The Generalized Lagrangian-Mean (GLM) Description in the General Coordinate System

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Abstract

The generalized Lagrangian-mean (GLM) description formulated by Andrews and McIntyre is extended to the general coordinate system. Four-dimensional Lagrangian coordinates are introduced to obtain a general relationship between the Lagrangian coordinate mean (LCM) and the GLM. It is shown that the choice of an initial hypersurface in space-time is essential in the determination of the relationship. The Eulerian mean and the GLM tensors are defined referring to a given coordinate system, so that mean quantities are dependent on the choice of the coordinate system. Symmetries in the Lagrangian density for fluids and related conservation laws are discussed for energy-momentum, pseudoenergy-pseudomomentum and wave-action. The extended GLM description provides a wider applicability in practice, owing to the less stringent assumptions for the initial conditions.

1. Introduction

The Generalized Lagrangian-mean (GLM) theory of Andrews and McIntyre (1978a,b; hereafter denoted as AMa and AMb, respectively) has been successfully applied to the study of the interaction between waves and mean flows (for recent reviews see McIntyre, 1980a; Dunkerton, 1980; Grimshaw, 1984a,b). The nonacceleration theorem, which had been derived for small amplitude waves by Eliassen and Palm (1961) and Charney and Drazin (1961), was generalized to an exact finite-amplitude theorem (AMa). The wave-action equation (Whitham, 1965, 1970; Bretherton and Garrett, 1968; Hayes, 1970) was also generalized to finite-amplitude, non-conservative cases (AMb). The GLM theory has also been used for theoretical studies of atmospheric phenomena, for example, stratospheric sudden warming (Matsumo and Nakamura, 1979; Dunkerton et al., 1981), trace gas transport (Matsumo, 1980; Rood and Schoeberl, 1983), mean meridional mass motions (Dunkerton, 1978) and wave-mean flow interaction (Schoeberl, 1981).

However, if the GLM theory is applied to observational studies, some difficulties arise. Following AMa, if a fluid particle is located at \( X^* \) in space at time \( t \), let \( x \) be the GLM point for \( X \) and let \( \phi(t, x) \) be the displacement defined such that

\[
X(t, x) = x + \xi(t, x). \tag{1.1}
\]

If \( \phi(t, x) \) is a quantity to be averaged and \( \phi(t, x) \) is its Eulerian mean at \( (t, x) \) in any of the usual senses (time, space, ensemble, etc.), then the corresponding GLM is defined by

\[
\phi(t, x) = \phi(t, x + \xi(t, x)), \tag{1.2a}
\]

and

\[
\xi(t, x) = 0. \tag{1.2b}
\]

Thus, one difficulty is how to solve the set of equations (1.2a,b) since it is neither \( x \) nor \( \xi(t, x) \) but \( \phi(t, X(t, x)) \) that is given as a function of \( X \).

* Boldfaced italic letters are used for spatial components in this section but boldfaced roman letters will be used for space-time components in the following sections.
and time $t$. The other difficulty stems from the postulate (viii) of AMa, i.e., there are no disturbances

$$|\xi|^2 = 0,$$  \hspace{1cm} (1.3a)

and

$$|\phi'|^2 = 0,$$ \hspace{1cm} (1.3b)

at an initial time $t = t_0$, where $\phi' = \phi - \bar{\phi}$ denotes the Eulerian disturbance field. Although (1.3a) is a matter of definition, (1.3b) cannot be satisfied if the GLM theory is to be applied to the real atmosphere, which is not uniform in any direction in the space-time domain. Thus, we must reformulate the GLM without assuming (1.3b). To overcome this difficulty, McIntyre (1980a,b), Dunkerton (1980) and Dunkerton *et al.* (1981) introduced a modified Lagrangian mean using potential temperature and Ertel's potential vorticity as the Lagrangian coordinates. However, the modified Lagrangian mean description cannot be applied to non-conservative fluids, because neither potential temperature nor Ertel's potential vorticity can be the Lagrangian or material coordinate for the non-conservative fluids.

There is another difficulty in defining GLM points in general situations. To clarify the argument, consider how to obtain a mean point between given points A and B on a plane as shown in Fig. 1. In the Cartesian coordinate system $(x, y)$, A and B take the coordinates $(0, 5)$ and $(5, 0)$, respectively. Hence, it is natural to assign a mean point C with the coordinates $(5/2, 5/2)$. On the other hand, A and B take the coordinates $(5, \pi/2)$ and $(5, 0)$, respectively, in the polar coordinate system $(r, \theta)$. Here, it is natural to assign a mean point $C'$ with the coordinates $(5/2, \pi/2)$.

![Fig. 1. An illustration showing how a mean point depends on the choice of the coordinate system. A mean point of A and B becomes C if the Cartesian coordinate system $(x, y)$ is used. However, a different point $C'$ is obtained if the polar coordinate system $(r, \theta)$ is used.](image)

$$r_C' = \frac{r_A + r_B}{2},$$

$$\theta_C' = \frac{\theta_A + \theta_B}{2}$$

$$x_C = \frac{x_A + x_B}{2},$$

$$y_C = \frac{y_A + y_B}{2}$$
coordinates \((5, \pi/4)\). However, \(C\) and \(C'\) indicate different points on the plane, showing that the mean point depends on the choice of the coordinate system. The coordinate-dependent averaging was taken by Kida (1977, 1983) and Dunkerton et al. (1981) to discuss the Lagrangian motion of the atmosphere in the meridional plane.

To avoid the coordinate dependence on defining the GLM, AMa, McIntyre (1980b) and Dunkerton (1980) presented a so-called true vector Lagrangian mean. They implicitly assumed Euclidean geometry so that an oriented segment of a straight line from a point \(p\) to a point \(q\) can be regarded as a vector \(\overrightarrow{pq}\). In contrast to the coordinate mean for the case shown in Fig. 1, the vector mean assigns the unique point \(C\) by \(\overrightarrow{OC} = (\overrightarrow{OA} + \overrightarrow{OB})/2\). However, the introduction of Euclidean geometry to define the displacement \(\xi\) as a vector will, in general, cause some strange results. For instance, consider a finite-amplitude, purely longitudinal wave on a thin, elastic ring which is always centered on the axis of rotation and is constrained to lie in a plane perpendicular to that axis. Then the velocity vector of any point on the ring always has null components transversal to the ring. Intuitively, this fact seems to imply that the GLM velocity vector also has null transversal component, and hence that the GLM points are on the ring and both of the initial and terminal points of the displacement vector \(\xi\) are on the same ring. However, if the azimuthal averaging operator defined in the footnote on page 614 of AMa or by Eq. 19 of Dunkerton (1980) is applied on \(\xi\), the condition (1.2b) would never be satisfied, because the vector \(\xi\) always has a non-vanishing component directed toward the center of the ring owing to the curvature of the ring. Therefore the GLM points should be inside of the ring to satisfy (1.2b). Indeed, Miyahara (private communication, 1987) and one of the referees have showed that the GLM velocity derived from the averaging operator has a radial component toward the center of the ring, and hence that the GLM points are on a ring inside of the material ring.

Moreover, Miyahara has pointed out that the inward GLM motion originates from the fact that the vector Lagrangian mean implicitly uses the concepts of parallel transport of velocity vector in the Euclidean space (e.g., Eq. 19b of Dunkerton, 1980).

In this paper, four-dimensional Lagrangian coordinates (or, for short, four-Lagrangian coordinates) are first introduced in Section 2 to extend the original GLM in a four-dimensionally symmetrical way and to give a practical method for calculating the GLM from observed data via the Lagrangian coordinate mean (LCM). As strange as it seems, no general relationship has yet been given between the LCM and the GLM, which has been often cited as the Lagrangian mean for short. Only Matsuno (1980) touched on this topic in the case of the zonal mean of a steady planetary wave. One of the reasons why the general relationship has not been given may lie in the fact that the conventional Lagrangian coordinates have only spatial components, while the GLM can be defined symmetrically in space and time. Indeed, a kinetic model of the time GLM presented by Dunkerton (1983) does not appear to be symmetrical with the spatial GLM model of AMa. (Compare Fig. 1 of AMa with Fig. 1 of Dunkerton.) However, the symmetry will be recovered in Section 2 by extending the conventional three-Lagrangian coordinates to four-Lagrangian coordinates. In Section 3, using a general coordinate system, the Eulerian mean and the GLM of tensors are defined in a coordinate dependent way to avoid such a strange phenomenon of the vector Lagrangian mean mentioned above. A theorem stating a general relationship between the LCM and the GLM is given in Section 4. Owing to this relationship, the first two difficulties can be overcome. Some examples of the Lagrangian means are found in Section 5. The covariant form of equations used for fluid mechanics and their GLM forms are presented in Section 6, where the conservation of (pseudo-) energy and (pseudo-) momentum is also discussed. Section 7 gives concluding remarks. An application of the extended GLM description in the present paper to a long term numerical simulation of atmospheric motion will be given in a separate paper (Noda, 1988a).

2. Four-Lagrangian coordinates

In the following, a four-dimensional notation is used to treat space and time symmetrically.
Greek indices take on the values 0, 1, 2, 3, Latin indices take on the values 1, 2, 3, and Einstein's summation convention is adopted. Fluid motions are considered as a transformation of the space-time manifold \( M \) into itself. The manifold \( M \) is locally identical with a region of \( R^4 \), i.e., the set of all 4-tuples of real numbers \((x^0, x^1, x^2, x^3)\) \((-\infty < x^\mu < +\infty)\), so that a point \( x \in M \) is denoted by \((x^0, x^1, x^2, x^3) = (\xi^0(x), \xi^1(x), \xi^2(x), \xi^3(x))\), using a general coordinate system \( \xi = (\xi^\mu) \), where \( x^0 \) and \( x^i \) denote the time and space coordinates, respectively. Only the coordinate basis \( \{\partial / \partial x^\mu\} \) for vectors and its dual basis \( \{dx^\mu\} \) are used for one-forms in this paper. Any tensor \( T(x) \) at \( x \) can be expressed in terms of the basis

\[
\omega^\mu_{\alpha\beta\cdot\cdot\cdot}(x) = d\xi^\mu \otimes d\xi^\nu \otimes \cdots \otimes \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta} \otimes \cdots
\]

(2.1)
as

\[
T(x) = T^\mu_{\beta\cdot\cdot\cdot}(x) \omega^\nu_{\alpha\beta\cdot\cdot\cdot}(x),
\]

(2.2)
where denotes the tensor product. Hence, the tensor \( T(x) \) is denoted by its components \( T^\mu_{\beta\cdot\cdot\cdot}(x) \) unless otherwise stated. Similarly, a notation such as \( x = (x^\mu) \) will be used for brevity. See e.g. Landau and Lifshitz (1971), Misner et al. (1973) or Schutz (1980) for relevant details.

Now suppose that a fluid particle traces a curve from \( a = (a^\mu) \to X = (X^\mu) \) in \( M \). The curve is mathematically defined as a map \( C \) of an interval on the real line \( R^1 \) into \( M \), i.e., \( \tau \in R^1 \to C(\tau) \in M \) with \( a = C(\tau_a) \) and \( X = C(\tau) \). According to the conventional Lagrangian description of fluid motions (e.g. Lamb, 1932) as shown in Fig. 2; 1) \( R^1 \) is identified with the time coordinate, i.e., \( \tau = t = \tau_0 \); 2) any fluid particle is assumed to start from a special hypersurface \( H_0 \) which is specified by the relation \( a^0 = \text{constant} \) in \( M \); 3) the space coordinates \( (X^\mu) \) are expressed as functions of \( t \) and \( a^i \), i.e., \( X^i = X^i(t, a^i) \), suppressing the time coordinate of \( a \). Thus, as shown in Fig. 3, it is a natural extension of the conventional Lagrangian description that 1) the parameter \( \tau \) is retained to distinguish the real line \( R^1 \) from the time coordinate \( t \); 2) any fluid particle is assumed to start from a hypersurface \( H_0 \) specified by the generalized relation

\[
h_\theta(a) = 0,
\]

(2.3)

* A hypersurface is a submanifold of \( M \) whose dimension is one less than that of \( M \).
where $\theta$ is a parameter independent of $a$ and $\tau$; 3) the space-time coordinates $(x^\mu)$ are expressed as functions of $\tau$ and $a$, i.e., $X^\mu = X^\mu(a; \tau)$, in which (2.3) is implicitly assumed. Alternatively, noting that $H_\theta$ is a three-dimensional manifold, which is locally identical to a region of $R^3$, the three-Lagrangian coordinates $(A^i)$ can be assigned to $a \in H_\theta$. In this case, $a^\mu$ are expressed as functions of $A^i$, such as $a^\mu = a^\mu(A^i)$, and the relation (2.3) must become an identity as a function of $A^i$ because $A^i$ are independent coordinates on $H_\theta$. The conventional Lagrangian description can be recovered by specifying

$$h_\theta(a) = a^0 - \theta = 0, \quad a^i = A^i.$$  

(2.4)

Now the four-dimensional velocity (four-velocity) vector field is defined by

$$\bar{U}(X(a; \tau)) = \frac{dC(\tau)}{d\tau}$$  

(2.5a)

In components,

$$\bar{U}^\mu(X(a; \tau)) = \frac{\partial X^\mu(a; \tau)}{\partial \tau}$$  

(2.5b)

Hence

$$X^\mu(a; \tau) = a^\mu + \int_0^{\tau} \bar{U}^\mu(X(a; \tau')) d\tau'.$$  

(2.6)

To complete the Lagrangian description, we must introduce a relation

$$\frac{\partial (X^\mu)}{\partial (A^i, \tau)} \neq 0,$$  

(2.7)

i.e., $X^\mu$ and $(A^i, \tau)$ correspond one-to-one. For the conventional Lagrangian description, (2.7) reduces to $-\det(\partial X^i/\partial A^i) \neq 0$ since $\partial X^\mu/\partial \tau = 1$ and $\partial X^\mu/\partial A^i = 0$. If a function of $a$ is introduced such that $A^\mu = h_\theta(a)$, then

$$\frac{\partial (A^\mu)}{\partial (a^\mu)} \neq 0,$$  

(2.8)

because otherwise $A^0$ is expressed as a function of $A^i$, which contradicts $H_\theta$ being three-dimensional. Thus, since

$$\frac{\partial (X^\mu)}{\partial (A^i, \tau)} = \frac{\partial (A^\mu, X^\mu)}{\partial (A^\mu, A^i, \tau)} = \frac{\partial (h_\theta, X^\mu)}{\partial (a^\mu, \tau)},$$  

(2.9)

(2.7) implies, with (2.8), that

$$\frac{\partial (X^\mu, h_\theta)}{\partial (a^\mu, \tau)} \neq 0,$$  

(2.10)

which further reduces to

$$-\det(\partial X^\mu/\partial a^\mu) U^i \frac{\partial h_\theta}{\partial X^i} \neq 0.$$  

(2.11)

Here $h_\tau(X(a; \tau)) = h_\theta(a(X))$ is given by solving $a^\mu$ with respect to $X^\mu$ from (2.6) for a fixed $\tau$; hence $\frac{\partial h_\tau}{\partial X^i} = (\partial a^\mu/\partial X^i) (\partial h_\theta/\partial a^\mu)$. Note that $h_\tau(X(a; \tau)) = 0$ determines the hypersurface $H_\tau$ on which the fluid particle started from $H_\theta$ at $\tau_0$ locates at $\tau$. Therefore, (2.11) shows that 1) $\det(\partial X^\mu/\partial a^\mu) \neq 0$, i.e., $X$ and $\tau$ correspond one-to-one for a fixed $\tau$, and 2) $\bar{U}^i \frac{\partial h_\tau}{\partial X^i} \neq 0$, i.e., $X$ and $\tau$ correspond one-to-one for a fixed $a$.

3. The Eulerian mean and the GLM

The Eulerian mean and the GLM of tensors are redefined using the general coordinate system $X = (x^\mu)$ introduced in the previous section.

3.1 The Eulerian mean of tensors

Let the coordinate system $X = (x^\mu)$ be fixed, and let $f(X; a)$ be a function of $X = (x^\mu)$ and a set of ensemble parameters $a = (a_1, a_2, \ldots)$. (Hereafter the $a$ dependence is omitted from the expression of the function for simplicity.) Then the Eulerian mean of $f(X)$, which is denoted by $\bar{f}(X)$, is defined in the usual sense (over time coordinate $X^0$, space coordinate $X^i$, ensemble parameter $a_n$, etc.) so as to satisfy the postulates (i) – (vi) of AMa*. Now consider the Eulerian mean of the tensor (2.2). Since the components $T_{\mu\nu\ldots}^{\beta\gamma\ldots}(X)$ are functions, their Eulerian means $\bar{T}_{\mu\nu\ldots}^{\beta\gamma\ldots}(X)$ can be given in the usual sense. Therefore, the Eulerian mean of $T(X)$ is defined as

$$\bar{T}(X) \equiv \bar{T}_{\mu\nu\ldots}^{\beta\gamma\ldots}(X) \omega_\mu^{\beta}(X) \omega_\nu^{\gamma}(X).$$  

(3.1)

* The postulate (v) of AMa*, i.e., $\bar{X}^\mu = x^\mu$, is incorrect if the average is taken over one of the coordinates, say $X^i$. In this case, the postulate is true only if $\mu \neq \lambda$. 

\[ \frac{\partial (a^\mu, \tau)}{\partial (A^0, A^i, \tau)} = \frac{\partial (h_\theta, X^\mu)}{\partial (a^\mu, \tau)} \]
Note that the Eulerian mean operator \( \overline{\cdot} \): \( T \rightarrow \overline{T} \) depends on the choice of the coordinate system. This kind of definition for the Eulerian mean of tensors was suggested by McIntyre (1980b) and has been implicitly used in practice for the spherical coordinate system (Kida, 1977, 1983; Dunkerton et al., 1981; Grimshaw, 1984b) and for the polar coordinate system (Grimshaw, 1984a).

The definition (3.1) can be seen from another point of view. Let \( T \) and \( T' \) be tensor fields on the space-time \( M \). Then since the summation of tensors is possible only for the tensors defined at the same point \( X \) in a form such as \( aT(X) + bT'(X) \), where \( a \) and \( b \) are scalars, forms such as \( aT(X) + bT'(X') \), \( aT(X) + bT'(X') \), \( \ldots \) (\( X \neq X' \)) make no sense without introducing a rule for transporting the tensors to a certain common point. The Eulerian mean of \( T \) along a curve therefore makes no sense as it stands because different points are involved. Thus one is led to the notion of parallel transport of tensors (see e.g. Schutz, 1980). However, since there is no global notion of parallelism, parallel transport can be defined for the convenience of the problem at hand. The notion of parallelism used in (3.1) is as follows. First, the general coordinate system \( \chi = (\chi^\nu) \) is fixed, and then the tensors are parallel-transported in their coordinate space \( R^4 \) such that the Christoffel symbols \( \Gamma^{\beta}_{\alpha\nu} \) vanish* in referring to the coordinate system \( \chi \). Consequently, the summation of \( T(X) \) and \( T'(X') \) is defined as

\[
(T+T')(X) = \left[ T^{\alpha\beta\cdots}_{\mu\nu\cdots}(X) + T'^{\alpha\beta\cdots}_{\mu\nu\cdots}(X') \right] \omega^{\rho \delta \cdots}_{\beta \lambda \cdots}(X),
\]

(3.2)

\[
(T+T')(X') = \left[ T^{\alpha\beta\cdots}_{\mu\nu\cdots}(X') + T'^{\alpha\beta\cdots}_{\mu\nu\cdots}(X') \right] \omega^{\rho \delta \cdots}_{\beta \lambda \cdots}(X'),
\]

which leads to the definition (3.1).

By using parallelism in Euclidean geometry, a different notion of parallel transport was suggested for spherical geometry by AMa, Dunkerton (1980) and McIntyre (1980b); however, in practice it gave a strange result as shown in Fig. 4 of McIntyre (1980b) — reference rings went underground, i.e., outside the atmosphere, as disturbances in the atmosphere grew. This result can be reproduced in the present formulation if the parallel transport is defined referring not to the spherical coordinate system but to the cylindrical coordinate system.

3.2. The GLM

Let \( g \) be a diffeomorphism** from the space-time to itself, i.e., \( g: N \rightarrow M; x \rightarrow X \) for \( x \in N \subset M \) and \( X \in M \). Here it is assumed that \( x \) and \( X \) are covered with the fixed coordinate system \( \chi = (\chi^\nu) \) used for the Eulerian mean. Then, if \( f \) is a function on \( M \), the transformation \( g \) defines the function \( g^*f \) on \( N \) such that

\[
g^*f(x) = f(g(x)).
\]

(3.3)

Similarly, if \( T \) is a tensor on \( M \), \( g \) defines the tensor \( g^*T = \overline{T} \) on \( N \) such that

\[
g^*T^\alpha_{\mu\nu\cdots}(x) = g^* T^\alpha_{\mu\nu\cdots}(x) \omega^{\rho \delta \cdots}_{\beta \lambda \cdots}(x)
\]

(3.4)

with

\[
g^*T^\alpha_{\mu\nu\cdots}(x) = T^\alpha_{\mu\nu\cdots}(g(x)),
\]

(3.5a)

\[
g^* \omega^{\rho \delta \cdots}_{\beta \lambda \cdots}(x) = \omega^{\rho \delta \cdots}_{\beta \lambda \cdots}(X^\alpha) X^\nu_{\rho\lambda\cdots} x^\mu_{\nu\delta\cdots} \cdots \omega^{\rho \delta \cdots}_{\beta \lambda \cdots}(x),
\]

(3.5b)

so that

\[
\overline{T}^\alpha_{\mu\nu\cdots}(x) = T^\alpha_{\mu\nu\cdots}(g(x)) X^\nu_{\rho\lambda\cdots} X^\mu_{\nu\delta\cdots} \cdots \omega^{\rho \delta \cdots}_{\beta \lambda \cdots} X^\alpha_{\nu\lambda\cdots} \cdots X^\alpha_{\nu\delta\cdots} ..., \cdots,
\]

(3.5c)

where \( g(x), g^{-1}(X) \) and their derivatives \( Dg(x), Dg^{-1}(X) \) are denoted by \( (X^\nu(x)), (\chi^\nu(X)), (X^\nu_{\rho \lambda \cdots}) = (\partial X^\nu/\partial x^\rho) \) and \( (X^\nu_{\rho \lambda \cdots}) = (\partial X^\nu/\partial X^\rho) \), respectively.

Next, introduce an operator \( (\overline{\cdot})^L \) which satisfies

\[
\overline{f}(X)^L = \overline{g^*f}(x) = \overline{f}(g(x)),
\]

(3.6)

\[
\overline{T}(X)^L = \overline{g^*T}(x) = \overline{T}^\alpha_{\mu\nu\cdots}(x) \omega^{\rho \delta \cdots}_{\beta \lambda \cdots}(x).
\]

(3.7)

Hence \( (\overline{\cdot})^L \) maps a function (tensor) on \( M \) to a function (tensor) on \( N \). In particular, \( (\overline{\cdot})^L \) maps

* On referring to the other coordinate system \( \chi' = (\chi'^\nu) \), the Christoffel symbols take the form \( \Gamma^{\beta}_{\alpha\nu} = (\partial X'^\alpha/\partial x^\nu)(\partial/\partial X'^\beta)(\partial X'^\gamma/\partial x^\nu) \), where the coordinates of \( X \in M \) are denoted by \( (X^\nu) \) and \( (X'^\nu) \) for the coordinate systems \( \chi \) and \( \chi' \), respectively.

** A mapping \( g \) is a diffeomorphism if \( g \) is a bijection (one-to-one and onto mapping) with \( g \) and its inverse \( g^{-1} \) being continuously differentiable (see e.g. Schutz, 1980).
the fluid velocity vector field $\tilde{U} = (\tilde{U}^\mu)$ on $M$ to $\tilde{u} = (\tilde{u}^\mu)$ on $N$ such that $\tilde{U}^\mu(X) = X\cdot \tilde{u}^\mu(x)$. Now the map $g$, the operator $(\cdot)^L$, and the mean associated with $(\cdot)^L$ can be called the GLM map, the GLM operator and the GLM, respectively, if the Eulerian mean of the displacement

$$\xi^\mu(x) = X^\mu(x) - x^\mu = X^\mu(x) - x^\mu = X^\mu(x) - x^\mu = X^\mu(a; \tau) = X^\mu(a; \tau) = X^\mu(a; \tau) = X^\mu(a; \tau).$$

vanishes, i.e.,

$$\xi^\mu(x) = 0,$$

and $\tilde{u}^\mu$ itself is a mean quantity, i.e.,

$$\tilde{u}^\mu(x) = \tilde{u}^\mu(x).$$

4. The relationship between the LCM and the GLM

Let the coordinate system $\chi = (X^\mu)$, the four-velocity vector field $\tilde{U} = (\tilde{U}^\mu)$, and the initial hypersurface $H_\theta (\bar{h}_\theta (a) = 0)$ be the same as in Section 2. The transformation $\tilde{g}$ is defined from $N \subset M$ to $M$ such that $\tilde{g}$ associates each $X \in M$ given by (2.6) with a point $x \in N$:

$$x^\mu(a; \tau) = a^\mu + \int_0^\tau u^\mu(x(a; \tau')) d\tau', \quad (4.1a)$$

$$u^\mu(x(a; \tau)) = \langle U^\mu(X(a; \tau)) \rangle_{b}, \quad (4.1b)$$

where $b$ stands for some of the Lagrangian coordinates $(a^\mu)$ or some of the ensemble parameters $a = (a\mu) \in (\cdot)_b$ and denotes the $b$-Lagrangian coordinate mean ($b$-LCM)* where the average is taken in the Eulerian sense over $b$ with $\tau$ and $H_\theta$ fixed. Also, introduce the Eulerian average operator $\langle (\cdot) \rangle_b$ instead of $\langle (\cdot) \rangle$ to explicitly show a set of variables used for the average. By operating $\langle (\cdot) \rangle_b$ on (2.6) and using (4.1b), (4.1a) can be rewritten as

$$x^\mu(a; \tau) = a^\mu + \langle X^\mu(a; \tau) \rangle_b - a^\mu = X^\mu(a; \tau) - a^\mu.$$ (4.2)

Since $H_\theta$ is fixed when taking the $b$-LCM, $d\bar{h}_\theta = (\partial \bar{h}_\theta / \partial a^\mu) da^\mu = 0$ but $da^\mu = 0$ if $b = a^\mu$. Therefore the condition $\partial \bar{h}_\theta / \partial a^\mu = 0$ is necessary for the $a^\mu$-LCM to be defined. Thus the $b$-LCM is called being compatible with $H_\theta$ if $d\bar{h}_\theta = 0$ holds on taking the average.

Next, define the displacement field $\xi^\mu(a; \tau)$ associated with the $b$-LCM by

$$\xi^\mu(a; \tau) = X^\mu(X(a; \tau)) - X^\mu(x(a; \tau)) = X^\mu(a; \tau) - x^\mu(a; \tau).$$ (4.3)

Hence $\xi^\mu(a; \tau) = 0$ on $H_\theta$ at $\tau = \tau_0$ and $\langle \xi^\mu(a; \tau) \rangle_b = 0$ by definition. The former relation satisfies the postulate (viii) of AMa. However, it is not assumed that the Eulerian disturbances $f(X) - f(X)$ or $T(X) - T(X)$ vanish on $H_\theta$ even though $\xi^\mu(a; \tau) = 0$ on $H_\theta$. This requires some modifications to the GLM equations of motion, continuity, thermodynamics etc., as will be shown in Section 6.

Now the relationship between the LCM and the GLM is given by the following theorem.

**Theorem.** Let $\tilde{g}$ be a map associated with a $b$-LCM which is compatible with a given initial hypersurface $H_\theta$. Then if

$$\frac{\partial(X^\mu, h_\theta)}{\partial(a^\rho, \tau)} \neq 0, \quad \frac{\partial(x^\mu, h_\theta)}{\partial(a^\rho, \tau)} \neq 0, \quad (4.4a, b)$$

the $b$-GLM is equivalent to a $b$-LCM under the given $H_\theta$, and $\tilde{g}$ is a GLM map.

**Proof.** The initial condition $\xi^\mu(a; \tau) = 0$ and $\langle \xi^\mu(a; \tau) \rangle_b = 0$ are satisfied by definition. It must be proved that $\tilde{g}: x \rightarrow X$ is a one-to-one map. This is evident from (4.4a,b):

$$\frac{\partial(X^\mu)}{\partial(x^\mu)} = \frac{\partial(X^\mu, h_\theta)}{\partial(a^\rho, \tau)} \neq 0.$$

(4.5)

Next, if $b = a^2$, then $da^2 = 0(\mu \neq \lambda)$ and $d\tau = 0$ on taking the $a^2$-LCM. This means $dx^\mu = 0(\mu \neq \lambda)$ and $dx^2 = da^2$ from (3.2) since $\langle X^\mu - a^\mu b \rangle_{b = a^2}$ is independent of $a^2$. Conversely, if $dx^\mu = 0(\mu \neq \lambda)$ and $dx^2 = 0$ on $H_\theta$, it follows from (4.4b) that $da^2 = 0(\mu \neq \lambda)$ and $d\tau = 0$; hence $dx^2 = da^2$. Thus the $a^2$-LCM is equivalent to the average over $x^2$. If $b$ stands for the ensemble parameter, the equivalence can be similarly proved.

It should be remarked that the compatibility condition for $H_\theta$ is necessary only for the $a^2$-LCM. Indeed, if some additional conditions are satisfied, the $x^2$-GLM can be defined for $H_\theta$ which is incompatible with the $a^2$-LCM. See the next section for details.

Now the following GLM property is evident.
Corollary. Under the same hypotheses, the LCM and GLM commute with the material derivative:

$$\frac{\partial}{\partial \tau} \langle f(\mathbf{a}; \tau) \rangle_{\mathbf{b}} = \left\langle \frac{\partial f(\mathbf{a}; \tau)}{\partial \tau} \right\rangle_{\mathbf{b}}$$

$$\tilde{U}^\mu \frac{\partial}{\partial x^\mu} f(\mathbf{X}) = \hat{u}^\mu \frac{\partial}{\partial x^\mu} f(g(\mathbf{x}))$$

$$= \hat{u}^\mu \frac{\partial}{\partial x^\mu} f(g(\mathbf{x})),$$

where $g(\mathbf{x}(\mathbf{a}; \tau)) = \tilde{g}(\mathbf{a}; \tau)$, and $f$ is an arbitrary function.

5. Examples of the Lagrangian mean

5.1. The Lagrangian spatial mean

As mentioned before, the conventional Lagrangian description assumes the initial hypersurface $H_0$ in the form $h_0(\mathbf{a}) = \mathbf{a}_0 - \mathbf{a} = 0$. Since $\partial h_0 / \partial \mathbf{a} = 0$, the $a^i$-LCM is compatible with $H_0$ and hence equivalent to the $x^i$-GLM if (4.4a,b) are satisfied. Thus the present GLM is consistent with the kinematical model shown in Fig. 1 of AMa.

5.2. The Lagrangian time mean

It is evident that the $a^0$-LCM is incompatible with the conventional initial hypersurface, so that the time-GLM cannot be defined in general within the framework of the conventional Lagrangian description. Indeed, Dunkerton (1983) introduced a new $H_0$, which can be regarded as $h_0(\mathbf{a}) = a^0 - \theta = 0$. Since $\partial h_0 / \partial a^0 = 0$, the $a^0$-LCM is compatible with $H_0$ and hence equivalent to the $x^0$-GLM if (4.4a,b) are satisfied. Thus the present GLM is consistent with the kinematical description shown in Fig. 1 of AMa.

5.3. The $\theta$-mean

So far the initial hypersurface $H_0$ ($h_0(\mathbf{a}) = 0$) is fixed. However, since $H_0$ is parameterized with $\theta$, a mean can be taken over $\theta$ (the $\theta$-mean). In this case, three-Lagrangian coordinates $(A^i)$ are assigned to each $H_0$, and then the $\theta$-mean is taken with $A^i$ and $\tau$ fixed. The GLM map is obtained from (4.1) or (4.2) with $b = \theta$ if the conditions

$$\frac{\partial (\mathbf{x}^\nu)}{\partial (A^i, \tau)} \neq 0, \quad \frac{\partial (\mathbf{x}^\nu)}{\partial (A^i, \tau)} \neq 0,$$

are satisfied. The $\theta$-mean, however, cannot be taken with $d a^\nu = 0$ for all $\mu$, because $d h_0 = (\partial h_0 / \partial a^\mu)d a^\mu + (\partial h_0 / \partial \theta) d \theta = 0$ and $\partial h_0 / \partial \theta \neq 0$ are incompatible. Nevertheless, since each $H_0$ is identified with a single hypersurface through the three-Lagrangian coordinates $(A^i)$, and since $(A^i)$ are independent of $\theta$, the relations $d A^i = 0$ and $d \tau = 0$ are equivalent to $d x^\nu = 0$, so that the $\theta$-mean can be regarded as the GLM over the ensemble parameter $\theta$.

In practice, this $\theta$-mean bears some resemblance to the LCM. As shown in Fig. 5, consider a case where the initial hypersurfaces $H_0$ for the $\theta$-mean and $\tilde{H}_0$ for the $a^0$-LCM ($x^0$-GLM) are given by $h_0 = a^0 - \theta = 0$ and $\tilde{h}_0 = a^0 - \theta' = 0$, respectively. Further assume that $A^i$
Fig. 5. The initial hypersurface \( H* \) \((i = 1, 2, 3, \ldots) \) \((h* (a) = a0 - \theta_i = 0)\) used for the \( \theta \)-mean and their identification to \( H* \) through \( a^i = A^i \), where \( A^i \) are the coordinates of \( H* \). The \( \theta \)-mean is taken with \( A^i \) and \( \tau \) fixed. If the \( \theta \)-mean is restricted to the fluid parcels which are located on \( H* \) \((h* = a1 - \theta = 0)\) initially, it gives the same mean quantities as the \( a0-LCM \).

For \( \theta \)-mean \( = a^i \) the \( \theta \)-mean. Then, since \( d \theta = da^o \) from \( dh_0 = 0 \), the \( \theta \)-mean and the \( a0-LCM \) give the same mean values for any quantity defined on \( H_0 \cap H* \). However, it should be noted that the \( \theta \)-mean is not the same as the \( x0-GLM \). Indeed, it can be seen from

\[
\frac{\partial X^u}{\partial a^o} \left( \frac{\partial h_0}{\partial a^o} \right) \left( \frac{\partial u^i}{\partial a^o} \right)^{-1} \left( \frac{\partial X^u}{\partial a^o} \right) \left( \frac{\partial h_0}{\partial a^o} \right)^{-1} \left( \frac{\partial h_0}{\partial a^o} \right) \left( \frac{\partial h_0}{\partial a^o} \right)
\]

that \( \partial X^u / \partial a^o \) are different between the \( \theta \)-mean and the \( a0-LCM \) \((x0-GLM)\) because they depend on the functional form of \( h_0 \) even though \( \partial X^u / \partial a^o \) and \( \partial u^i / \partial a^o \) are the same.

The above example clearly shows that it cannot be determined whether the \( a^i-LCM \) is the \( x^i-GLM \) or the \( \theta \)-ensemble GLM until the initial hypersurface is specified. Thus it can be seen that the kinematical models of AMa and Dunkerton (1983) are insufficient to illustrate the GLM since the initial hypersurface was not shown in their figures.

5.4. The GLM incompatible with the initial hypersurface for the corresponding LCM

Let \( H_0 \) be the initial hypersurface which is compatible with the \( b-LCM \) \((b \neq a^i)\) but incompatible with the \( a^i-LCM \). Then, by using the \( b-LCM \), a GLM map \( gb: x \rightarrow X = gb(x) \) is obtained. \( \xi^b_0 (x) \) and \( \bar{u}^b_0 (x) \) are denoted, respectively, as the displacement \( X^u - x^b \) and the mean velocity field associated with the \( b-LCM \). Now, suppose that \( \xi^b_0 (x) \) and \( \bar{u}^b_0 (x) \) have the following symmetry:

\[
\langle \xi^b_0 (x) \rangle - x^b = 0
\]

\[
\langle \bar{u}^b_0 (x) \rangle = \bar{u}^b_0 (x)
\]

Since (5.3) and (5.4) are no more than the relations (3.9) and (3.10), respectively, the \( b-GLM \) map \( gb \) can be regarded as the \( x^i-GLM \).

To illustrate the above argument more clearly, the following simple example is presented. Suppose that a flow is given in one-dimensional space and time by

\[
X^a (a^0; \tau) = a^0 + \tau
\]

\[
X^i (a^0; \tau) = a^i + u \tau + A \sin (\omega (a^0 + \tau) - k (a^i + u \tau))
\]

and the initial hypersurface \( H_0 \) is assigned as
\[ h_\theta(a) = a^0 - \theta = 0. \] (5.6)

Here \( u, \omega, A \) and \( k \) \((|kA| < 1)\) are constants. Since \( \frac{\partial h_\theta}{\partial a^0} \neq 0 \), the \( a^0 \)-LCM is incompatible with \( H_\theta \). Therefore, the \( \theta \)-mean of (5.5) is taken after replacing \( a^0 \) by \( \theta \). Then the GLM map is obtained

\[
\begin{align*}
X^0 &= x^0 \\
X^1 &= x^1 + A \sin(\omega x^0 - kx^1). 
\end{align*}
\] (5.7)

Hence

\[
\begin{align*}
\bar{U}^0(X) &= \bar{u}^0(x) = 1 \\
\bar{U}^1(X) &\equiv u + (\omega - kA) \cos(\omega x^0 - kx^1).
\end{align*}
\] (5.8)

Thus, as far as (5.7) and (5.8) are concerned, the \( \theta \)-dependence is implicit in the GLM map and velocity field, so that only the \( x^0 \)-GLM or \( x^1 \)-GLM appears relevant to the mean velocity field.

6. **GLM equations**

From the definitions of the Eulerian mean and the GLM given in Section 3, a GLM equation associated with the GLM map \( g: x \rightarrow X \in M \) is obtained for a given equation as follows. First, the given equation on \( M \) is rewritten in the covariant form to find a transformation property under the map \( g \), then the equation on \( M \) is transformed to that on \( N \), and finally the Eulerian mean of the transformed equation is taken. Since the covariant form of the following equations will be given elsewhere (Noda, 1988b), their derivation is omitted.

6.1. **The equation of continuity**

Let \( \rho(X) \) and \( \bar{\rho}(x) \) be the densities associated with the volume elements \( dX^1dX^2dX^3 \) on \( M \) and \( dx^1dx^2dx^3 \) on \( N \), respectively. If mass is conserved under the GLM map \( g \), then the equation on \( M \) is transformed to that on \( N \), and finally the Eulerian mean of the transformed equation is taken. Since the covariant form of the following equations will be given elsewhere (Noda, 1988b), their derivation is omitted.

\[
\frac{\rho(X)}{\bar{U}^0(X)} \det(X,\tau) = \frac{\bar{\rho}(x)}{\bar{u}^0(x)},
\] (6.1)

where the initial hypersurface and the parameter \( \tau \) are chosen such that \( \bar{U}^0(X) \neq 0 \) and \( \bar{u}^0(x) \neq 0 \). On the other hand, Flanders (1963, §10.6) gave the equation of continuity in the differential form:

\[
d\rho(X)(dX^1-U^1(X)dX^0) \\
\cdot (dX^2-U^2(X)dX^0)(dX^3-U^3(X)dX^0) = 0,
\] (6.2)

where \( U^i(X) = \bar{U}^i(X)/\bar{U}^0(X) \). From the definition (3.7), the GLM equation of continuity becomes

\[
\frac{d}{d\tau} \left( \frac{\bar{\rho}(x)(dx^1-u^1(x)dx^0)}{\bar{u}^0(x)} \right) \\
\cdot \left( (dx^2-u^2(x)dx^0)(dx^3-u^3(x)dx^0) \right) = 0,
\] (6.3)

where \( u^i(x) = \bar{u}^i(x)/\bar{u}^0(x) \) and \( \bar{U}^\mu(X) = X^\sigma_{\mu} \bar{u}^\nu(x) \). By using elementary properties of the exterior derivative, (6.3) reduces to

\[
\frac{\partial}{\partial x^\mu} \left( \left( \frac{\bar{\rho}(x)}{\bar{u}^0(x)} \right) \bar{u}^\nu(x) \right) = 0.
\] (6.4)

In the non-relativistic case, take \( \bar{U}^0(X) = 1, X^\sigma_0 = \delta^\sigma_0 \) and \( \bar{u}^0(x) = 1 \), where \( \delta^\sigma_0 \) is the Kronecker delta. A.M. proved that if \( \bar{\rho}(X) = \bar{\rho}(X) \) is satisfied initially, then \( \bar{\rho}(x) = \bar{\rho}(x) \) subsequently holds. However, since it was not assumed that \( \bar{\rho}(X) = \bar{\rho}(X) \) as an initial condition, \( \bar{\rho}(x) = \bar{\rho}(x) \) need not generally hold.

6.2. **The thermodynamic equation**

Since the specific entropy \( S(X) \) is a scalar quantity (e.g., Misner et al., 1973, Chapter 22), the source term \( Q_S(X) \) in the thermodynamic equation,

\[
\bar{U}^\mu \frac{\partial}{\partial x^\mu} S(X) = Q_S(X),
\] (6.5)

is also a scalar quantity. By applying the GLM operator \( (\bar{U}^\mu)^\nabla \), this reduces to the GLM thermodynamic equation

\[
\bar{u}^\nu \frac{\partial}{\partial x^\mu} g^\mu S(x) = g^\nu Q_S(x).
\] (6.6)

Here \( g^*S(x) = S(g(x)) \) is not a mean quantity in general, by the same token as that for \( \bar{\rho}(x) \).

6.3. **The equations of motion**

Noda (1988b) obtained the equations of motion for viscous and diabatic fluids from Hamilton's principle

\[
\delta \int L(g(x), Dg(x), x) dx = 0,
\] (6.7)
by regarding the diffeomorphism \( g : x = (x^\alpha) \rightarrow X = (X^\mu) \) as a field, where the Lagrangian density is given by

\[
\mathcal{L} = -\frac{\dot{\rho}(x)}{\dot{u}^\alpha(x)} \frac{1}{2} \tilde{G}_{ij} (X) \left( X^\alpha_i X^\beta_j \tilde{u}^\alpha(x) \tilde{u}^\beta(x) \right) - V^i(X) V^j(X) \left( X^{\mu}_p \tilde{u}^\mu(x) (x) - 1 \right) + \left\{ \tilde{E}(x) + \frac{1}{2} \tilde{G}_{ij} (X) \right\} \left[ X^{\mu}_p \tilde{u}^\mu(x) (x) - 1 \right] + \tilde{G}_{ij} (X) X^{\mu}_i X^{\mu}_j X^{\mu}_p \left\{ \tilde{u}^\alpha(x) \tilde{P}^\alpha\mu\nu(x) + \tilde{Q}^\alpha\beta\mu\nu(x) \right\} (6.8)
\]

Any quantity denoted as a function of \( x \) is fixed under the variation \( g \rightarrow g + \delta g \left( X^\mu \rightarrow X^\mu + \delta X^\mu \right) \). The constraints imposed on the variational principle are

\[
X^{\mu}_p \tilde{u}^\mu(x) = 1, \quad \tilde{P}^{\alpha\beta\mu\nu}(x) = 0, \quad \tilde{Q}^{\alpha\beta\mu\nu}(x) = 0. \quad (6.9)
\]

all of which come from the assumption \( dX^\mu/d\tau = 1 \) for the constituent particles of the non-relativistic fluid. Here the following notation and transformation properties are used.

\( \tilde{G}_{ij} (X) \): metric on the space and \( \tilde{G}^{ij} (X) \) its reciprocal metric. These metrics are related to the space-time metric \( G_{\mu\nu} \), and its reciprocal metric \( G^{\mu\nu} \) by \( \tilde{G}_{ij} = G_{ij} / G^{00}_{1/2} \). The metrics induced by \( g \) are \( \tilde{G}^{\alpha\beta\mu\nu}(x) = X^\alpha_i X^\beta_j G^{ij}(X) \) and \( \tilde{G}_{\alpha\beta\mu\nu}(x) = X^\alpha_i X^\beta_j \tilde{G}_{ij}(X) \).

\( V^i(X) \): shift vector (\( = G^{ij} / G^{00} \))

\( \tilde{E}(x) \): geopotential

\( \tilde{P}^{\alpha\beta\mu\nu}(x) \): proper density, \( \tilde{P}^{\alpha\beta\mu\nu}(x) = \rho(x) \rho(x) \)

\( \tilde{Q}^{\alpha\beta\mu\nu}(x) \): specific internal energy which is a functional of \( X^\mu, X^\nu, \) and \( x^\mu \) but not \( X^\nu \). However, because of the factor \( X^\mu \tilde{u}^\mu(x) = 1, \delta \tilde{E} \) makes no contribution to the Euler-Lagrange equations, so that \( \tilde{E}(x) = g^\ast \tilde{E}(x) \) can be set in \( \mathcal{L} \).

\( P^{ij}(X) \): stress tensor, \( P^{ij}(X) = \rho(x) \tilde{G}^{ij}(X) + \tilde{\tau}^{ij}(X) \) \( X^\mu_i X^\mu_j \left( \tilde{P}^{\alpha\beta\mu\nu}(x) + \tilde{Q}^{\alpha\beta\mu\nu}(x) \right) = X^\mu_i X^\mu_j \tilde{P}^{\alpha\beta\mu\nu}(x) \), and hence \( \tilde{P}^{\alpha\beta\mu\nu}(x) = X^\mu_i X^\mu_j \tilde{S}^{ij}(X) \).

\( P(x) \): scalar pressure, \( \tilde{P}(x) = g^\ast \rho(x) \).

\( \tilde{Q}^{\alpha\beta}(x) \): viscous tensor,

\( \tilde{\tau}^{ij}(X) \): trace of viscous tensor \( = \tilde{G}_{ij}(X) \tilde{\tau}^{ij}(X) \)

\( \tilde{\tau}(X) \): trace of viscous tensor \( = \tilde{G}_{ij}(X) \tilde{\tau}^{ij}(X) \).

\( Q^{ijk}(X) \): heat conduction tensor, \( \tilde{Q}^{ijk}(X) = x^\alpha_i x^\alpha_j x^\alpha_k Q^{ijk}(X) \).

The Euler-Lagrange equations

\[
\frac{D}{Dx^\nu} \frac{\partial \mathcal{L}}{\partial X^\mu} - \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = 0 \quad (6.10)
\]

yield the energy equation for \( \mu = 0 \) and the momentum equations for \( \mu = i \neq 0 \), where \( D/Dx^\nu = X^\mu_i \partial / \partial X^\mu_i + X^\mu_0 \partial / \partial X^\mu_0 + \partial / \partial x^\nu \). It should be noted that the first term enclosed by brackets on the right hand side of (6.8) makes no contribution to the momentum equations owing to the constrains of (6.9). Since (6.10) are regarded as the components of a one-form with the basis \( dx^\mu(X) \) at \( X \), the corresponding GLM equations become, from (3.7),

\[
X^\mu_i \left( \frac{D}{Dx^\nu} \frac{\partial \mathcal{L}}{\partial X^\mu} - \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \right) dx^i = 0. \quad (6.11)
\]

Thus the multiplication of \( X^\mu_i \) to (6.10), which is also found in AMa, is the consequence of the definition of the GLM. With the use of the canonical energy-momentum tensor (e.g., Landau and Lifshitz, 1975, § 32)

\[
T^\mu_i = X^\mu_i \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} - \mathcal{L} \delta^\mu_i, \quad (6.12)
\]

(6.11) can be rewritten as

\[
\phi(X), \rho_0(X), \rho_0(X), \rho(X), \rho(x) = |\det(G_{ij})|^{1/2} \det(X^\mu_\ast) \tilde{\rho}(x)
\]

\( E(X) \): specific internal energy which is a functional of \( X^\mu, X^\nu, \) and \( x^\mu \) but not \( X^\nu \).
\begin{equation}
\left( T_{\mu, \nu}^\alpha + \frac{\partial_L}{\partial X^\nu} \right) dX^\mu = 0,
\end{equation}
where \( T_{\mu, \nu}^\alpha = DT_{\mu, \nu}^\alpha / D x^\nu \).

For the ensemble-GLM with the ensemble parameter \( \tilde{a} \in a \), AMa derived a scalar relation called the wave-action equation:

\begin{equation}
\left\langle \frac{\partial X^\nu}{\partial \tilde{a}} \left( \frac{D}{D x^\nu} \frac{\partial L}{\partial X^\nu} - \frac{\partial L}{\partial X^\alpha} \right) \right\rangle_{\tilde{a}} = 0,
\end{equation}
from which it follows that

\begin{equation}
\left\langle T_{\alpha, \nu}^\mu \frac{D}{D x^\nu} + \frac{\partial L}{\partial \tilde{a}} \right\rangle_{\tilde{a}} = 0,
\end{equation}
with the modified canonical energy-momentum tensor

\begin{equation}
T_{\alpha}^\mu = \frac{\partial X^\nu}{\partial \tilde{a}} \frac{\partial L}{\partial X^\nu},
\end{equation}
where \( D/D \tilde{a} = (\partial X^\nu/\partial \tilde{a}) \partial / \partial X^\nu + (\partial X^\nu/\partial \tilde{a}) \partial / \partial X^\alpha + \partial / \partial \tilde{a} \). It can be seen from the mass conservation (6.4) and \( \left\langle D_L / D \tilde{a} \right\rangle_{\tilde{a}} = 0 \) that the wave-action \( T_{\alpha}^\mu \) is conserved unless \( L \) depends on \( \tilde{a} \) explicitly.

### 6.4. Conservation laws associated with symmetries in the Lagrangian density

It can be noted from the above argument that there are two kinds of conservation laws. If \( L \) is independent of a certain \( X^\mu \), say, \( \mu = \lambda \), then it follows from (6.10) that

\begin{equation}
\frac{D}{D x^\nu} \frac{\partial L}{\partial X^\nu} = 0 \quad (\lambda \text{ fixed}),
\end{equation}
which expresses the conservation of energy if \( \lambda = 0 \), and the conservation of momentum in the \( X^\lambda \) direction if \( \lambda = \lambda \neq 0 \). Similarly, if \( L \) is independent of \( x^\nu \), then (6.13) gives

\begin{equation}
T_{x^\nu} = 0 \quad (\lambda \text{ fixed}),
\end{equation}
which AMb called the conservation of pseudoenergy if \( \lambda = 0 \), and the conservation of pseudomomentum if \( \lambda = \lambda \neq 0 \). Referring to the concrete form of the Lagrangian density (6.8), it can be seen that the conservation of energy and momentum is related to the symmetry in the metric tensor and external fields alone and therefore it is irrelevant to the fluid quantities such as fluid velocity, pressure, internal energy and viscous tensor. Similarly, the conservation of pseudoenergy and pseudomomentum is related to the symmetry in the fluid quantities alone but is irrelevant to the symmetry in the metric tensor and external fields. The same argument can be applied to the conservation laws in the Eulerian form and those in the Lagrangian form if referring to the footnote in the previous subsection.

Finally, consider the case that \( \xi \) is independent of both \( x^\lambda \) and \( X^\lambda \). The Eulerian average of (6.17) over \( x^\lambda \) gives

\begin{equation}
\frac{D}{D x^\nu} \frac{\partial L}{\partial X^\nu} = 0 \quad (\lambda \text{ fixed}).
\end{equation}
Hence (6.18) becomes

\begin{equation}
\frac{D}{D x^\nu} \left( \xi^\nu \frac{\partial L}{\partial X^\nu} \right) = 0 \quad (\lambda \text{ fixed}).
\end{equation}
In the almost-plane wave limit \( \xi^\nu \propto \exp \left( ik_\mu d x^\nu + \tilde{a} \right) \), this reduces to the conservation of the wave-action \( \left\langle T_{\alpha, \nu}^\mu \right\rangle_{\tilde{a}} \) (AMb, § 4). Moreover, if appropriate conditions are satisfied, \( F^\nu / F^\lambda \), where \( F^\nu = \xi^\nu \partial L / \partial X^\nu \), represents the group velocity (Hayes, 1977; AMb; Noda, 1986). Thus, in the Lagrangian mean formalism, the governing equation for wave-mean flow interaction is decoupled into (6.19) and (6.20) when the fluid and the external fields are symmetric in the \( \lambda \)-direction. This result is quite different from that in the Eulerian mean formalism (Edmon et al., 1980; Holopainen et al., 1982; Hoskins et al., 1983; Plumb, 1986; Trenberth, 1986), where the spatial divergence of the flux of wave activity forces the mean flow.

### 7. Concluding remarks

The generalized Lagrangian-mean (GLM) theory of AMa (Andrews and McIntyre, 1978a) and AMb (Andrews and McIntyre, 1978b) has been reformulated in a symmetrical way in space and time. The main extensions to the original GLM theory are as follows.

1) The Eulerian mean and the GLM tensors have been defined, referring to the general coordinate system. Both means depend on the choice of the coordinate system as do the mean
tensors. In this sense, the present GLM is different from the so-called true vector Lagrangian mean (AMa; McIntyre, 1980b; Dunkerton, 1980).

2) Four-dimensional Lagrangian coordinates \((a^\alpha)\) have been introduced to show the relationship between the Lagrangian coordinate mean (LCM) and the GLM. The main result has been given by a theorem in Section 4. It is shown that the choice of the initial hypersurface in the space-time manifold is essential in the determination of the relationship. The \(a^\lambda\)-LCM can be either the \(x^\lambda\)-GLM or the ensemble-GLM with the ensemble parameter \(a^\lambda\), depending on the choice of the initial hypersurface. Owing to this relationship, the GLM can be obtained via the corresponding LCM, which can be calculated more easily in practice than the GLM.

3) The GLM map \(g : x* \rightarrow X\) has been defined without referring to Eulerian disturbance fields. This means that the present GLM theory abandons the unique correspondence between the displacement field \(\xi^\alpha(x) = X^\alpha(x) - x^\alpha\) and the Eulerian disturbance field. For theoretical application the uniqueness may be guaranteed to some extent for conservative motion by introducing an additional assumption that no Eulerian disturbance exists if \(\xi^\alpha = 0\) (AMb). However, this assumption will be of no practical use in the analysis of atmospheric data (real or simulated) where perpetual Eulerian disturbances prevail.

Thus, rather than to strive to recover the uniqueness, it seems to be more natural to admit the non-uniqueness. In fact, the theoretical example in Section 5.4 suggests the non-uniqueness between the disturbances in the velocity field and the displacement field, because (5.7) determines the GLM map \(g\) but arbitrary constant values can be assigned to \(u\) for the same \(g\).

4) GLM equations used for fluid mechanics have been given in the covariant form in Section 6. The GLM equations of motion have been derived by using Hamilton’s principle. The GLM map \(g\) (denoted by \(X^\alpha\) in the Lagrangian density) was chosen as the field to be varied by Hamilton’s principle, while AMb chose the displacement field \(\xi^\alpha\). It can be shown that both give the same resulting equations of motion, since the variation with respect to \(X^\alpha\) and \(X^\alpha_\xi\) is equivalent to the variation with respect to \(\xi_\alpha\) and \(\xi_\xi^\alpha\). However, the canonical energy-momentum tensor for each one is different; Eq. (5.7) of AMb contains only the wave component, while (6.13) in this paper contains both the mean flow and the wave components. Moreover, it has been shown that the two components are decoupled if the Lagrangian density is symmetric in a certain direction in the space-time manifold. This may be compared to Charney and Drazin’s (1961) non-acceleration theorem in the Eulerian framework (AMa, § 5), although the exact correspondence is not necessarily expected between the Lagrangian and Eulerian frameworks due to the non-uniqueness mentioned above.

5) Conservation laws associated with symmetries in the Lagrangian density have been derived for energy-momentum, pseudoenergy-pseudomomentum and wave-action. Recently, Ripa (1981) and Salmon (1982) derived the conservation law of Ertel’s potential vorticity from the symmetry of the Lagrangian density in labeling fluid particles under the constraint of mass conservation. As to whether or not this argument can be applied to the present Lagrangian density (6.8) is under study. For, as inferred from the footnote in Section 6.3, the GLM map \(g\) is a special case for labeling of fluid particles, just as fluid particles are labeled with their initial coordinates in the Lagrangian description and with their present coordinates in the Eulerian description.

6) The GLM theory extended in this paper can be applied to the relativistic case, since neither a special metric nor a special coordinate system is assumed in the formulation and because the difference between the relativistic and nonrelativistic cases stems from the difference of the dispersion relation (or energy-momentum relation) of constituent particles of a fluid (Noda, 1982). The Lagrangian density for relativistic fluids will be given in a separate paper (Noda 1988b).

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## Appendix

### Expressions in the Spherical Coordinate System

In the following, an atmospheric flow on a spherical earth is assumed, where \( \mu = 1, 2 \) and 3 denote the radial (vertical), colatitudinal and longitudinal components, respectively. Hence

\[
(x^\mu) = (r, \theta, \phi),
\]

\[
(X^\mu) = (T, R, \Theta, \Phi),
\]

\[
(\xi^\mu) = (T - t, R - r, \Theta - \theta, \Phi - \varphi),
\]

\[
(\bar{u}^\mu) = (i, \hat{r}, \hat{\theta}, \hat{\phi}),
\]

\[
(\bar{U}^\mu) = (\bar{T}, \bar{R}, \bar{\Theta}, \bar{\Phi}),
\]

\[
(x^\mu) = (t, r, \theta, \phi),
\]

\[
(X^\mu) = (T, R, \Theta, \Phi),
\]

\[
(\xi^\mu) = (T - t, R - r, \Theta - \theta, \Phi - \varphi),
\]

\[
(\bar{u}^\mu) = (i, \hat{r}, \hat{\theta}, \hat{\phi}),
\]

\[
(\bar{U}^\mu) = (\bar{T}, \bar{R}, \bar{\Theta}, \bar{\Phi}),
\]

\[
\rho = \text{det}(\bar{G}_{ij})^{1/2} = R^2 \sin^2 \theta.
\]

2) The kinetic energy due to the vertical motion is much smaller than that due to the horizontal motion. Hence

\[
\bar{G}_{i1} \bar{U}^1 \bar{U}^1 = \bar{G}_{i1} X_{1,\mu} Y_{1,\nu} \bar{u}^\mu \bar{u}^\nu = 0.
\]

This approximation eliminates the material derivative of \( \bar{U}^1 \) from the equation of motion for the vertical component but does not eliminate \( \bar{u}^1 \) from the material derivative, so that it is equivalent to the hydrostatic approximation for the primitive equations.

### References


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一般座標系における GLM（一般化されたラグランジ平均）記述

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Andrews・McIntyre（1978）によって定式化されたGLM（一般化されたラグランジ平均）記述を一般的な座標系に拡張した。四元ラグランジ座標を導入してLCM（ラグランジ座標平均）とGLMの間に成り立つ一般的関係を求めた。両者の関係を求めるうえで、初期の曲面の取り方が重要であることが示された。テンソルのオイラー平均とGLMを、与えられた座標系に準拠して定義した。従って、平均化されたテンソルは座標系の選び方に依存する。流体のラグランジアン密度が持つ対称性と、それに関連した保存則は、エネルギー・運動量と擬エネルギー・擬運動量、波の作用について議論した。ここで拡張されたGLMは初期条件に対する制限が緩いため、現実の現象に対して、より広い応用が期待される。