Problems on Nonlinear Computational Instability in NWP

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Abstract

In last years, several problems about nonlinear computational stability have been studied. Many nonlinear computational unstable examples of Richtmyer-type and Pernberg-type are obtained. It is shown that the instability may occur for leap-frog Lilly's scheme and leap-frog Arakawa's scheme which are used in the meteorology and keep energy-conservation instantaneously. In Galerkin finite element model and spectral model with leap-frog timedifference, similar examples of nonlinear instability also exist. The general mechanisms of nonlinear computational instability are discussed, particularly the effects of initial conditions on the computational stability are analysed. Furthermore, some theorems on the sufficient conditions for computational stability have been proved and some schemes with non-negativity of numerical operator which can guarantee the nonlinear computational stability are given.

1. Introduction

The computational stability is not only an important problem in computational mathematics, but also a common problem in numerical weather prediction, computational physics and computational mechanics and so on. It is well known that if the unsteady problem is described by the linear partial differential equations, its computational stability can be analysed by means of Von Neumann method or somewhat improved. As the finite difference method is used, and the ratio of time step to space step satisfies certain condition, the scheme is computational stable, otherwise it is unstable. However, in the nonlinear cases, such as Navier-Stokes equations or the numerical weather prediction equations, the stability criteria obtained by linear equation can only be used as a reference, the nonlinear difference equations possess particular unstable phenomenon which is called as nonlinear computational instability. Such computational instability was first discovered by Phillips (1959). After that Arakawa (1966), Lilly (1965) and Zeng (1982a) have constructed some schemes free from the nonlinear computational instability in the constrains of energy conservation. The newest schemes of such kinds are the so called energy and enstrophy conservation scheme (Arakawa, 1981), the perfect energy conservative scheme (Zeng et al., 1982b, 1985; Ji, 1980, 1982). The mathematical aspects of nonlinear computational instability have been summerized by Gary (1979) and Zeng and Ji (1983). Our recent results shown that only the perfect but not the instantaneous non-negativity of the numerical operator could guarantee the nonlinear computational stability. Perfect energy-conservative scheme (Zeng et al., 1982b; Zeng, 1985; Ji, 1980; Ji and Zeng, 1982; in the sense of time-space differences) belongs to the perfect non-negative scheme. We have found that the every scheme which is used in meteorology and keeps energy-conservation instantaneously (i.e. when ∂/∂t is not replaced by Δ/Δt) has an unstable numerical solution.

2. Examples of nonlinear computational stability in the advection equation

Rewriting the advection equation
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0
\]  
(2.1)

into the following
\[
\frac{\partial u}{\partial t} + (1 - \theta)u \frac{\partial u}{\partial x} + \theta \frac{\partial uu}{\partial x} = 0,
\]  
(2.2)

where \(0 \leq \theta \leq 1\) is a parameter, and approximating it by a leap-frog difference scheme
\[
\frac{u_{j}^{n+1} - u_{j}^{n-1}}{2\Delta t} + (1 - \theta)u_{j}^{n} \frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x} + \frac{\theta}{2} \frac{(u_{j+1}^{n})^{2} - (u_{j-1}^{n})^{2}}{2\Delta x} = 0.
\]  
(2.3)

we have a special solution in the form
\[
u_{j}^{n} = C_{n} \cos \frac{\pi j}{2} + S_{n} \sin \frac{\pi j}{2} + U_{n} \cos \frac{\pi j}{2} + V_{n}
\]  
(2.4)

where \(C_{n}, S_{n}, U_{n}\) and \(V_{n}\) satisfy the following difference equations
\[
\begin{align*}
C_{n+1} &= \frac{2\Delta t}{\Delta x} [(2\theta - 1) U_{n} - V_{n}] S_{n} \\
S_{n+1} &= \frac{2\Delta t}{\Delta x} [(2\theta - 1) U_{n} + V_{n}] C_{n} \\
U_{n+1} &= \frac{2\Delta t}{\Delta x} (\theta - 1) S_{n} C_{n} \\
V_{n+1} &= V_{n} = 0
\end{align*}
\]  
(2.5)

It can be easily shown that equations (2.5) have the special solution
\[
\begin{align*}
S_{n} &= 0 \quad \text{(when } n \text{ is odd)} \\
C_{n} &= 0 \quad \text{(when } n \text{ is even)}
\end{align*}
\]  
(2.6)

consequently, as a special case to (2.5), we get
\[
\begin{align*}
C_{n+1} &= \frac{2\Delta t}{\Delta x} [(2\theta - 1) U_{n} - V_{n}] S_{n} \\
S_{n+1} &= \frac{2\Delta t}{\Delta x} [(2\theta - 1) U_{n} + V_{n}] C_{n} \\
U_{n+1} &= \frac{2\Delta t}{\Delta x} (\theta - 1) S_{n} C_{n} \\
V_{n+1} &= V_{n} = 0
\end{align*}
\]  
(2.7)

From (2.7), we obtain
\[
C_{n+2} - (2 + \mu) C_{n} + C_{n-2} = 0
\]  
(2.8)

and
\[
S_{n+2} - (2 + \mu) S_{n} + S_{n-2} = 0
\]  
(2.9)

where
\[
\mu = \left(\frac{2\Delta t}{\Delta x}\right) [((2\theta - 1) \bar{U}_{1} + \bar{V}_{1} \cos \pi n) - ((2\theta - 1) \bar{U}_{1} \cos \pi n + \bar{V}_{1})]\]
\]  
(2.10)

and
\[
\bar{U}_{1} = (U^{n} + U^{n+1})/2, \quad \bar{V}_{1} = (U^{n} - U^{n+1})/2,
\]
\[
\bar{V}_{1} = (V^{n} + V^{n+1})/2, \quad \bar{V}_{1} = (V^{n} - V^{n+1})/2.
\]

Hence, if \(|1 + (\mu/2)| < 1\), then \(C_{n}\) and \(S_{n}\) are bounded; if \(|1 + (\mu/2)| > 1\), then \(C_{n}\) and \(S_{n}\) grown exponentially with \(n\). It is found that if
\[
\bar{V}_{1} \pm (2\theta - 1) \bar{U}_{1} \geq \left((2\theta - 1) \bar{U}_{1} \pm \bar{V}_{1}\right)
\]  
(2.11)

and
\[
S_{n} = C_{n+1} = 0
\]  
(2.12)

(or \(S_{n+1} = C_{n} = 0\), \(n = 0, 2, 4, 6, \cdots\))

then the solution (2.4) is always unstable. This is a unstable example of Richtmyer-type of generalization.

Besides this many nonlinear computational unstable examples of Fornberg type are obtained in Ji (1980), and Zeng and Ji (1981).

3. Example of nonlinear computational instability in leap-frog lilly's and arakawa's schemes

Lilly's scheme and Arakawa's scheme both keep energy conservation instantaneously. However, as the leap-frog time difference is used, the computational stability of these schemes can't be guaranteed.

Approximating the vorticity equation
\[
\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} = 0,
\]  
(3.1)

where \(\zeta = \psi_{f} \phi, u = - (\partial \phi / \partial y)\) and \(v = (\partial \phi / \partial x)\), by the leap frog Lilly's scheme
\[
\frac{\zeta_{j,k}^{n+1} - \zeta_{j,k}^{n-1}}{2\Delta t} + (\zeta^{2} u)_{x} + (\zeta v v)_{y} = 0.
\]  
(3.2)

we have a special solution to (3.2) in the form
\[
\psi_{j,k}^{n} = \left(\frac{C_{n} \cos \frac{\pi j}{2} + S_{n} \sin \frac{\pi j}{2} + U_{n} \cos \frac{\pi j}{2} + V_{n}}{2\Delta t} \right) \sin \frac{2\pi}{3} k
\]  
(3.3)
where $C^n, S^n$ and $U^n$ satisfy some difference equations, one of them is
\[ C^{n+2} - 2\mu C^n + C^{n+2} = 0 \quad (3.4) \]
where $\mu = 1 + (\sigma^2 U^n U^n)/2$ and $\sigma = 7\sqrt{3} \Delta t / 20 (\Delta x)^3$. Obvious, the solution is unstable as $\sigma^2 U^n U^n > 0$.

Approximating (3.1) by leap-frog Arakawa’s scheme, some unstable solutions have been found in Ji (1981a).

4. Examples of nonlinear computational instability in spectral model and galerkin finite element model with leap-frog time difference

Generally speaking, the spectral model is a quite effective and accurate model, and keeps instantaneously conservation of energy. However, when leap-frog time difference approximation is used, the conservation of energy may be violated, and hence nonlinear computational instability may also exist.

Approximating (3.1) by the spectral method with leap-frog time-difference, we have a special solution
\[ \phi = -\frac{A}{2} \cos ky - \frac{F}{k^2} \cos ky + 2G \sin ky \sin ky \quad (4.1) \]
where $A, F, G$ are determined by
\[
\begin{align*}
A^{n+1} &= A^n - 2\Delta t \left( \frac{1}{k^2} - \frac{1}{k^2 + l^2} \right) k l F^n G^n \\
F^{n+1} &= F^n + 2\Delta t \left( \frac{1}{k^2} - \frac{1}{k^2 + l^2} \right) k l A^n G^n \\
G^{n+1} &= G^n - 2\Delta t \left( \frac{1}{l^2} - \frac{1}{k^2} \right) k l A^n F^n
\end{align*}
\]
(4.2)
The solution is always unstable if the following condition
\[ C^* G^* > 0 \quad (4.3) \]
and
\[ A^n = F^{n+1} = 0 \quad (or \quad A^{n+1} = F^n = 0), \]
\[ \alpha = 0, 2, 4, 6, \ldots \quad (4.4) \]
is satisfied.

Similarly, if a leap-frog scheme is used to approximate the time derivatives of Galerkin finite element scheme, the scheme may be subject to nonlinear instability Ji (1984).

5. The relation between the computational stability and the non-negativity of operator or the energy conservation

Numerical weather prediction equations as well as the fluid dynamical equations can be formally written in an operator equation, the “evolution equation”
\[ \frac{\partial F}{\partial t} + \mathcal{A}_F F = 0 \quad (5.1) \]
where $F = F(X, t)$ is the function to be solved; $\mathcal{A}$ is a parameter, $\mathcal{A} = \mathcal{A}(F, X, t)$ is an operator, usually nonlinear, $X = X(x_1, \ldots, x_k)$ is the space coordinate, $k$ is the space dimension and $t$ is the time coordinate. Later, we will restrict ourselves to the “energy conservative” case, that is $(\mathcal{A}_a F, F) = 0$ and $\|F(0)\|^2 = \|F(0)\|^2$, where $(\mathcal{A}_a F, F)$ is the inner product (see below). In this case the mechanism of computational instability is more simple and clear.

Let the finite difference approximation for (5.1) be taken as follows
\[ \frac{F^{n+1} - F^n}{\tau} + A_a(F^n) [\theta F^{n+1} + (1 - \theta) F^n] = 0, \quad 0 \leq \theta \leq 1 \quad (5.2) \]
where $A_a$ is the difference analogue of the operator $\mathcal{A}_a$, and $F^*$ is some smoothed value of $F$.

The inner product of mesh functions $F$ and $G$ are defined as
\[ (G, F) = \sum_m F_m G_m A_m \]
In the case of one-dimensional problem $A_m = h$, and in two-dimensional case $A_m = h^2$.

The norm is taken as $\|F\| = (F, F)^{1/2}$. If $(AF, F) \geq 0$, the operator $A$ is called as non-negative; and if the equal sign is satisfied, it is called as anti-symmetric.

If every solution of the difference scheme (5.2) satisfies $\|F^n\| < C$, the scheme (5.2) is stable, where $C$ is a constant depending on $\|F^0\|$ but independent of $n$. Obviously, if $\|F^{n+1}\| < \|F^n\|$, then the scheme is stable.

In Zeng and Ji (1981), Ji and Zeng (1982), and Zeng and Ji (1983), the following theorems have been proved.
Theorem 1. If \( A \) is a non-negative operator, the scheme \((5.2)\) is absolutely stable as \(1 \geq \theta \geq (1/2)\); if \( A \) is anti-symmetric and \( 0 \leq \theta < (1/2) \) then the scheme \((5.2)\) is absolutely unstable.

Theorem 2. If \((AF, F) = 0\) and \( \theta = (1/2) \), then the scheme \((5.2)\) keeps the following conservations:

1. Energy conservation:
   \[
   \| F^{n+1} \|^2 \equiv \| F^n \|^2 = \| F^0 \|^2 ;
   \]
   \( (5.3) \)

2. Generalized energy conservation
   \[
   \| F^{n+1} \|^2 + \frac{\tau^2}{4} \| AF^{n+1} \|^2 = \| F^n \|^2 + \frac{\tau^2}{4} \| AF^n \|^2 ,
   \]
   \( (5.4) \)

3. "Mean scale" conservation
   \[
   \frac{\| F^{n+1} \|^2}{\| AF^{n+1} \|^2} = \frac{\| F^n \|^2}{\| AF^n \|^2} \]

Note: If the Crank-Nicolson scheme
   \[
   \frac{F^{n+1} - F^n}{\tau} + \theta AF^{n+1}F^{n+1} + \theta (1 - \theta) AF^n F^n = 0
   \]
   \( (5.6) \)
is used instead of \((5.2)\), then only \((5.4)\) can be obtained, but the other two conservation are lost. However, \((5.4)\) restricts the total energy and the mean scale, so that \(\| F^n \|^2\) and \(\| AF^n \|^2\) are still always bounded.

In summary, the advantages of conservation-type scheme which satisfy \((AF, F) = 0\) are evident, and \((5.2)\) is much better than \((5.6)\).

6. A discussion of the mechanism to nonlinear computational instability

In order to make the scheme stable, keeping the physical properties of the original differential equation is important. So far as meteorological problems are concerned, the importance of keeping energy conservation so as to eliminate false source has been emphasised in Lorenz (1960), Li and Zeng (1978). As what has been analysed above, the occurrence of nonlinear computational instability has the following common features. First, there is no explicit conservation of energy, or for the time-space difference schemes, some spurious sources of energy exist. Second, the false transfer process in the energy spectrum makes some special disturbances grow up quickly. The common feature of these special disturbances is that their wavelengths are comparable with the space step, the amplitude is large enough, and the discrete functions alter its signs very quickly both in the space and the time.

In practice, the initial field which leads to nonlinear instability contains more disturbance than the ideal case mentioned above. An example which is capable for illustrating this is given in Ji (1980). Using schemes \((2.3)\) with \( \theta = (1/2) \) and \( \theta = (2/3) \), taking \( u(x, \theta) = V + \sin 2\pi x \), \( h = 0.01 \), \( \tau = 0.004 \), and setting \( V = 1.5 \), the total energy is always finite (Fig. 2), although not conserved, but oscillates with time; If \( V = 0 \), then the signs
of initial disturbance chang from point to point, the total energy increase suddenly after some time, and explosive instability is occured (Fig. 1). This is a representative for phenomeon of nonlinear computational instability. If the scheme of perfect energy conservation is used, no matter what value of V is taken, the computational result is always of energy conservative. As can be seen from the solid straight line in Figs. 1 and 2. Mathematically, why are some difference schemes computational stable for constant coefficient differential equations, but computational unstable for the corresponding nonlinear equations?

Note that the above mentioned schemes are the time and space central difference schemes. If the function is continuous and the mesh tends to zero, it seems that the difference equation would approach the differential equation if the scheme meets the condition of consistency, consequently, the solution of the difference equation should have the same properties as the solution of the original differential equation. Unfortunately, the problem just lies on that the solution of nonlinear equation may not be a continuous function. General speaking, for the nonlinear equation, even if initial condition is smooth and differentiable, the solution may not be a smooth function as t tends to a certain value. The nonlinear advection equation (2.1) is a simplest example. Usually, we can obtain only some generalized or weak solutions for the nonlinear equations. But for the weak solution the conservation of energy perhaps does not exist. When using the difference method to solve the nonlinear problem, in general, we obtain a weak solution as \( h \to 0 \) and \( \tau \to 0 \). For example, as \( h \to 0 \) the solution (2.4) and (3.3) are everywhere discontinuous functions, so the consistency of difference and differential equations is violated. This shows that there exist quite large differences between linear and nonlinear equations, peculiar computational instability occurs for nonlinear equations.

8. On the construction of the computational stable schemes

As what is stated above, the scheme of energy conservation has some advantageous for long-term integration. We have pointed out in section 5, that energy conservation or boundness is closely related to the non-negativity of operator. In this respect, the problem of construction of difference scheme is transformed into the construction of non-negative operators. How can we construct a scheme with non-negative operator? In Ji (1982) some concrete methods have been given. Now, we give some examples.

For the advective equation

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \tag{7.1}
\]

one of the time-space finite difference scheme which perfectly conserves the energy is as follows

\[
\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{1}{3} \left[ \frac{u_j^* \tilde{u}_{j+1} - \tilde{u}_{j+1}}{2h} \right]
\]

\[
+ \frac{u_{j+1}^* \tilde{u}_{j+1} - u_j^* \tilde{u}_{j+1}}{2h} = 0 \tag{7.2}
\]

where \( \tilde{u}_j = (u_j^{n+1} + u_j^n)/2 \), and \( h \) are the grid sizes, and \( u_j^* \) is flexible, for example, \( u_j^* \) can be taken as \( \tilde{u}_j \). The solution of equation (7.2) is refered to Zeng et al. (1982b).

For the general evolution equation

\[
\frac{\partial F}{\partial t} + \mathcal{A}_a F = 0 \tag{7.3}
\]

if the operator \( \mathcal{A}_a \) and the inner production \( (F, \mathcal{A}_a F) \) are approximated by a finite difference operator \( A \) and summation \( (F, A^a F) \) respectively, and if \( (F, A^a F) = 0 \), then

\[
\frac{F^{n+1} - F^n}{\tau} + A_a (F^*) F = 0 \tag{7.4}
\]

is a scheme conserving the energy perfectly, where \( F = (F^{n+1} + F^n)/2 \), the subscript "*" is omitted, and the nonlinear finite difference operator \( A^a (F^*) \) depends on a flexible field \( F^* \) which can be taken as \( F \). Note that (7.4) is not the Crank-Nicholson scheme, it does not conserve the energy.

It is easy to know that the advective equation (7.1) belongs to (7.3), and (7.2) to (7.4). After a transformation of variables

\[
(U \equiv \sqrt{\phi} u, V \equiv \sqrt{\phi} v, \phi)
\]

(7.5)
the shallow water equations

\[
\begin{aligned}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v + \frac{\partial \phi}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u + \frac{\partial \phi}{\partial y} &= 0 \\
\frac{\partial \phi}{\partial t} + \frac{\partial \phi u}{\partial x} + \frac{\partial \phi v}{\partial y} &= 0
\end{aligned}
\]  

(7.6)

are transformed into the followings

\[
\frac{\partial U}{\partial t} + \left\{ \frac{1}{2} \left( \frac{\partial u U}{\partial x} + u \frac{\partial U}{\partial x} \right) + \frac{1}{2} \left( \frac{\partial u V}{\partial y} + u \frac{\partial V}{\partial y} \right) \right\} - f V + \Phi \frac{\partial \Phi}{\partial x} = 0
\]

\[
\frac{\partial V}{\partial t} + \left\{ \frac{1}{2} \left( \frac{\partial u V}{\partial x} + u \frac{\partial V}{\partial x} \right) + \frac{1}{2} \left( \frac{\partial u V}{\partial y} + u \frac{\partial V}{\partial y} \right) \right\} + f U + \Phi \frac{\partial \Phi}{\partial y} = 0
\]

\[
\frac{\partial \Phi}{\partial t} + \left\{ \frac{\partial \Phi u}{\partial x} + \frac{\partial \Phi v}{\partial y} \right\} = 0
\]

where \( \Phi = \sqrt{\phi} \). Obviously, (7.7) belongs to (7.3) with a vector \( F = (U, V, \phi) \) and the energy

\[
\frac{1}{2} \left\| F \right\|^2 = \frac{1}{2} \iint (U^2 + V^2 + \phi^2) \, dx \, dy
\]

\[
= \frac{1}{2} \iint [\phi (u^2 + v^2) + \phi^2] \, dx \, dy
\]

One of the perfectly energy-conservative schemes is as follows

\[
\begin{aligned}
\frac{U^{n+1} - U^n}{\Delta t} &= \frac{1}{2} \left[ \frac{\partial \delta u u^* U}{\partial x} + u^* \frac{\partial \Phi}{\partial x} \right] \\
&\quad + \frac{1}{2} \left[ \frac{\partial \delta v v^* U}{\partial y} + v^* \frac{\partial \Phi}{\partial y} \right] \\
&\quad - f \frac{\partial \delta \phi}{\partial y} + \beta_1 \frac{\partial \delta \phi}{\partial x} = 0
\end{aligned}
\]

\[
\begin{aligned}
\frac{V^{n+1} - V^n}{\Delta t} &= \frac{1}{2} \left[ \frac{\partial \delta u u^* V}{\partial x} + u^* \frac{\partial \Phi}{\partial x} \right] \\
&\quad + \frac{1}{2} \left[ \frac{\partial \delta v v^* V}{\partial y} + v^* \frac{\partial \Phi}{\partial y} \right] \\
&\quad + f \frac{\partial \delta \phi}{\partial x} + \beta_1 \frac{\partial \delta \phi}{\partial y} = 0
\end{aligned}
\]

\[
\begin{aligned}
\frac{\Phi^{n+1} - \Phi^n}{\Delta t} &= \frac{1}{2} \left[ \frac{\partial \delta \phi \phi^* U}{\partial x} + \phi^* \frac{\partial \Phi}{\partial x} \right] \\
&\quad + \frac{1}{2} \left[ \frac{\partial \delta \phi \phi^* V}{\partial y} + \phi^* \frac{\partial \Phi}{\partial y} \right] \\
&\quad + \gamma \frac{\partial \delta \phi}{\partial x} + \beta_2 \frac{\partial \delta \phi}{\partial y} = 0
\end{aligned}
\]

where \( \gamma, \alpha, \beta_1, \) and \( \beta_2 \) are some flexible parameters, \( u^*, v^*, \Phi^* \) are flexible fields, and the \( \delta_x(\cdot)/\delta x \) and \( \delta_y(\cdot)/\delta y \) are the central differences. The solution of (7.8) is referred to Zeng et al. (1982b).

The operators \( (\partial u F/\partial x) + u (\partial F/\partial x) \sqrt{2} \) can be approximated also by the following finite scheme

\[
\frac{1}{2} \left[ \frac{\partial u F}{\partial x} + u \frac{\partial F}{\partial x} \right] + \left( \frac{u^*_+ - u^*_-}{2} \right) F_{i+1}
\]

\[
\approx \frac{1}{2\Delta x} (u^*_+ + u^*_-) F_{i+1} - (u^*_+ - u^*_-) F_{i-1} (7.9)
\]

Similarly

\[
\frac{1}{2} \left[ \frac{\partial v F}{\partial y} + v \frac{\partial F}{\partial y} \right] + \left( v^*_+ - v^*_- \right) F_{j+1} - (v^*_+ - v^*_-) F_{j-1} (7.10)
\]

Then, we get another perfectly energy-conservative scheme, which is more economical but with staggered grid. Such scheme has been extended to spherical grid, as well as to baroclinic atmosphere and coupled ocean-atmosphere-land model in Zeng (1983), and has been used in Zeng et al. (1984). Note, that for the baroclinic atmosphere, the transformation of variables is as follows

\[
\begin{align*}
V_x &= \sqrt{\rho} U, & V_y &= \sqrt{\rho} V, \\
H &= \sqrt{\rho} T, & p_v &=
\end{align*}
\]

(7.11)

where \( p_v \) is the root of surface pressure, and can be also approximated by its standard.

References


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