\ \tilde{\underline{S}}_N + \underline{S}_0 \\ \tilde{\underline{Q}}_N + \underline{Q}_0 \\ \tilde{\underline{\Pi}}_N + \underline{\Pi}_0 \end{pmatrix} \quad (5.2)$$

We finally obtain the spectral prediction equations:

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} \underline{U}_0 \\ \underline{V}_0 \\ \underline{S}_0 \\ \underline{Q}_0 \\ \underline{\Pi}_0 \end{pmatrix} \\ = \mathcal{F}_0 \frac{\partial}{\partial t} \begin{pmatrix} \underline{u}^* \\ \underline{v}^* \\ \underline{T} \\ \underline{q} \\ \underline{\pi}^* \end{pmatrix} - \mathcal{F}_0 \mathcal{F}^{-1} \frac{\partial}{\partial t} \begin{pmatrix} \tilde{\underline{U}}_N \\ \tilde{\underline{V}}_N \\ \tilde{\underline{S}}_N \\ \tilde{\underline{Q}}_N \\ \tilde{\underline{\Pi}}_N \end{pmatrix} \end{aligned} \quad (5.3)$$

These equations are re-written as

$$\frac{\partial \underline{A}}{\partial t} = \mathcal{F}_0 \frac{\partial \underline{a}}{\partial t} - \mathcal{F}_0 \mathcal{F}^{-1} \frac{\partial \underline{A}_N}{\partial t} \quad (5.4)$$

where $\partial \underline{a} / \partial t$ is obtained by substituting (5.2) into the basic equations in the physical space (2.2)~(2.8). The vertical derivatives of $\underline{a}$ in the basic equations are evaluated by the finite-difference schemes described in the next section.

In addition to the variables given by (5.2) the basic equations include the parameters,

$f, m^2, \partial m^2/\partial x$ and $\partial m^2/\partial y$. In the present model, f and m^2 are converted to the spectral representation F and M_2 , respectively, then are converted back to the values at grid points. Here m^2 is represented by the modified Fourier $\{\cos^*k\hat{x}\cos^*l\hat{y}\}$ series and f by the $\{\sin^*k\hat{x}\sin^*l\hat{y}\}$ series. We truncate F and M_2 accordingly.

5.2 Outline of the spectral 12L-FLM

(1) Forecast area and horizontal grid

The forecast area of the spectral model is the same as the grid-point 12L-FLM (JMA operational limited area Asia model). It is shown in Fig. 3 together with the topography of the grid-point 12L-FLM. Both the grid-point and the spectral 12L-FLMs introduce the same horizontal grid net of 73×55 ($I=73, J=55$) with the grid distance of $d=127\text{km}$ at 60N . The spectral 12L-FLM uses the horizontal grid in order to calculate the nonlinear terms by the transform method as well as to calculate physical processes.

(2) Vertical levels

The vertical structure of the model is shown in Fig. 4. The odd numbered levels in Fig. 4 are defined at $p=1000, 960, 900, 800, \dots, 300, 200, 100, 50$ and 0 hPa provided $P_s=1000\text{hPa}$. The vertical difference scheme is based on Arakawa and Lamb (1977). We describe the vertical difference scheme in Appendix B.

(3) Time integration scheme

The semi-implicit scheme is introduced in

Fig. 3 Forecast area and the topography (m) of the 12L-FLM.

Fig. 4 Vertical grid structure of the grid point 12L-FLM. The variables u, v, θ are substituted by u^*, v^*, T in the spectral 12L-FLM.

order to reduce total amount of calculation. We give a detailed description of the semi-implicit formulation in Appendix C.

(4) Wavenumber truncation

The “elliptic tuncation” at wavenumber 46 (E46: the shortest resolvable wavelength by this truncation is 320km at 30N and 400km at 60N) is applied to all prognostic variables. By this truncation the wavenumber is restricted to $k\leq 46$, and $l\leq 34$. The map factor m^2 is truncated at $k=5$ and $l=5$, i.e., it may be called as “square truncation”. This truncation keeps the relative error of map factor less than 10^{-3} , and offers good economical advantage. It should be noted that the present selection ($I=73, J=55, K=46, L=34, K_M=L_M=5$) satisfies the condition (4.21) i.e., no aliasing error occur for quadratic nonlinear terms.

(5) Physical processes

The physical processes included in the spectral 12L-FLM are the same as those in the grid point 12L-FLM, i.e.,

- horizontal diffusion by linear fourth-order Laplacian
- surface fluxes of momentum, heat and moisture over sea, and fluxes of momentum over land
- vertical diffusion: eddy viscosity coefficients being assumed to be a function of

vertical shear and stability

- convective adjustment similar to that proposed by Gadd and Keers (1970)
- topography shown in Fig. 3.

No radiative process is included. They are calculated on the 73×55 grid points on which the transform method is applied.

The topography of the spectral model is obtained by transforming the topography of the grid model shown in Fig. 3 by $\cos^* k \hat{x} \cos^* l \hat{y}$ series and applying wave number truncation and weak smoothing. Thus the topography of the spectral model is slightly different from that of the grid model.

(6) Initial field

Initial field of the grid point 12L-FLM is transformed to the modified double Fourier series. Note that the prognostic variables of the spectral and the grid point 12L-FLMs are (u^*, v^*, T, q, π^*) and (u, v, θ, q, p_s) , respectively. Therefore after converting the prognostic variables we apply the modified double Fourier transform.

(7) Boundary relaxation

Boundary relaxation (4.26) and (4.27) is applied using the following parameters which is the same as used in the linear advection experiments.

$$I_B = 7, \alpha^* = 1/[3\Delta t(\pi K/L_x)^2],$$

$$\beta^* = 1/4 \text{ hour}^{-1}, K = 46$$

where I_B is the width of the boundary relaxation area, and K represents wavenumber truncation. The damping rate at the boundary which is measured by e -folding time defined by α^* is $3\Delta t$ (Δt changes from 7 to 15 minutes as a function of the maximum wind speed in the model) for the wave with the maximum wave number and by β^* 4 hours which does not depend on the wavenumber. The lateral boundary data $\hat{u}$ are linearly interpolated from 6 hour interval forecasts of the JMA operational 12 level northern hemispherical spectral model.

6. Comparison between the spectral and the grid-point models

The forecasts of the spectral model are compared with the corresponding grid-point model for several situations to see overall performances

of the present limited-area spectral method with time-dependent boundary conditions. Before showing the forecast comparison we outline the grid point model which is now used operationally at JMA.

6.1 Outline of the grid-point 12L-FLM

Forecast area and vertical grid structure of the grid-point 12L-FLM are the same as those of the spectral 12L-FLM which is described in the previous section. The dry, dynamical part of the model is constructed following the Arakawa schemes (Arakawa and Lamb, 1977) with some modifications. The horizontal grid structure of the model is the B-type staggered system by Arakawa's classification. The momentum advection in the horizontal is evaluated by the fourth-order Arakawa Jacobian, while the other horizontal differentials are evaluated by second-order schemes. The economical explicit time-integration scheme proposed by Tatsumi (1983) is used for the time integration. The model is designed to conserve total energy, momentum, potential temperature and the square of potential temperature when no diabatic process and no flux across the boundary exist.

The grid-point model is nested in one way in the JMA operational 12 level northern hemispherical spectral model by the formulation originally proposed by Hovermale (personal communication) which is same as the boundary relaxation method described in section 4.3 with some modification for $\alpha(i_B)$ and $\beta(i_B)$.

The physical processes of the spectral and the grid-point 12L-FLMs are substantially the same, however there are some difference between them: the grid-point 12L-FLM evaluates the physical processes using the B-type staggered grid, while the spectral model does using the simplest non-staggered A-type grid since the grid used in transform method reduces to A-type.

The objectively analyzed fields at p -level are interpolated to the σ -level of the model. No special initialization scheme is used in the grid 12L-FLM. Further details of the operational 12L-FLM may be found in Electronic Computation Center (1983).

6.2 Case 1 – dry model –

The experiment described here include only

horizontal diffusion and all the other physical processes are excluded. The diffusion coefficient of the spectral 12L-FLM is a little smaller than that of the grid point 12L-FLM.

Fig. 5 shows the 36 hour forecast of the 500hPa geopotential height and vorticity (left column) and the sea level pressure (right column). Initial state is chosen as May 5 00Z, 1984. The forecasts of the grid model and the spectral model are shown in the upper and the lower panels. Small scales are dominating in the spectral model forecast presumably due to the different horizontal diffusion effect.

Forecast patterns of the sea level pressure of the two models (Fig. 5) are almost identical except for small scale features. This is also observed for geopotential height and vorticity at 500hPa. However, if we carefully look into the

detailed feature of the 500hPa vorticity forecast, the phase differences of the small-scale vorticities shown by the letter A, B and C can be detected between the two models. As is described in the linear advection experiments, phase speed of the small-scale (short wavelength) disturbances by horizontal advection is expected to be underestimated by the grid-point model, while error is very small for the spectral model. The experiment shown in Fig. 5 supports this fact. No phase difference can be seen for those disturbances of which horizontal scale is larger than those shown by the letter A, B and C. This may also have been expected by the linear advection experiments.

6.3 Case 2 – moist model (Sept. 26, 1983; Typhoon FOREST or T8310) –

Prediction of typhoon position and intensity

Fig. 5 Forecast comparison of the grid point 12L-FLM (top panel) and the spectral 12L-FLM (bottom panel). The two models do not have any physical processes except horizontal diffusion. The 36 hour forecast of 500hPa geopotential height (gpm) and vorticity (positive area is shaded; until 10^{-6} sec^{-1}) and sea level pressure (hPa) are shown in the left and right columns, respectively. The initial state of the forecast is May 5 00Z, 1984.

are one of the most essential requirements at JMA. In the operational grid-point 12L-FLM, however, the maintenance of the typhoon vortex is not satisfactory. The initial typhoon usually quickly disappears in the forecast due to (a) improper tuning of parameterization of convection that is necessary for the maintenance of typhoon vortex (operational model is tuned for middle latitude synoptic scale disturbances) and (b) large horizontal diffusion (e -folding time for 3.2d wave length disturbances is 2.5 hours). In the present experiment, we tuned the current operational convective-adjustment scheme to allow more heating and reduce the

horizontal diffusion coefficient to 1/5 (e -folding time for 3.2d wave length disturbance becomes 12 hours) for the two models. By these modifications the typhoon vortex (FORREST) is well maintained by both the grid-point and the spectral 12L-FLM. Fig. 6 shows the verification analysis at every 12 hour interval from Sep. 26 00Z, 1983.

(1) Comparison between the grid and the spectral model forecasts

In this experiment, initial field is produced by introducing a climatological typhoon vortex into the objective analysis. This procedure is essential in producing initial typhoon vortex

Fig. 6 The 12 hour interval synoptic charts from Sep. 26 00Z, 1983 to Sep. 28 00Z, 1983, and 500hPa analysis at Sep. 27 12Z, 1983, for typhoon FORREST (T8310). The integer number in the bracket corresponds to the forecast time (hour) shown in Fig. 7 ~ Fig. 9.

from very little observations. Note that initial divergence accompanying the typhoon is assumed zero.

Fig. 7 shows the 36 hour forecasts of the two models. In these figures the 12 hour accumulated precipitation (24~36 hour) is superposed on the sea-level pressure maps. Both the spectral and the grid-point models predict the typhoon movement well. However the predicted vortex by the spectral model is clearly superior to that by the grid point model in the following points:

- surface pressure at the center of the typhoon,
- the concentration of the vorticity at 500hPa,
- the degree of axial symmetry of the typhoon vortex.

It is also noted that the precipitation of the spectral model seems well organized for maintaining typhoon vortex compared with that of the grid-point model.

Fig. 8a compares vorticity of the two models at $\sigma=0.980$ (the lowest level) at every 12 hour interval from the initial state. Fig. 8b compares vorticity at $\sigma=0.550$ (about 550hPa level). The predicted position of the vorticity center of the two models are almost the same. However, the vortex of the spectral model is well organized and axially symmetric pattern is well conserved until 36 hours in the spectral model. By comparing the forecasts of the two models with the analysis shown in Fig. 6, the predicted typhoon vortex by the spectral model is superior to that of the grid point model.

The transition from typhoon vortex to an extratropical cyclone starts at about Oct. 28 00Z ($t=48$ hour in the present experiment). This process is fairly well predicted by the spectral model, however it begins about 12 hours earlier than the observed transition in the grid-point model.

Fig. 7 Comparison of the 36 hour forecasts of the spectral 12L-FLM (left column) and the grid-point 12L-FLM (right column) for the typhoon FORREST (T8310). Top panels show sea level pressure and 12 hour ($t=24\sim36$ hour) accumulated precipitation amount (mm), and the bottom panels 500hPa geopotential height and vorticity.

Fig. 8a The forecast comparison of the vorticity of typhoon FORREST (T8310) at the lowest σ -level ($\sigma=0.98$) of the spectral 12L-FLM with E46 truncation (left column) and the grid-point 12L-FLM (right column), from initial (top panels) to 48 hours (bottom panel). Contour line of the negative vorticity is shown by broken lines. The integer numbers in the middle column show the forecast time (hour).

Fig. 8b Same as Fig. 8a except for the forecast comparison of the vorticity at $\sigma=0.55$ level.

(2) Resolution impact on typhoon forecasts

In order to see the impact of resolution to the predicted typhoon vortex, we compare the forecasts of the four spectral models of different resolution and of the grid point model. They are the original E46 (elliptic truncation at wave number 46), E31, E23 and E16 truncation spectral models.

The models with the four different resolutions are built on the same grid net on which trans-

form technique is applied, therefore E31, E23 and E16 models used much larger number of grid points required for aliasing free calculation of nonlinear terms. This is mainly for simplicity of the experiments.

Fig. 9 compares the 500hPa geopotential height of the five models for every 12 hour interval. The E46, E31, E23 and E16 truncated spectral model forecasts are shown in the four columns from the left to right, and the grid point model forecasts are shown in the right most column.

We observe clear impact of resolution to the prediction of typhoons in Fig. 9. The E46 spectral model produces the best forecast including the intensity of typhoon vortex and its

Fig. 9 Forecast comparison of 500hPa geopotential height of typhoon FORREST (T8310) by the four different resolution spectral 12L-FLMs with the truncation of E46, E31, E23, E16, and the grid-point 12L-FLM, for every 12 hour interval from 12 hour forecasts to 48 hour forecasts (top panels). The forecasts of the five different models are shown from the left to the right by the order given above. The initial date is at Sep. 26 00Z, 1983.

movement (predicted vorticity and divergence fields, geopotential height of the other pressure levels, and the amount of precipitation also show that the E46 model is the best although we do not show these figures).

The E31 model predicts well the movement of typhoon, so do the grid point model. However the forecast of the E23 model shows that the typhoon moves slower than observed and the predicted track shifts to the south of the actual track. The E16 truncated model completely fails to maintain the typhoon vortex itself.

We conclude from the above experiments that: (a) when we use the same horizontal grid to both the grid-point and the spectral model, the spectral model produces better forecast for small scale disturbances whose horizontal scale is near the resolution limit, and that (b) the forecast performance of the grid point model are nearly equivalent to the spectral model whose horizontal resolution is reduced to two third of the maximum possible resolution defined by the given grid. These results suggest that the present spectral formulation offers powerful tool to improve forecasts of usual grid point models with time dependent lateral boundary condition. These improvements are presumably due to improved estimation of horizontal differential which is brought by the spectral representation of the horizontal distribution of the prognostic variables.

6.4 *Comparison of computational economy of the grid and the spectral model*

The computation time of the present spectral 12L-FLM is compared with that of the grid point 12L-FLM in order to see the possibility of applying the present spectral formulation to operational models. We compare the computation time using HITAC M200H (which is under operational use at JMA). This computer has an additional processor called "integrated array processor" and sometimes called "semi-super computer", and has a peak speed of 60 MFLOPS (million floating operation per second).

The computer code for the grid-point 12L-FLM is highly economized to utilize the array processor efficiently. In this sense, the code for the spectral model is not yet highly economized. The grid-point 12L-FLM is also economized by

the economical explicit scheme proposed by Tatsumi (1983) for its operational use.

The spectral 12L-FLM with E46 truncation, which uses the same grid as the grid point model for transform method, takes 1.2~1.3 times as much computation time as the grid point 12L-FLM to complete the same forecast period. About half the computer time of the spectral model is consumed in double Fourier transform although well optimized FFT scheme is used.

We conclude that the computational economy of the spectral 12L-FLM is effectively superior to the grid-point 12L-FLM, since, as is described before, the grid point 12L-FLM and the E31 spectral model have about the same accuracy and resolution. The computer time required by the E31 model is estimated about $(1/1.5)^3=0.3$ times of E46 model. Therefore, the spectral 12L-FLM is practically quite economical. It should be noted that such computational economy of the spectral model is crucially attained by the successful and efficient application of semi-implicit scheme to the model. The spectral formulation of the semi-implicit scheme is shown in Appendix C.

7. Conclusions

A spectral formulation of a limited-area model with time-dependent lateral boundary conditions is proposed and applied to a multi-level primitive equation model.

The proposed spectral limited-area model is constructed by modifying the spectral model with a free-slip wall-boundary condition in order to satisfy a time-dependent lateral boundary condition. The spectral representation of the prognostic variables can be made using Fourier sine or cosine series in both the x and y directions. We have four different sets of bases by different boundary conditions in the x and y direction. Each set is a complete set of orthogonal system only for the particular boundary condition. The selection from the four different sets of bases for each variable is determined by the consideration of an appropriate wall-boundary condition.

In order to extend the model with the wall boundary condition to the one with a time-dependent boundary condition, we propose to introduce a few additional bases to each set

of the orthogonal bases. In the one-dimensional case, we add $\sin \hat{x}$ and $\sin 2\hat{x}$ for the Fourier cosine series, and $\cos 0\hat{x}$ and $\cos \hat{x}$ for the Fourier sine series, where the forecast area is $0 \leq \hat{x} \leq \pi$. In the two dimensional, the additional bases are also necessary to represent the lateral boundary forcing from the outside of the forecast area. The amplitude of the additional bases is determined by specifying non-zero fluxes of mass, momentum, energy etc. across the lateral boundary.

In the spectral limited-area model, these external forcing terms are time-dependent. We prescribe the time evolution of the amplitude of the additional bases at every forecast time, using the historical data from an available coarse-mesh forecast.

This lateral boundary condition does not guarantee the well-posedness condition. Moreover basically, it provides reflective boundary since the model is constructed on the one with a wall-boundary condition. In order to reduce spurious solutions resulting from ill-posed boundary conditions (over- or under-specified boundary) and from reflective nature of the boundary condition, we introduce a "boundary relaxation technique"

We applied the present spectral method to the following models in order to see overall performances

- (a) one dimensional linear advection model with a constant general flow.
  - with zero-boundary condition
  - with time dependent lateral boundary condition
- (b) multi-level primitive equation model (application to the JMA operational grid-point 12L-FLM)
  - with dry situation
  - with moist situation which includes an intense typhoon vortex

The following are the results:

- The computational dispersion and the systematic phase error associated with the grid-point models are completely removed using the spectral method. This contributes to the increased accuracy and resolution of the spectral model.
- The boundary relaxation technique works quite well in the linear advection model as

well as in the multi-level primitive equation model.

- The resolution of the spectral model practically exceeds that of the grid point model when the two models use the same horizontal grid.
- The spectral method marks effectively better computational economy than the grid point method in application to the JMA operational 12 level fine mesh Asia model, when the improved accuracy of the spectral method is taken into account.

In addition to these advantages the spectral method simplifies the design and structure of the model in its dry dynamical part as well as its physical processes: we need not worry about complicated selection of horizontal differencing schemes, and the simplest non-staggered transform grid of the spectral model also simplifies the treatment of recent sophisticated parameterization schemes of physical processes. There are some disadvantages in the present spectral method:

- difficulty in applying to a two-way interaction boundary model,
- difficulty in introducing a well-posed boundary condition such as a radiation condition.

However we do not think they are major shortcomings since most of the recently developed limited area model do not rely on these lateral boundary conditions.

It should be noted that our selection of the functional form of the additional bases is only one possible way, and there are other possible selection. Even so, we are confident that our choice is one of the best solutions of this problem since we have satisfactory forecast performance and practically sufficient economy from our selection of the additional bases.

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Appendix A

Description and forecast performance of the spectral limited area model with wall-boundary condition

Here is an application and the results of the spectral limited area model with the wall-boundary condition; the application to a divergent barotropic, northern hemispherical model. We refer the model by DFM (double Fourier spectral model) in order to distinguish from usual spherical harmonic spectral model (SHM).

Basic equations

The prognostic variables of the DFM is u^* , v^* and ϕ . The u and v momentum equations are the same as in (2.2) and (2.3), respectively, except that the vertical advection terms and the terms include $\dot{\sigma}$ are excluded. The continuity equation is written as

$$\begin{aligned} \frac{\partial \phi}{\partial t} = & -m^2 \left(u^* \frac{\partial \phi}{\partial x} + v^* \frac{\partial \phi}{\partial y} \right) \\ & -m^2 \phi \left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \right) \end{aligned} \quad (\text{A.1})$$

The spectral representation of u^* and v^* are used in (3.1). The spectral expression of ϕ is given by the $\{\cos k\hat{x} \cos l\hat{y}\}$ series.

Description of the model

A square forecast region which covers whole northern hemisphere is defined on the 60N polar stereographic projection map. The forecast area is divided into 65×65 grid-points with 381km interval (at 60N), and a transform method is applied on this grid.

The square of map factor, m^2 , is expanded into the $\{\cos k\hat{x} \cos l\hat{y}\}$ series with the wave number truncation at $K_m = L_m = 19$. For the prognostic variables we apply E36 truncation. The wave length of the shortest resolvable wave in the E36 truncation is about 1,350km at 60N and 1,090km at 30N (see section 4 as for wave number truncation and the transform method).

A semi-implicit scheme together with a time filter scheme proposed by Asselin (1972) is used for time integration of the model. The semi-implicit formulation of the present model is the same as those described in Appendix C which is for the multi-level primitive spectral limited area model.

Results and discussion of the forecast experiment

Two day forecasts of the two northern hemispherical barotropic models, DFM and SHM are compared using 500hPa geopotential field at 00 GMT 17 November, 1983 as an initial field. Horizontal resolution of the SHM is triangular truncation at wave number 42. The shortest resolvable wave length of this model is about 950km. This model is developed based on the JMA operational 12-level northern hemispherical spectral model (see Kanamitsu *et al.*, 1983). It is noted that the resolutions of the two models are approximately the same at around 30N.

The DFM is different from the SHM in the following points:

- The shape of the lateral boundary: the SHM assumes free slip boundary along the equator, while DFM has rectangular shaped free slip boundary that enclose equator.
- The horizontal resolution is uniform over the whole northern hemisphere for SHM, while it depends on the map factor for DFM.

Fig. A compares the forecasts of the 500hPa geopotential height of the two models. The SHM and DFM forecasts are shown in the left and right columns at $t=48$ hours, respectively.

The two forecasts are considered to have some differences due to the above mentioned differences. Particularly large differences are expected in the tropical and subtropical regions because of the proximity to the boundaries. Therefore, we compare the forecasts only in the extratropical region. In the extratropical regions, the 48 hour forecasts of two models are almost identical. We observe no systematic phase differences in the predicted disturbances of the two models. Furthermore, the amplitude of the predicted disturbances of the two models is almost the same. The forecast by the SHM is little smoother presumably due to slightly larger horizontal diffusion coefficients of the SHM. From the synoptic point of view, however, the two models

Fig. A Forecast comparison of the 500hPa geopotential height of the two spectral models; the 48 hour forecasts of the spectral model with the spherical harmonics (SHM) and that of the double Fourier series (DFM) are shown in the left and the right panels, respectively.

produces substantially the same forecast over the extratropical region. This experiment suggests that spectral model with the double Fourier series on the polar stereographic projection produces forecast comparable to that with the spherical harmonics over the hemisphere, at least away from the boundaries, and that the Fourier series method is shown to work reasonably well.

Appendix B

Vertical differencing scheme of the spectral 12L-FLM

The vertical differencing schemes of the spectral 12L-FLM is identical to those of the JMA operational 12 level northern hemispherical spectral model (Kanamitsu; 1983). The vertical advection of momentum and moisture, that appear in (2.2) (2.3) (2.8), are evaluated by the finite difference form:

$$\left(\dot{\sigma} \frac{\partial f}{\partial \sigma}\right)_{2k} = \frac{1}{2\Delta\sigma_{2k}} [\dot{\sigma}_{2k-1}(f_{2k-2} - f_{2k}) + \dot{\sigma}_{2k+1}(f_{2k} - f_{2k+2})] \quad (\text{B.1})$$

where k represents vertical level index shown in Fig. 4. For the spectral limited area model θ (potential temperature) in Fig. 4 is substituted by T (temperature). This scheme conserves f and f^2 . The vertical advection of temperature appeared in (2.4) is calculated by the finite

difference form:

$$\left(\sigma^k \dot{\sigma} \frac{\partial T \sigma^{-k}}{\partial \sigma}\right)_{2k} = a_{2k} \dot{\sigma}_{2k-1} + b_{2k} \dot{\sigma}_{2k+1} \quad (\text{B.2})$$

here

$$a_{2k} = \frac{1}{2\Delta\sigma_{2k}} \left[T_{2k-2} \left(\frac{\sigma_{2k}}{\sigma_{2k-2}} \right)^k - T_{2k} \right] \\ b_{2k} = \frac{1}{2\Delta\sigma_{2k}} \left[T_{2k} - T_{2k+2} \left(\frac{\sigma_{2k}}{\sigma_{2k+2}} \right)^k \right] \quad (\text{B.3})$$

The vertical integration of the continuity equation in (2.5) and (2.6) is

$$\int_0^1 \nabla \cdot V^* = \sum_{k=1}^{k_{\max}} \nabla \cdot V_{2k}^* \Delta\sigma_{2k} \quad (\text{B.4})$$

$$\dot{\sigma}_{2k-1} - \dot{\sigma}_{2k+1} = m^2 (\nabla \cdot V_{2k}^* + V_{2k}^* \nabla \cdot \pi^*) \Delta\sigma_{2k}$$

$$+ \frac{\partial \pi}{\partial t} \Delta\sigma_{2k} \quad (\text{B.5})$$

here $\dot{\sigma}$ is zero at the top ($\sigma=0.0$) and the bottom ($\sigma=1.0$) boundaries. The hydrostatic equation (2.7) is integrated by

$$\phi_2 = -RT_2 \ln \sigma_2 \\ \phi_{2k} = \phi_{2k-2} + (\alpha_{2k} T_{2k} + \beta_{2k-2} T_{2k-2}) \quad (\text{B.6})$$

here

$$\alpha_{2k} = (C_P/2) [(\sigma_{2k}/\sigma_{2k-2})^k - 1] \\ \beta_{2k} = (C_P/2) [1 - (\sigma_{2k}/\sigma_{2k+2})^k] \quad (\text{B.7})$$

The finite differencing scheme applied here (Arakawa, 1977) conserves potential temperature and its square. Furthermore, it guarantees the conservation of total energy.

Appendix C

Time differencing scheme — semi-implicit formulation of the spectral 12L-FLM —

In order to save computation time, semi-implicit time integration scheme of the form described below is applied. The semi-implicit scheme applied here to the spectral limited area model is principally the same as those proposed by Hoskins and Simmons (1975), and Kanamitsu *et al.* (1983). Here we introduce the following notation in order to simplify discussions. We assume that Q be arbitrary variables.

$$Q(t) = Q_t$$

$$\bar{Q}^t = \frac{1}{2}(Q_{t+\Delta t} + Q_{t-\Delta t}) \quad (C.1)$$

$$\delta_t Q = \frac{1}{2\Delta t}(Q_{t+\Delta t} - Q_{t-\Delta t})$$

We define the wind vector, divergence and vorticity as

$$V^* = u^*i + v^*j$$

$$D = \nabla \cdot V^* = \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} \quad (C.2)$$

$$\zeta = \mathbf{k} \cdot \nabla \times V^* = \frac{\partial v^*}{\partial x} - \frac{\partial u^*}{\partial y}$$

where $\mathbf{k}$ is the unit vector in the vertical. We introduce the following semi-implicit formulation of the governing equations using the notation (C.1) and (C.2):

$$\delta_t D + \nabla \cdot (\overline{PGF})^t = \nabla \cdot \left(\frac{\partial V^*}{\partial t} \right)_t + \nabla \cdot (PGF)_t$$

$$\delta_t T + (\overline{TVA})^t = \left(\frac{\partial T}{\partial t} \right)_t + (TVA)_t \quad (C.3)$$

$$\delta_t \pi^* + (\overline{DIV})^t = \left(\frac{\partial \pi^*}{\partial t} \right)_t + (DIV)_t$$

$$\delta_t \zeta = \mathbf{k} \cdot \nabla \times \left(\frac{\partial V^*}{\partial t} \right)_t$$

where $(\partial Q / \partial t)_t$ represents time tendency of Q calculated using the prognostic variables at t and the governing equations (2.2)~(2.8), and

$$(PGF) \equiv \nabla \phi + RT_0 \nabla \pi^*$$

$$(TVA) \equiv \sigma^* \frac{\partial}{\partial \sigma} (T_0 \sigma^{-\kappa}) - \frac{RT_0}{C_p} \left(\frac{\partial \pi^*}{\partial t} \right)_D$$

$$(DIV) \equiv m_0^2 \int_0^1 D d\sigma \quad (C.4)$$

$$\left(\frac{\partial \pi^*}{\partial t} \right)_D \equiv m_0^2 \int_0^1 D d\sigma = -(DIV)$$

$$\frac{\partial \bar{\sigma}_D}{\partial \sigma} = -m_0^2 D - \left(\frac{\partial \pi^*}{\partial t} \right)_D$$

The lateral boundary relaxation terms defined by (4.26) is also included in $(\partial Q / \partial t)_t$ using $(Q - \bar{Q})$ at $t - \Delta t$ (i.e., by the forward scheme) in order to avoid the linear computational instability (see Mesinger, 1976 p. 19). The variable T_0 , that is a function of σ only, is the reference temperature profile, and m_0^2 (constant) is a reference square map factor. Introducing these reference values we linearize the gravity wave terms and integrate them by the trapezoidal implicit scheme.

Introducing the matrix notation of the vertical differencing schemes of the hydrostatic equation (B.6) and the thermodynamic equation (B.2) by the following form:

$$\underline{\phi} = \underline{P}T \quad (C.6)$$

$$(\underline{TVA}) = \underline{H}D \quad (C.7)$$

where $\underline{Q} = (Q_1, Q_2, \dots, Q_K)^T$ is the column vector and $k=l-K$ is the level index, and the k -th element of the vector $\underline{Q}$ is the value of Q at level $2k$ in Fig. 4. The constant matrix $\underline{P}$ and $\underline{H}$ are uniquely determined by the vertical difference schemes in Appendix B. Using the matrix notation and (C.1), the eqs. (C.3) becomes

$$\delta_t \underline{D} + \Delta t \underline{P} \nabla^2 \delta_t \underline{T} + \Delta t R T_0 \nabla^2 \delta_t \pi^* = \underline{N}D$$

$$\delta_t \underline{T} + \Delta t \underline{H} \delta_t \underline{D} = \underline{N}T \quad (C.8)$$

$$\delta_t \pi^* + \Delta t m_0^2 \underline{\Delta \sigma}^T \delta_t \underline{D} = N \pi^*$$

here $\underline{N}D$, $\underline{N}T$ and $N \pi^*$ include all terms except linearized gravity wave terms which are written as

$$\begin{aligned} \underline{ND} &= \left(\frac{\partial}{\partial t} \nabla \cdot \underline{V}^* \right)_t + \underline{P} \nabla^2 (\underline{T}_t - \underline{T}_{t-\Delta t}) \\ &\quad + \underline{RT}_0 \nabla^2 (\pi_t^* - \pi_{t-\Delta t}^*) \\ \underline{NT} &= \left(\frac{\partial}{\partial t} \underline{T} \right) + \underline{H} (\underline{D}_t - \underline{D}_{t-\Delta t}) \end{aligned} \quad (\text{C.8})$$

$$\begin{aligned} N\pi^* &= \left(\frac{\partial}{\partial t} \pi^* \right)_t + m_0^2 \underline{A} \sigma^T (\underline{D}_t - \underline{D}_{t-\Delta t}) \\ \text{Eliminating } \delta_t \underline{T} \text{ and } \delta_t \pi^* \text{ from (C.7) we have} \\ \text{the following Helmholtz equation with respect} \\ \text{to } \delta_t \underline{D}. \\ -\Delta t^2 (\underline{P} \underline{H} + m_0^2 \underline{RT}_0 \underline{A} \sigma^T) \nabla^2 \delta_t \underline{D} + \delta_t \underline{D} \\ = \underline{ND} - \Delta t \underline{P} \nabla^2 (\underline{NT}) - \Delta t \underline{RT}_0 \nabla^2 (N\pi^*) \end{aligned} \quad (\text{C.9})$$

Next we convert the Helmholtz equation (C.9) to the spectral form. We can get the spectral expression of the r.h.s. of (C.9) by introducing the spectral expression of $\underline{ND}$, $\underline{NT}$ and $N\pi^*$ as follows:

- We calculate the spectral expression of $(\partial Q / \partial t)_t$ in the r.h.s. of (C.8) using the spectral prognostic equation (4.17). Here we convert the tendency of u^* and v^* to the tendency of divergence D and vorticity ζ in the spectral form by applying D_x and D_y operator.
- We calculate $(Q_t - Q_{t-\Delta t})$ in the r.h.s. of (C.8) using the amplitude of the orthogonal bases at t and $t-\Delta t$, here we need not take into account the amplitude of the additional bases since we apply the implicit scheme to time evolution of the amplitude of the orthogonal bases.

Substituting spectral representation of $\underline{ND}$, $\underline{NT}$ and $N\pi^*$ into (C.9) and using the following equation

$$\nabla^2 Q_{kl} = -(k^{*2} + l^{*2}) Q_{kl} \quad (\text{C.10})$$

we have the spectral form Helmholtz equation as

$$\{ \underline{I} + \Delta t^2 (k^{*2} + l^{*2}) \underline{C} \} \delta_t \underline{D}_{kl} = (\underline{NL})_{kl} \quad (\text{C.11})$$

where Q_{kl} represents the amplitude of Q of the orthogonal bases with the wave number k and l , $(\underline{NL})_{kl}$ the spectral representation of the r.h.s. of (C.9), and $\underline{I}$ the unit matrix. The matrix $\underline{C}$ is also a constant matrix defined by

$$\underline{C} = \underline{P} \underline{H} + m_0^2 \underline{RT}_0 \underline{A} \sigma^T \quad (\text{C.12})$$

In order to solve (C.11), we prepare the matrix $\underline{E}$ which consists of the eigenvector of the

matrix $\underline{C}$ i.e.,

$$\underline{E}^{-1} \underline{C} \underline{E} = \begin{bmatrix} C_1^2 & & 0 \\ & \ddots & \\ & & C_i^2 & \\ 0 & & & \ddots & \\ & & & & C_k^2 \end{bmatrix}, \quad \underline{E}^{-1} \underline{E} = \underline{E} \underline{E}^{-1} = \underline{I} \quad (\text{C.13})$$

where C_i^2 is the eigenvalue of $\underline{C}$; C_i represents the phase speed of the gravity wave of the i -th vertical mode. Then (C.11) can be written as

$$\begin{aligned} \underline{E}^{-1} [\Delta t^2 (k^{*2} + l^{*2}) \underline{C} + \underline{I}] \underline{E} \underline{E}^{-1} \delta_t \underline{D}_{kl} \\ = \underline{E}^{-1} (\underline{NL})_{kl} \end{aligned} \quad (\text{C.14})$$

and finally $\delta_t \underline{D}_{kl}$ can be solved as

$$\begin{aligned} \delta_t \underline{D}_{kl} &= \underline{E} \begin{bmatrix} (\omega_1^2 + 1)^{-1} & & \\ & \ddots & \\ & & (\omega_i^2 + 1)^{-1} & \\ 0 & & & \ddots & \\ & & & & (\omega_k^2 + 1)^{-1} \end{bmatrix} \\ &\quad \cdot \underline{E}^{-1} (\underline{NL})_{kl} \end{aligned} \quad (\text{C.15})$$

where $\omega_i^2 = (c \Delta t)^2 (k^{*2} + l^{*2})$ is the frequency normalized by Δt of the i -th vertical mode with wave number k and l . Substituting $\delta_t \underline{D}_{kl}$ into (C.7) we have

$$\delta_t \underline{T}_{kl} = (\underline{NT})_{kl} - \Delta t \underline{H} \delta_t \underline{D}_{kl} \quad (\text{C.16})$$

$$\delta_t \pi_{kl}^* = (N\pi^*)_{kl} - \Delta t m_0^2 \underline{A} \sigma^T \delta_t \underline{D}_{kl}$$

and using (C.10) we have

$$\begin{aligned} \delta_t \underline{u}_{kl}^* &= D_x [(k^{*2} + l^{*2})^{-1} \delta_t \underline{D}_{kl}] \\ &\quad - D_y [(k^{*2} + l^{*2})^{-1} \delta_t \underline{\zeta}_{kl}] \\ \delta_t \underline{v}_{kl}^* &= D_y [(k^{*2} + l^{*2})^{-1} \delta_t \underline{D}_{kl}] \\ &\quad + D_x [(k^{*2} + l^{*2})^{-1} \delta_t \underline{\zeta}_{kl}] \end{aligned} \quad (\text{C.17})$$

where $\delta_t \underline{\zeta}_{kl}$ is the spectral representation of $\delta_t \zeta$ in (C.3). The leapfrog scheme is used for the rotational part of wind. Eqs. (C.15), (C.16) and (C.17) give us the final spectral form time tendency given by the semi-implicit formulation (C.3).

We set the reference temperature as $T_0 = 300\text{K}$ (isothermal) and the reference map factor $m_0 = 1.5$ which corresponds to the value of m at about 14N in the spectral 12L-FLM. Using these constants we achieved stable integrations. The time integration of specific humidity q is performed by the leapfrog scheme.

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時間的に変化する境界を持つ局地スペクトルモデルと その多層プリミティブモデルへの応用

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気象庁数値予報課

時間的に変化する側面境界を持つ局地スペクトルモデルを開発した。モデルは、広域低分解能モデルに one way nesting で接続される。場のスペクトル展開は、“拡張2重フーリエ基底”を導入し、これを用いて行った。この基底は、矩形の予報領域で、mirror 境界条件 (free slip wall 境界の一種) を満たす2重フーリエ直交基底と、これには属さない新たに導入した準基底で構成する。準基底の関数形は一意的には決まらない。本モデルでは準基底も x 、 y 方向はそれぞれ $\sin$ または $\cos$ で構成した。準基底は先の直交基底とは直交しない。

拡張2重フーリエ基底による場の展開では、準基底の振幅は mirror 境界に属さない側面境界条件 (値または境界に直角方向の微分) のみに依存し、内部領域の場にはよらない。局地モデルの外界との相互作用は境界を横切る質量、運動量、エネルギー等のフラックスで決まる。このフラックスは準基底の振幅を与えることによるのみ指定できる。本モデルでは、スペクトル予報方程式を導くための境界条件として、準基底に属する場合 (境界値またはその微分値) を広域モデルの予報値を用いて指定した。この境界条件を用いると、mirror 境界に属する直交基底の時間変化率を与えるスペクトル予報方程式が支配方程式から一意的に導かれ、局地スペクトルモデルが成立する。この境界条件は well posedness の条件を満たさないで物理的に意味のない反射やノイズの発生を招く。このため、boundary relaxation の手法を導入し、境界付近で発生する乱れを抑制している。

本局地スペクトルモデルを一次元線形移流方程式に応用し、従来の差分モデル (2次および4次精度) と数値解を比較した。格子モデルでは不可避である切断誤差に伴う位相の遅れと計算上の分散性がスペクトルモデルの数値解には全く現れず、スペクトル化によって、特に小スケールのじょう乱の予報精度が著しく改善されることが分かった。

多層プリミティブモデルへの応用として、気象庁ルーチン12層ファインメッシュモデル (12L-FLM: 格子間隔約 100 km の格子モデル) をスペクトル化し、両者を比較した。モデルの分解能に比べて比較的大きなスケールの現象に対しては両モデルの予報には差が認められなかった。しかし、分解能から見て小スケールの現象の例として選んだ強い台風の予報比較では、スペクトルモデルが格子モデルに比較してその渦の強さ、軸対称性や温帯低気圧への移行の点では著しく優れていた。格子とスペクトルの12L-FLM の HITAC M200H による計算時間は、格子モデルを1とすると、1.2~1.3でややスペクトルモデルが劣っている。しかし、スペクトル化による実質的な分解能の向上が上記の台風の場合およそ1.5倍という結果を得ているので、本局地モデルは格子モデルよりも高い経済性を持つことがわかった。

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