Accuracy and Initialization of a Two-Time-Level Split Semi-Lagrangian Model

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Abstract

A two-time-level semi-Lagrangian primitive equations model has been constructed using the efficient alternating direction implicit (ADI) procedure of Bates (1984), together with split horizontal advection and geostrophic adjustment processes. If the simplest splitting method is used, the solution suffers from large neutral-amplitude oscillations. However a more accurate splitting arrangement may be used in which the adjustment processes are staggered in time. In this case a smoothly developing solution is obtained, even without explicit diffusion.

Further improvements are obtained if the accuracy of the ADI procedure is increased by a repetition ensuring spatial symmetry. Time traces are used to show the similarity of the solutions for various timestep sizes. Only for the largest timesteps are there noticeable differences, mainly in the form of extra gravity wave activity. Comparisons using a variety of horizontal advection schemes indicate that residual truncation errors from the splitting procedure are also made with a limited-area semi-implicit model. The vertical mode initialization scheme of Bourke and McGregor (1983) was successfully applied to the semi-Lagrangian model; convergence was facilitated by the use of centered advection formulae during the initialization iterations.

1. Introduction

A recent advance in numerical weather prediction is the development of semi-Lagrangian models (Robert, 1981, 1982; Bates and McDonald, 1982; Bates, 1984; McDonald, 1986) in which timesteps significantly larger than the usual CFL criterion may be employed. The attraction of such methods lies both in the computational savings permitted by the use of large timesteps, and alternatively the possible reduction of time truncation errors with conventional timesteps. The schemes may be performed carrying variables either for three-time-levels (Robert, 1981, 1982) or two-time-levels (Bates and McDonald, 1982; Bates and McDonald, 1986b). The primary attraction of two-time-level schemes is the reduction in advective truncation errors for

a given timestep size. Also there is a reduction in computer storage requirements.

Marchuk (1974) was the first to note that a complex meteorological problem may be solved by splitting it into smaller component problems, each of which has a stable numerical solution, and then performing the component solutions sequentially. Several primitive equations models have since been developed in which the splitting technique is used to separate the horizontal advection and geostrophic adjustment processes, for example Mesinger (1977) and Gadd (1978). These models, however, were not attempting to permit the large timesteps of semi-Lagrangian advection.

In the context of semi-Lagrangian schemes, the splitting approach was adopted in the models of Bates and McDonald (1982) and

Bates (1984). Both models were constructed on an Arakawa E-grid and employed divergence damping. The multi-level model of Bates and McDonald (1982) used multiple explicit steps for the adjustment phase. The Bates (1984) shallow water model included an attractive alternating direction implicit (ADI) treatment for the adjustment processes. This ADI procedure has recently been extended by Bates and McDonald (1986b) to the multi-level equations, but in a non-split manner.

Recently McDonald (1986) has handled the geostrophic adjustment terms semi-implicitly on the Arakawa C-grid, with an approximate implicit treatment for the staggered Coriolis terms. He notes that in a multi-level model the advective tendency of surface pressure is required for the calculation of vertical velocity, and that this requirement presents a conceptual problem if the surface pressure changes have been first split off. He gives this as his reason for departing from the splitting approach.

The present study demonstrates in fact that the two-time-level split ADI approach of Bates (1984) can be successfully extended to multi-level primitive equations models. The present model was constructed on an Arakawa C-grid in order to optimize the geostrophic adjustment performance (Schoenstadt, 1980). This choice of grid also avoids the horizontal decoupling problems evident with semi-implicit schemes on other grids (McGregor and Leslie, 1977).

During the course of this study it was found however that the splitting procedure itself can contribute significantly to the problems of spurious noise. In fact it was found necessary to modify the time-splitting algorithm by staggering in time to achieve second order accuracy, as proposed by Strang (1968). A relevant discussion on the nature of splitting truncation errors for systems containing both fast and slow modes is provided by Le Veque and Olinger (1983).

The present method also involves horizontal splitting for the ADI calculations. Here, also, increased accuracy was found advantageous. Some similarities are observed with the behaviour of the tridiagonal pro-

cedure used in the fully implicit shallow water model of Cohn *et al.* (1985).

In section 2 the basic equations are presented, divided into advective and adjustment processes. Section 3 proposes a choice of splitting options, with corresponding results in section 4. In section 5 results are presented for various modifications to the horizontal advection scheme. None of the previous semi-Lagrangian models have addressed the problem of initialization. Section 6 demonstrates that the vertical mode initialization scheme of Bourke and McGregor (1983) may in fact be utilized without difficulty. Concluding comments are provided in section 7.

2. Primitive Equations in Semi-Lagrangian Form

Throughout this study, the splitting approach of Marchuk (1974) is used. Each of the processes of horizontal advection and geostrophic adjustment is calculated separately for timestep Δt with all fields updated after each process. The various splitting options will be described in Section 3. The two sets of basic equations for horizontal advection and geostrophic adjustment are written and solved separately. The incorporation of extra physical parameterizations is not considered here.

(H) Horizontal advection

$$\frac{\partial u}{\partial t} + mu \frac{\partial u}{\partial x} + mv \frac{\partial u}{\partial y} = 0 \quad (1)$$

$$\frac{\partial v}{\partial t} + mu \frac{\partial v}{\partial x} + mv \frac{\partial v}{\partial y} = 0 \quad (2)$$

$$\frac{\partial T}{\partial t} + mu \frac{\partial T}{\partial x} + mv \frac{\partial T}{\partial y} = 0 \quad (3)$$

where all variables have their usual meaning, and m denotes the map factor of the Lambert conformal projection. Usually cubic upstream advection (Leonard, 1979) is used for each equation, split into x advection then y advection (the order is reversed for alternate timesteps). As shown in the appendix, this horizontal splitting improves the accuracy compared to ordinary bicubic interpolation. The reverse trajectory from each grid point

uses in turn the current u and v velocity components at each grid point. Some alternative advection schemes are considered in section 5.

The advective change to the surface pressure, p^* , is also calculated by the same upstream advection technique, and the advective time rate of change of $\ln p^*$, denoted by N_{p^*} , is saved for each model level,

$$N_{p^*} = -mu \frac{\partial \ln p^*}{\partial x} - mv \frac{\partial \ln p^*}{\partial y} \quad (4)$$

Similarly to the split treatment used by Gadd (1978), p^* itself is not updated until during the geostrophic adjustment phase. This allows its advective time rate of change to be consistent with the diagnostic calculation of vertical velocity, for use in the temperature equation. It is possible, alternatively, to split off both p^* and T for all the N_{p^*} terms within the horizontal advection stage. This latter option is completely analogous to the shallow water procedure of Bates (1984) and his analyses apply immediately; however the present treatment for p^* advection was found to be significantly superior in regard to noise generation.

(G) Geostrophic adjustment processes

The following derivation uses the same linearization as Bourke and McGregor (1983), and is suitable for the present solution procedures. Once horizontal advection has been performed, the remaining equations of the adiabatic primitive equations are separated into their linear and nonlinear components. It is convenient to define the divergence

$$D = m^2 \left\{ \frac{\partial(u/m)}{\partial x} + \frac{\partial(v/m)}{\partial y} \right\} \quad (5)$$

and to use the vertical integral notation

$$(-^\sigma) = \int_0^\sigma () d\sigma$$

Continuity equation

$$\frac{\partial \ln p^*}{\partial t} = -D - \frac{\partial \delta}{\partial \sigma} + N_{p^*} \quad (6)$$

where N_{p^*} is the nonlinear component derived

earlier from the horizontal advection of the scalar $\ln p^*$ at each level. This is integrated vertically to yield

$$\frac{\partial \ln p^*}{\partial t} = -\bar{D}^1 + \bar{N}_{p^*}^1 \quad (7)$$

Vertical velocity diagnostic equations

$$\delta = \sigma \bar{D}^1 - \bar{D}^\sigma - (\sigma \bar{N}_{p^*}^1 - \bar{N}_{p^*}^\sigma) \quad (8)$$

$$\frac{\omega}{p^*} = -\bar{D}^\sigma - (\sigma N_{p^*} - \bar{N}_{p^*}^\sigma) \quad (9)$$

Horizontal velocity equations

$$\begin{aligned} \frac{\partial u}{\partial t} + m \frac{\partial \phi}{\partial x} + mRT \frac{\partial \ln p^*}{\partial x} + \delta \frac{\partial u}{\partial \sigma} \\ = fv \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + m \frac{\partial \phi}{\partial y} + mRT \frac{\partial \ln p^*}{\partial y} + \delta \frac{\partial v}{\partial \sigma} \\ = -fu \end{aligned} \quad (11)$$

Defining the augmented geopotential

$$P = \phi + R\bar{T} \ln p^* \quad (12)$$

with reference vertical temperature profile $\bar{T}$, (10) and (11) may be separated into their linear and nonlinear components

$$\frac{\partial u}{\partial t} + m \frac{\partial P}{\partial x} = fv + N_u \quad (13)$$

$$\frac{\partial v}{\partial t} + m \frac{\partial P}{\partial y} = -fu + N_v \quad (14)$$

where

$$N_u = mR(\bar{T} - T) \frac{\partial \ln p^*}{\partial x} - \delta \frac{\partial u}{\partial \sigma} \quad (14)$$

$$N_v = mR(\bar{T} - T) \frac{\partial \ln p^*}{\partial y} - \delta \frac{\partial v}{\partial \sigma} \quad (16)$$

Temperature equation

$$\frac{\partial T}{\partial t} + \delta \frac{\partial T}{\partial \sigma} - \frac{RT\omega}{c_p \sigma p^*} = 0 \quad (17)$$

Substitution of the vertical velocity formulae (8) and (9) produces

$$\begin{aligned} \frac{\partial T}{\partial t} + (\sigma \bar{D}^1 - \bar{D}^\sigma) \frac{\partial \bar{T}}{\partial \sigma} + \frac{R\bar{T}\bar{D}^\sigma}{c_p \sigma} \\ = N_T \end{aligned} \quad (18)$$

where

$$N_T = (\sigma \bar{D}^1 - \bar{D}^\sigma) \frac{\partial \bar{T}}{\partial \sigma} - \dot{\sigma} \frac{\partial T}{\partial \sigma} + \frac{R \bar{T} \bar{D}^\sigma}{c_p \sigma} + \frac{R T \omega}{c_p \sigma p^*} \quad (19)$$

Geopotential equation

Assuming a log-linear temperature interpolation with respect to σ as in McGregor *et al.* (1978), the geopotential may be expressed in terms of temperature as

$$\phi_k = \phi^* + \sum_{j=1}^K B_{kj} T_j \quad (20)$$

where ϕ^* is the surface geopotential, and K denotes the number of vertical levels. Combining the time derivatives of (12) and (20), and substituting (7) and (18) yields an equation for the augmented height in the form

$$\frac{\partial P_k}{\partial t} + \sum_{j=1}^K C_{kj} D_j = N_P \quad (21)$$

where the constant matrix C is a function of $\bar{T}$ the reference temperature profile, the choice of σ levels, and the choice of discretization for vertical integration and differentiation. N_P is available by substitution from (4) and (19). The nonlinear terms are all evaluated at the most recent timestep using known quantities.

As in Bourke and McGregor (1983) it is possible to decouple the equations in the vertical by determining the eigenvalues λ of C and the corresponding eigenvector matrix E to satisfy

$$\sum_{j=1}^K C_{kj} E_{jm} = E_{km} \lambda_m \quad (22)$$

Now $u, v, D, P, N_u, N_v, N_P$ are expanded in terms of their vertical modes $U, V, \mathcal{D}, \mathcal{P}, N_U, N_V, N_{\mathcal{P}}$, for example

$$u_k = \sum_{j=1}^K E_{kj} U_j \quad (23)$$

The final equations in eigenvector space corresponding to (10), (11) and (21) are then

$$\frac{\partial U}{\partial t} + m \frac{\partial \mathcal{P}}{\partial x} = fV + N_U \quad (24)$$

$$\frac{\partial V}{\partial t} + m \frac{\partial \mathcal{P}}{\partial y} = -fU + N_V \quad (25)$$

$$\frac{\partial \mathcal{P}_k}{\partial t} + \lambda_k \mathcal{D}_k = N_{\mathcal{P}} \quad (26)$$

ADI solution procedure

Bates (1984) ADI algorithm is now used to advance the solution of (24), (25) and (26) $\Delta t/2$ in time by first treating the x derivative terms implicitly

$$V^{n+1/2} = V^n - \bar{m}^j a (\mathcal{P}_{i,j+1}^n - \mathcal{P}^n) + \Delta t (N_V - f \bar{U}^n) / 2 \quad (27)$$

$$U^{n+1/2} = U^n - \bar{m}^i a (\mathcal{P}_{i+1,j}^{n+1/2} - \mathcal{P}^{n+1/2}) + \Delta t (N_U - f \bar{V}^{n+1/2}) / 2 \quad (28)$$

$$\mathcal{P}^{n+1/2} = \mathcal{P}^n - m^2 \lambda a \left(\frac{U^{n+1/2}}{\bar{m}^i} - \frac{U_{i-1,j}^{n+1/2}}{\bar{m}^{i-1}} + \frac{V^n}{\bar{m}^j} - \frac{V_{i,j-1}^n}{\bar{m}^{j-1}} \right) + \Delta t N_{\mathcal{P}} / 2 \quad (29)$$

Vector quantities without subscripts are assumed to be subscripted (i, j) , and $a = \Delta t / (2 \Delta x)$. $\bar{V}$ and $\bar{U}$ denote velocity components averaged respectively to U and V grid points, and $\bar{m}^i$ and $\bar{m}^j$ denote the corresponding averaged map factors. Combining these three equations, a single tridiagonal equation is obtained for $\mathcal{P}^{n+1/2}$ in terms of known *RHS* quantities

$$m^2 \lambda a^2 \mathcal{P}_{i+1,j}^{n+1/2} - (1 + 2m^2 \lambda a^2) \mathcal{P}^{n+1/2} + m^2 \lambda a^2 \mathcal{P}_{i-1,j}^{n+1/2} = \text{RHS} \quad (30)$$

This equation is solved by a standard tridiagonal x -sweep with fixed end-points to yield $\mathcal{P}^{n+1/2}$, and then $U^{n+1/2}$ and $V^{n+1/2}$ are obtained by substitution in (27) and (28).

Analogously to Bates (1984) the second half timestep is solved by treating implicitly the corresponding y derivative terms and solving the resulting tridiagonal equation with a y -sweep. The above pair of ADI solutions is denoted as X in section 4. The alternative procedure of performing the y -sweep first, then the x -sweep is denoted by Y .

Inversion to physical space

The velocities invert immediately using (23), and similarly P is obtained from $\mathcal{P}$. Subtracting the $\Delta t N_P$ contribution, the resulting linear change to P over one timestep, δP , may be used to give the linear changes

to T and $\ln p^*$

$$\delta T_k = \sum_{j=1}^K A_{kj} \sum_{l=1}^K C_{jl}^{-1} \delta P_l \quad (31)$$

$$\delta \ln p^* = \sum_{j=1}^K \Delta \sigma_j \sum_{l=1}^K C_{jl}^{-1} \delta P_l \quad (32)$$

where A is the matrix of coefficients of D in (18). The update procedure is completed by the addition of $\Delta t N_T$ and $\Delta t N_{p^*}$ respectively. This inversion procedure is the same as used in the initialization study of Bourke and McGregor (1983), and may be shown to be internally consistent with the present ADI method.

3. ADI Scheme with various splitting options

In this section comparisons are provided between various splitting options. Operators H and G are used to denote the Horizontal advection and Geostrophic adjustment processes respectively. Two basic schemes are presented:

(NON) *Non-staggered-in-time scheme*

$$(HG_1)(HG_2)(HG_1)(HG_2) \dots (HG_1)(HG_2)(HG_1)(HG_2)$$

The subscripts allow for extra options to be described later in this section. Each full timestep consists of a pair of H and G operations. This splitting is of the same type as used by Gadd (1978), Bates and McDonald (1982) and Bates (1984). As discussed by Le Veque and Olinger (1983), this scheme has splitting errors $O(\Delta t)^2$, when the H and G operators do not commute.

(STAG) *Staggered-in-time scheme*

Strang (1968) showed that the splitting method may be improved for systems having fast and slow processes G and H respectively. He showed that the time splitting errors are reduced to $O(\Delta t)^3$ by the following arrangement

$$(G_1/2)HG_2HG \dots G_2HG_1HG_2H(G_1/2)$$

where the notation “/2” occurs only for the first and last timesteps of the prognosis, and denotes that the operator is applied only for

the time interval $\Delta t/2$. In Strang's original version of this splitting each G was replaced by a pair of sub operators $(G/2)(G/2)$ with each completed timestep considered to occur midway between them. For plotting the time traces in the following section, each completed timestep is most conveniently taken to be the average of the fields before and after each full G operation.

As implied in the above subscript notation, for each of the non-staggered (NON) and staggered (STAG) schemes, there are several options for the treatment of the G operators. One can perform the ADI sweeps first in the x and then the y direction (denoted by X), or alternately perform first the y then the x sweep (denoted by Y). Four variations on this theme are investigated:

(SYM) The *symmetric* G operator

$$G_1 = G_2 = 0.5*(X+Y)$$

The meaning of this notation is that the X and Y procedures are performed separately for the time interval Δt using the same starting data, and the result averaged. As proposed by Strang (1963) the symmetry of the combined operation will reduce the truncation errors. Note that only the actual ADI calculations are repeated by this method. This is not a great overhead, considering that there are no repeated calculations needed for horizontal advection, nonlinear terms, conversion to and from eigenvector space, or physical parameterizations. Note that this symmetric arrangement can also be applied to the H and G ordering, but if this is done an expensive doubling of all computations occurs.

(ALT) The *alternating* G operators

$$G_1 = X, \quad G_2 = Y$$

provide an approximately symmetric sequence of operations. Results corresponding to the combination (NON.ALT) were presented by Bates (1984) for the shallow water equations, but included divergence damping which helped to control any oscillatory behaviour.

(X) The *X* G operator

$$G_1 = G_2 = X$$

in which the x sweep is always performed first.

(Y) The $\bar{Y} G$ operator

$$G_1 = G_2 = Y$$

in which the y sweep is always performed first.

4. Results for the various splitting options

Comparative 24-hour prognoses were carried out for combinations of the above splitting options. Firstly a typical Australian Bureau of Meteorology regional data set was initialized (the same case that was used in the initialization study of Bourke and McGregor, 1983) as described in section 6. The prognoses use 39 by 24 grid points on a 250 km mesh with 6 equally spaced levels in the vertical. Taking the map factor into account, the minimum physical grid spacing is 212 km near the southern boundary. Three vertical modes were initialized. The results for this section are mainly for the split-cubic upstream advection scheme, using as advection velocities the velocities from the most recent geostrophic adjustment step. Some results using alternative advection velocity procedures will be presented in section 5.

The only physical parameterization included here is surface drag. No other explicit diffusion is included. Boundary values are not altered during any of these simulations. The computations were carried out on the C.S.I.R.O. Cyber 205 computer; all code was vectorized apart from the ADI sweeps. Some comparisons will also be presented in section 5 for the Australian Region Primitive Equations (ARPE) model, as described by McGregor *et al.* (1978), a semi-implicit centered-advection model, which routinely uses a 36 minute timestep at this resolution.

It is instructive to display surface pressure traces for selected grid points during the 24 hour forecasts. Two points near Adelaide and Melbourne were chosen, as for Bourke and McGregor (1983). Typical of the non-staggered-in-time schemes, Fig. 1 shows the pressure traces for (NON.ALT) for $\Delta t = 18, 36, 72$ and 120 min. Despite starting from an initialized state, the spurious oscillations are

dauntingly large even for the 36 min. time-step, although remaining stable.

The key improvement resulted from using the staggered-in-time arrangement. Fig. 2 displays the pressure traces for (STAG.ALT) for the same choice of timesteps, and clearly the noise problem has been dramatically improved. However for $\Delta t = 72$ min. the MSL chart (Fig. 3) shows boundary problems particularly on the southern and eastern boundaries. This is probably due to some form of aliasing of the nonlinear terms for this

Fig. 1 Surface pressure 24 hour time traces for grid points near Adelaide and Melbourne. For plotting convenience Melbourne pressures are reduced by 8 mb. Non-staggered-in-time with the alternating ADI option. Solid lines: $\Delta t = 18$ min.; dashes: $\Delta t = 36$ min.; dots: $\Delta t = 72$ min.; dot-dashes: $\Delta t = 120$ min.

Fig. 2 Surface pressure traces as for Fig. 1, but staggered-in-time with the alternating ADI option.

particular sequence of alternating adjustment steps. Cohn *et al.* (1985) experienced a possibly related problem in solving implicitly the shallow water equations, where their tri-diagonal procedure required the x calculation to be always performed first. A similar, but more severe, problem occurs for the (NON. ALT) prognosis; the chart is not shown.

The boundary problems do not occur for the (STAG.X) and (STAG.Y) schemes, but the extra truncation errors lead to a significant loss of accuracy for the larger timesteps. For example, using (STAG.X) with $\Delta t=72$

min., Fig. 4 shows that the boundary problems are absent, but the MSL pressure pattern near Adelaide differed significantly from that of prognoses for smaller timesteps.

The optimal solution was to use a combination of the most accurate schemes, namely (STAG.SYM). Here satisfactory behaviour for the pressure traces was obtained, for timesteps up to 120 min. (Fig. 5). The corresponding MSL prognoses for timesteps of 18, 36, 72 and 120 min. are shown in Figs. 6, 7, 8 respectively. The pressure patterns are all quite similar. Maximum differences from

Fig. 3 24 hour MSL prognosis, staggered-in-time with the alternating ADI option for $\Delta t=72$ min.

Fig. 4 As for Fig. 3 but using the X ADI option.

Fig. 5 Surface pressure traces as for Fig. 1, but staggered-in-time with the symmetric ADI option.

Fig. 6 24 hour MSL prognosis, staggered-in-time with the symmetric ADI option, for $\Delta t = 36$ min.

Fig. 7 As for Fig. 6 but $\Delta t = 72$ min.

Fig. 8 As for Fig. 6 but $\Delta t = 120$ min.

the 36 min. timestep prognosis occur over the Southern Ocean, and are 1.9 mb and 4.7 mb respectively for 72 min. and 120 min. time-steps. The differences resemble superimposed gravity wave oscillations.

Prognoses using the split-quintic upstream advection scheme (Davies, 1980) for the same range of timesteps were also performed. The pressure traces and MSL prognoses were almost indistinguishable from the corresponding cubic simulations; separate figures are thus not shown.

Two types of splitting errors are evident in the above. Firstly the splitting of horizontal advection and geostrophic adjustment processes produces time truncation errors that can generate spurious gravity wave noise. In particular note that the simple non-staggered-in-time splittings all produce stable numerical solutions, but with large neutral-amplitude oscillations. However by modifying the time-splitting procedure to achieve second order accuracy, this problem is overcome.

The second type of splitting error is produced by the slightly asymmetrical nature of the ADI scheme with respect to x and y . This is much less severe than the first type of error, but can lead to boundary problems unless its accuracy is improved.

There is a discernible reduction in accuracy for large timesteps, which reflects some

form of time truncation error. A likely source is residual time-splitting errors. Another possibility arises from the use of effectively simple forward time stepping for the nonlinear terms. The experience of McDonald (1986) with a two-time-level non-split model suggests this is not a primary concern, although his method does include an upstream advective contribution to these terms. Thirdly, the advection scheme should possibly use information about a more recent velocity field along each trajectory; it will be demonstrated in the following section that this does not affect the residual truncation errors.

5. Results for alternative advection procedures

The previous section established that the preferred splitting option was staggered-in-time together with the symmetric ADI scheme. In this section, various modifications to the advection scheme are considered whilst preserving the same splitting framework.

The first modification considered was to use velocities taken as the average of values before and after the staggered geostrophic adjustment step. These are more representative of the velocities valid at the beginning of the horizontal advection step. Fig. 9 displays the corresponding set of pressure traces, again with split-cubic advection. Insignifi-

Fig. 9 Surface pressure traces as for Fig. 5, but using time averaged advection velocities.

cant differences are seen from the earlier traces of Fig. 5.

The next refinement considered was to linearly time extrapolate these averaged velocities, to locate them midway through the horizontal advection process. Little impact was observed, with the pressure traces being similar to those of Figs. 5 and 9; separate figures are not shown. This is at variance with the results for non-split schemes, where a noticeable improvement is found (Bates and McDonald, 1986a).

The final modification considered for the horizontal advection, was again to use time extrapolated velocities, but to replace the split-cubic advection scheme with a two-dimensional bicubic upstream interpolation scheme as used by Robert (1981) and McDonald (1986). This procedure also included an iterative computation for the advecting velocity at the mid-point of the trajectory as suggested by Robert (1981). The pressure traces in this case are provided in Fig. 10, and again are similar to the previous traces.

The conclusion from the above comparisons is that all the horizontal advection schemes show adequate accuracy for reasonably large timesteps. None of the modifications had a significant impact on improving the accuracy of the simulations with 120 min. timesteps. The implication is that the residual errors for the largest timesteps are caused by other small truncation errors.

Fig. 10 Surface pressure traces as for Fig. 5, but using a bicubic advection scheme with iteratively derived advection velocities.

To provide a further comparison some results are presented for the Australian Region Primitive Equations (ARPE) model, using the same resolution and input data. Fig. 11 shows the time traces of surface pressure for $\Delta t=18$ min. and 36 min. As for the (STAG.SYM) model, the ARPE model shows negligible time truncation errors for these timesteps. The MSL pressure field (Fig. 12 for $\Delta t=36$ min.) is seen to differ from the semi-Lagrangian results. This is not surprising considering that the ARPE model uses a less accurate second order (in space) advection scheme. Also the ARPE semi-implicit equations are in flux form and

Fig. 11 Surface pressure traces as for Fig. 1, but for the ARPE model for $\Delta t=18$ min. and 36 min.

Fig. 12 24 hour MSL prognosis using the ARPE model for $\Delta t = 36$ min.

produce a different linearization in terms of the mass-weighted divergence. In that case the nonlinear N_p terms disappear from the equations corresponding to (7) and (8) but extra nonlinearities appear in the height gradient terms of (10) and (11).

Given that the only physical parameterization included is for surface drag, and that the models are not nested it is not surprising that the MSL prognoses have deficiencies, albeit somewhat similar. In this instance, the semi-Lagrangian model has more adequately retained the south-east Australia low pressure system, whilst the ARPE model has better retained the central pressures of the anti-cyclonic regions. Given the similar level of accuracy, the semi-Lagrangian formulation has computational advantages even for the same timestep; these include the avoidance of a Helmholtz equation, and reduced storage requirements from use of a two-time-level scheme.

The upper-wind prognoses from the various models were also examined to check the impact of the various advection schemes. Only marginal differences in jet stream intensity were found. However, the ARPE prognosis did show a suggestion of boundary aliasing with growth of the wind maximum near the southern and eastern boundaries. The ARPE model uses an Asselin time filter

and boundary 1-2-1 filtering each timestep. If the 1-2-1 filter is omitted from the ARPE model, then serious aliasing problems were evident in the winds near the lateral boundaries. By contrast, although the semi-Lagrangian (STAG.SYM) prognoses use no boundary smoothing, they exhibit no boundary difficulties.

6. Initialization

The vertical mode initialization scheme of Bourke and McGregor (1983) was employed. This procedure is very similar to nonlinear normal mode initialization in its ability to remove initial data imbalances, but may also be applied to limited area models. Estimates of the instantaneous time derivatives are required for divergence, vorticity and augmented height. This is most conveniently achieved by performing a single model timestep, and using the tendencies of all variables over that timestep. If simple explicit time differencing with space-centred horizontal advection is used for the calculation of these tendencies, then they will be independent of the chosen timestep.

If an implicit time differencing scheme is used during the initialization iterations, then a very small timestep should be used. In the limit as the timestep tends to zero, the tendencies will approach those of the explicit

calculation, apart from computer precision limitations.

If an upstream advection formula is used during these calculations, then estimates of the time derivatives depend on the size of the timestep. Moreover, in such calculations it was observed that the sign of the velocity components was switching at some mountainous grid points during the initialization iterations, inhibiting convergence. Use of the iterative procedure of Rasch (1985) did not assist in this convergence problem. If centred advection formulae were used for p^* and T (but only during initialization), then convergence resulted, basically as occurred for Bourke and McGregor (1983). With upstream advection the switching velocities select alternate points for the upstream derivatives, so that some advective height tendencies may remain relatively unchanged from one iteration to the next; an important feedback mechanism may then be absent from the convergence process.

Alternatively it was found that slow convergence could be attained with upstream advection if all fields were altered each iteration, but only half the usual increment used. However, it was found that the model state initialized in this manner suppressed spurious oscillations no more effectively than the simple explicit centred method of initialization. The simple centred advection method

for all variables, initializing 3 modes, was used to provide the initial state for all the results shown in the preceding sections. It was found to be completely satisfactory as can be seen from Fig. 13, which displays both uninitialized and initialized surface pressure traces for staggered-in-time symmetric (STAG.SYM) prognoses with $\Delta t = 36$ min.

7. Concluding Comments

It has been demonstrated in this paper that the time-split semi-Lagrangian ADI technique of Bates (1984) can be successfully modified to integrate the multi-level primitive equations. It was possible to use time steps larger than the usual CFL criterion, with little loss of accuracy. In order to achieve this accuracy it was necessary to reduce the splitting errors as follows:

- the geostrophic adjustment processes were staggered in time with respect to the horizontal advection processes, following Strang (1968),
- the horizontally-split ADI calculation was repeated to ensure x - y symmetry and thus increase its accuracy, following Strang (1963).

A horizontally-split cubic upstream advection scheme was utilized. It performed more efficiently and accurately than simple upstream bicubic interpolation for the cone test. However when used in the present split primitive equations model, little sensitivity was found to choice of horizontal advection scheme. This included comparisons using a split-quintic scheme, and other schemes using time-extrapolated and iteratively derived advection velocities.

This primitive equations model is stable even for very large timesteps, but some loss of accuracy with related gravity wave noise does become apparent in comparative pressure traces. The lack of sensitivity of the results to refinements of the horizontal advection procedure indicates that this problem for very large timesteps is symptomatic of small residual truncation errors, from either the splitting procedure or the treatment of the nonlinear terms.

Fig. 13 Surface pressure traces for $\Delta t = 36$ min. as for Fig. 5, but also including the uninitialized traces.

Because it avoids the solution of a Helmholtz equation, the ADI approach should produce computational savings over the schemes of Robert (1982), and McDonald (1986), although it appears that the latter schemes avoid the occurrence of extra gravity wave noise for very large timesteps. In practice, of course, the nature of the physical parameterizations may restrict one's choice of timestep.

The present model has advantages over standard centred-advection semi-implicit models such as the ARPE model, in terms of computational efficiency (even for the same size timestep), and ease of inclusion of more accurate horizontal advection formulae. Upstream advection schemes also avoid aliasing problems near the boundaries, which were detectable with the centred-advection model.

Very recently, Bates and McDonald (1986b) have successfully applied the ADI semi-Lagrangian approach to a multi-level primitive equations model, also using the Arakawa C-grid. By departing from the split approach, they have avoided the noise generation of simple splitting, although a computational penalty has resulted from the need for extra advective calculations at the half time levels.

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Appendix. Calculation of Horizontal Advection

An upstream cubic (Leonard, 1979) or quintic (Davies, 1980) advection scheme was usually used, split alternately in the x and y directions. As noted by Crowley (1968), this type of horizontal splitting ensures stability, where each component one-dimensional scheme is stable. The standard cone test of Orszag (1971) with an initial maximum of 100 units was performed for each scheme and several timestep sizes. In each case if the beginning of the trajectory was more than one grid unit away during a timestep,

then the origin for the interpolation formula was moved the appropriate number of grid points in that direction. It should be noted that if this translation process is not done, the schemes are mildly unstable for advection exceeding one grid length per timestep.

Table 1 compares the accuracy of the split-cubic and split-quintic schemes with results similar to McDonald (1984) for bicubic and biquartic interpolation. Diagnosis of the extreme values uses the bisection method together with bicubic interpolation. It can be seen that the split scheme always has better accuracy than the two-dimensional interpolation scheme of similar order. The split cubic scheme is also usually even more accurate than the non-split quartic scheme.

Table 1 Maximum, minimum, radial error in units of grid lengths, and angular error in degrees after one revolution of the cone test for various timestep sizes.

Number of steps	Maximum	Minimum	Radial error	Angular error
a. Bicubic				
288	55	-2	-0.7	-1°
48	60	-2	-2.7	-1°
24	56	-2	-4.3	-7°
16	44	-2	-5.5	-20°
b. Split-cubic				
288	57	-2	-0.1	-1°
48	75	-2	0	1°
24	79	-1	0	4°
16	81	-1	0.1	11°
c. Biquartic				
288	80	-10	-0.7	-5°
48	80	-6	-2.7	0°
24	71	-5	-4.3	-5°
16	58	-5	-5.3	-22°
d. Split-quintic				
288	80	-3	0	0°
48	86	-2	0	1°
24	86	-2	0	4°
16	86	-1	0.1	11°

As discussed by McDonald (1984), the iterative procedure of Robert (1981) may be used to provide corrections to the departure point of the trajectory each timestep. Whilst this

technique improves the accuracy for the two-dimensional interpolation method, no similar technique was found to improve the accuracy of the horizontally split advection schemes.

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