Evolution of Disturbance and Geostrophic Adjustment on the Sphere

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Abstract

In order to examine the applicability of the concept, "geostrophic adjustment" on an f -plane to the atmosphere on the sphere, we study systematically the evolution of localized non-geostrophic disturbances in mid- and tropical latitudes. We solve numerically the linearized shallow water equations as an initial value problem for various initial conditions. For this calculation we use eigenfunctions of the shallow water equations on the sphere (Hough functions).

For an initial disturbance given in mid-latitudes, when $\epsilon = \{(\text{rotational velocity of the planet})/(\text{velocity of gravity wave})\}^2$ is large (i.e., the deformation radius is smaller than the spatial scale of the disturbance which is comparable with the earth's radius), the inertia-gravity waves emitted from the disturbance in a process of geostrophic adjustment propagate equatorward and disperse over the tropical region. When ϵ is small (i.e., the deformation radius is larger than the spatial scale of the disturbance), the excited gravity waves oscillate between the location of the disturbance and its antipodean point without dispersion.

When an initial disturbance is given in the tropics and ϵ is large, the evolution of the disturbance can be understood as that of equatorial waves; if the initial disturbance is symmetric about the equator, then the disturbance splits out into Kelvin and Rossby waves. When ϵ is small, the evolution of tropical disturbance is similar to that in mid-latitudes, i.e., a geostrophic adjustment is made in a usual sense.

The evolution of each disturbance after an establishment of the geostrophic relation is also discussed.

1. Introduction

The evolution of the non-geostrophic disturbance has been understood as a geostrophic adjustment, since Rossby's pioneering contribution (1938). So far, a number of works have been published by various authors. (For the review, see Blumen (1972) and Gill (1982).) The results obtained by these authors show that in the geostrophic adjustment problem "the Rossby radius of deformation" is a length scale of fundamental importance: for scales larger than the radius of deformation, the velocity field adjusts to be in equilibrium with the pressure field, and for scales smaller than the radius of deformation the pressure field adjusts to be in equilibrium with velocity field. This conclusion may be derived from the conservation of potential vorticity. Many authors have calculated the geostrophic equilibrium states in practice for various initial non-geostrophic states. Most of these authors treated the adjustment problem with a one-dimensional model on an f -plane: the distribution of an initial anomaly was

assumed to be uniform in one horizontal direction in a fluid of infinite lateral extent (e.g., Rossby, 1938) or to be contained in a channel infinitely extended in one direction (e.g., Gill, 1976). It is evident that there is a large gap between such simplified models and the real atmosphere on the earth; the latter is neither one-dimensional nor of infinite lateral extent and, further, a disturbance in it is not usually uniform in any direction. Hence in this study we attempt to fill this gap by examining the geostrophic adjustment problem *on the sphere* using the shallow water equations. When the geostrophic adjustment is successfully examined in a fluid on the sphere, according to the results obtained we can estimate the validity of the above one-dimensional models on an f -plane. Further, in the light of the results on the sphere we can determine whether or not two-dimensional models on an f -plane or β -plane can produce correctly the whole history of the evolution of non-geostrophic initial fields. Thus, the first purpose of this study is the examination of geostrophic adjustment of an extratropical disturbance on the sphere and the estimation of validity of the models on an infinite plane.

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In addition, in this study, by treating the shallow water equations on the sphere, we intend to examine the evolution of initial disturbances in the tropics. Since the effect of the Coriolis force vanishes on the equator, the geostrophic adjustment in the same sense as that in the extratropical region is not supposed to be made there. Then, how does the evolution of a non-geostrophic state proceed in the tropics? Does the concept of "geostrophic adjustment" on an f -plane become invalid there? Can the evolution of a tropical disturbance be adequately treated on an equatorial β -plane? In order to respond to these questions we examine the evolution of an initial disturbance using the linearized shallow water equations for a fluid at rest on the sphere in this study.

Wiin-Nielsen (1976) treated the same questions on the sphere in order to examine β -effects on the geostrophic adjustment and obtained some interesting results. However, his treatment is limited to the case of no longitudinal dependence. In the present study we consider the evolution of the initial disturbances which are not only latitudinally but also longitudinally concentrated.

We calculate systematically evolutions of the fields for various non-geostrophic initial states. The initial states treated in this study are specified by three aspects: (1) the initial disturbance lies in mid-latitudes or in the tropics, (2) the initial field is symmetric or antisymmetric with respect to the equator, (3) the initial field is perturbed only in the height field, in the vorticity field or in the divergence field. Calculations are made for the initial states of all combinations of these alternatives. The perturbation in each initial field is specified to have a Gaussian spatial distribution with a zonal scale of 30° – 60° and a meridional scale of 20° – 40° . Our problem contains the parameter representing the rotational velocity of the planet measured by the velocity of gravity wave. We perform calculations of some values of this parameter.

The mathematical formulation of our problem and the procedure to obtain the numerical solutions are described in Section 2. In Section 3 and 4 we examine the evolution of the disturbances which are initially given in mid-latitudes and in the tropics respectively. Section 5 is a summary and comments.

2. Formulation of the problem

In this section we describe the mathematical formulation of our problem and the procedure to obtain the solutions numerically. The formulation and mathematical method of the present problem, to some extent, overlap those of Matsuda and Kato (1987) which consider the linear stationary response of the atmosphere to localized heating with the shallow water equations on the sphere. Hence, here we describe the formulation and procedure only briefly.

First we introduce the dimensionless variables with circumflex defined by

$$\left. \begin{aligned} \hat{u} &= u/\sqrt{gH} \\ \hat{v} &= v/\sqrt{gH} \\ \hat{h} &= h/H \\ \hat{t} &= t/(a/\sqrt{gH}) \end{aligned} \right\}, \quad (2.1)$$

where u and v are eastward and northward components of horizontal velocity respectively; h is the deviation of isobaric height from the basic state; t is time; H is the equivalent depth. Then, the shallow water equations are written as

$$\left. \begin{aligned} \frac{\partial \hat{u}}{\partial \hat{t}} - \sqrt{\varepsilon} \cos \theta \hat{v} + \frac{1}{\sin \theta} \frac{\partial \hat{h}}{\partial \phi} &= 0 \\ \frac{\partial \hat{v}}{\partial \hat{t}} + \sqrt{\varepsilon} \cos \theta \hat{u} - \frac{\partial \hat{h}}{\partial \theta} &= 0 \\ \frac{\partial \hat{h}}{\partial \hat{t}} + \frac{1}{\sin \theta} \left[\frac{\partial(-\sin \theta \hat{v})}{\partial \theta} + \frac{\partial \hat{u}}{\partial \phi} \right] &= 0 \end{aligned} \right\}, \quad (2.2)$$

where $\varepsilon = 4a^2\Omega^2/gH$; θ is colatitude and ϕ is longitude. Hereafter we omit circumflexes which are attached to dimensionless variables. Further, introducing the velocity potential Φ and stream function Ψ , such that

$$\left. \begin{aligned} u &= \frac{1}{\sin \theta} \frac{\partial \Phi}{\partial \phi} - \frac{\partial \Psi}{\partial \theta} \\ v &= -\frac{\partial \Phi}{\partial \theta} - \frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \phi} \end{aligned} \right\}, \quad (2.3)$$

(2.2) is rewritten by the variables, Φ , Ψ and h .

We expand Φ , Ψ and h as a series of normalized spherical harmonics:

$$\left. \begin{aligned} \Phi &= \sum A_n^s P_n^s(\mu) \exp(i(s\phi - \sigma t)) \\ \Psi &= \sum B_n^s P_n^s(\mu) \exp(i(s\phi - \sigma t)) \\ h &= \sum C_n^s P_n^s(\mu) \exp(i(s\phi - \sigma t)) \end{aligned} \right\}, \quad (2.4)$$

where s is a non-negative integer and σ denotes the dimensionless radian frequency. Substituting (2.4) into the equations we obtain two systems of homogeneous algebraic equations. The two systems correspond to motions which are respectively symmetric and antisymmetric about the equator. The first system may be written as

$$(\sigma I + (J_s + C_s))\xi^s = 0, \quad (2.5)$$

where $\xi^s = (A_s^s, C_s^s, B_{s+1}^s, A_{s+2}^s, C_{s+2}^s, B_{s+3}^s, \dots)$ and I denotes the unit matrix. J_s and C_s are the matrices (see Matsuda and Kato (1987) for details). For the antisymmetric motions the system may be written in a similar way, so that we shall omit the presentation of the antisymmetric case. (2.5) constitutes an eigenvalue problem with respect to σ . We indicate eigenfrequencies of the symmetric system by $\sigma_m^s (m = 0, 1, \dots)$; $\xi_m^s (m = 0, 1, \dots)$ denotes the corresponding eigenvectors of (2.5).

Next, we formulate the numerical procedure for obtaining solutions of *initial value problems*. First, we expand Φ , Ψ and h as:

$$\left. \begin{aligned} \Phi(\mu, \phi, t) &= Re \sum_{s=0}^{\infty} \left(\sum_{n=s}^{\infty} a_n^s(t) \right. \\ &\quad \times P_n^s(\mu) \Big) \exp(is\phi) \\ \Psi(\mu, \phi, t) &= Re \sum_{s=0}^{\infty} \left(\sum_{n=s}^{\infty} ib_n^s(t) \right. \\ &\quad \times P_n^s(\mu) \Big) \exp(is\phi) \\ h(\mu, \phi, t) &= Re \sum_{s=0}^{\infty} \left(\sum_{n=s}^{\infty} ic_n^s(t) \right. \\ &\quad \times P_n^s(\mu) \Big) \exp(is\phi) \end{aligned} \right\} \quad (2.6)$$

The vector $\xi'^s(t) = (a_s^s(t), c_s^s(t), b_s^{s+1}(t), \dots)$ satisfies the equation

$$\left(\frac{\partial}{\partial t} - i(J^s + C^s) \right) \xi'^s = 0. \quad (2.7)$$

Then, the solution of (2.7) can be written in terms of the eigenfrequencies and eigenvectors of (2.5):

$$\xi'^s(t) = \sum_m h_m^s \xi_m^s \exp(-i\sigma_m t), \quad (2.8)$$

where h_m^s is a coefficient independent of time. From initial fields $\Phi(\mu, \phi, t=0)$, $\Psi(\mu, \phi, t=0)$ and $h(\mu, \phi, t=0)$ we can determine the value $a_n^s(0)$, $b_n^s(0)$, $c_n^s(0)$, i.e. $\xi'^s(0)$ with (2.6). From (2.8) at $t=0$ we obtain

$$\xi'^s(0) = \sum_m h_m^s \xi_m^s. \quad (2.9)$$

By (2.9) we can determine h_m^s ($m=0, 1, \dots$). With these values of h_m^s ($m=0, 1, \dots$), (2.8) gives values of $\xi'^s(t)$ at an arbitrary time. Finally, with $\xi'^s(t)$ we obtain Φ , Ψ and h at an arbitrary time from (2.6). In this way, we can obtain the evolution of fields from arbitrary initial fields. It should be noted that, unlike numerical time-integration methods based on a finite difference scheme, in our method the fields at an arbitrary time are obtained independently of those at the preceding time steps; hence, our solutions do not suffer from the accumulation of numerical error in time integration.

3. Geostrophic adjustment in mid-latitudes

By the numerical procedures described in the preceding section, we calculate systematically the evolution of various initial disturbances given in mid-latitudes for $\epsilon=10$ (or 1) and 1000.² Throughout

Table 1. $\epsilon=4a^2\Omega^2/gH$, L_R is the radius of deformation at 40° latitude and L_β is the equatorial radius of deformation. L_R and L_β are measured by the meridional degree.

ϵ	1	10	1000
L_R	89°	28°	2.8°
L_β	57°	32°	10.2°

this section the centre of the initial disturbance is assumed to lie at 40°N and 40°S. The zonal scale of the assumed disturbance is 30° or 36° and its meridional scale is approximately 20°. The radius of deformation at 40° of latitude and the equatorial radius of deformation for some values of ϵ are given in Table 1. This table indicates that for $\epsilon=10$ (or 1) the assumed spatial scale of the initial disturbance is smaller than the deformation radius, while for $\epsilon=1000$ the former is larger than the latter. We can regard (2.2) with ϵ as the horizontal structure equations derived from the linearized equations describing motions in the three-dimensional atmosphere at rest. Then, in the atmosphere of the earth solutions of (2.2) for $\epsilon=10$ (or 1) express approximately horizontal structures corresponding to the barotropic mode, because the equivalent depth H for the barotropic mode is about 10 km; on the other hand, solutions for $\epsilon=1000$ express approximately horizontal structures corresponding to the first baroclinic mode, because the equivalent depth H for this baroclinic mode is roughly 100 m. (See, e.g., Kasahara (1984).)

Calculations are made both for initial disturbances symmetric and antisymmetric about the equator. According to the results obtained, apart from some points described below, the evolution of the antisymmetric initial state is similar to that of the symmetric one except for the sign of vorticity, divergence and deviation height in the Southern Hemisphere. This fact indicates that a disturbance located in mid-latitudes on one hemisphere evolves independently of one in another hemisphere. Hence, the evolution of the disturbance of one hemisphere in the symmetric and antisymmetric cases conforms with that of the isolated disturbance having no counterpart in another hemisphere; it is not necessary to combine the symmetric and antisymmetric results to examine the evolution of the isolated disturbance in one hemisphere. Thus, in this section we discuss mainly the evolution of symmetric initial states.

a. solutions for $\epsilon=1000$

For $\epsilon=1000$, first we examine the evolution of the initial state with perturbation in the height field only: its initial fields are

²Since we cannot expect that the geostrophic adjustment process in high-latitudes is fundamentally different from that in mid-latitudes, in this study we do not examine the adjustment problem in high-latitudes.

$$h(\mu, \phi, 0) = \exp(-(\phi/\Delta\phi)^2) \left\{ \exp(-(\mu - \mu_o)^2 / (\Delta\mu)^2) + \exp(-(\mu + \mu_o)^2 / (\Delta\mu)^2) \right\}$$

$$\Phi(\mu, \phi, 0) = \Psi(\mu, \phi, 0) = 0,$$

where $\Delta\mu = \sqrt{2}/10$, $\Delta\phi = \pi/5$ and $\mu_o = \cos 50^\circ$. Together with the velocity field, the evolution of the deviation height and that of the divergence field are depicted in Fig. 1a and 1b respectively. Fig. 1a shows that the initial disturbance largely retains its pattern and location in the height field even at $t = 8.0$. Note that time is measured by $a/\sqrt{gH}$ in this study. For $\varepsilon = 1000$ (*i.e.* $H = 88$ m), $a/\sqrt{gH} = 2.2 \times 10^5$ s. The geostrophic wind to be balanced with this given height field is gradually formed from an equatorward part of the disturbance; a geostrophic relation becomes established there at $t = 0.5$ or 1.0 . At $t = 2.0$ a geostrophic relation is almost established for the whole region of the disturbance.

Figure 1b depicts the evolution of the divergence field. The divergence field mainly reflects activity of inertia-gravity waves. This figure shows that the inertia-gravity waves excited by the non-geostrophic initial condition propagate equatorward from the disturbance. It should be noted that the emission of the waves seen at $t = 0.5$ and 1.0 ceased after $t = 2.0$. This shows that the geostrophic adjustment is almost completed by $t = 2.0$ by the radiation of the inertia-gravity waves from the disturbance. This corresponds to the establishment of a geostrophic relation at $t = 2.0$ seen in Fig. 1a. After $t = 2.0$ the inertia-gravity waves which propagated equatorward expand in a longitudinal direction. At $t = 4.0$ or 8.0 the inertia-gravity waves become dispersed in the regions between 40°N and 40°S . Hence, these waves cannot disturb the geostrophic balance established in the disturbance. It should be noted that the excited inertia-gravity waves are neither propagated poleward from the disturbance nor concentrated in the narrow region adjacent to the equator (whose latitudinal width is determined by the equatorial radius of deformation).³

Next, we discuss the evolution of the disturbance after the establishment of the geostrophic relation. Fig. 1 shows that the disturbance propagates westward as a whole. Further, it is shown that the northeast-southwest (southeast-northwest) tilt of the disturbance seen in the height field on the Northern (Southern) Hemisphere becomes pronounced after $t = 4.0$. The disturbance seen in the height field can be regarded as a superposition of Rossby waves. Since the westward phase velocity of

Rossby wave decreases with latitude, the phase of disturbance in a lower latitude precedes that in a higher latitude. Hence, the tilt seen in Fig. 1a occurs. Except for this tilt the pattern of the disturbance in the height field is not much changed with time. This is because for large ε the phase velocity of Rossby waves weakly depends on the wavenumber, as a result the Rossby wave is less dispersive in this case.

By inspecting the amplitudes of the waves (*i.e.*, eigenfunctions of the shallow water equations on the sphere) which compose the initial disturbance we can predict the evolution of the initial field. In Fig. 2 we show amplitudes of the main waves composing an initial state for three cases: the case which has an initial perturbation only in the height field (*i.e.*, the case presented in Fig. 1), the case which has the perturbation only in the divergence field and the case which has the perturbation only in the vorticity field. The figure shows that the initial fields of the first case are mainly composed of Rossby waves; from this fact we can predict that the initial state of the height field remains or evolves only slowly because of low-frequency of Rossby waves.

We calculate also the evolution of the initial disturbance with a height anomaly antisymmetric about the equator. The result of this case (not shown) is similar to that for the symmetric disturbance except that, in the former case, a cross-equatorial flow made up from inertia-gravity waves emitted from the initial disturbance exists. It is interesting that some characteristics in the evolution of the symmetric and antisymmetric initial disturbance with height anomaly are seen also in the study by Lim and Chang (1981); they study the dynamic response of the tropical atmosphere to mid-latitude pressure surge in the Northern Hemisphere using the linear shallow water equations on an equatorial β -plane. In their study, the motion of the tropical atmosphere is produced by forcing in the height field. However, the period of their forcing is rather short, so that their solution is similar to ours for an initial value problem rather than that for a steady forcing problem. An equatorward flow of the gravity-wave type and a northeast-southwest tilt of the disturbance are produced in their study also. However, the excitation of equatorial waves obtained in their study is not clearly seen in ours. Probably, this discrepancy is due to the fact that the centre of the area of forcing in their study is located at 30°N , while that of our initial disturbance is at 40°N .

Further, we have obtained solutions for the two initial disturbances which have divergence and vorticity anomaly respectively. Fig. 2 indicates that the initial state with anomaly only in the divergence field is mainly composed of inertia-gravity waves. From this fact we can predict that the initial disturbance will be almost dispersed as inertia-gravity

³The equatorward propagation of inertia-gravity waves is discussed by Anderson and Gill (1979) who treat inertial oscillations produced by a wind stress in an ocean. In particular, they examine propagation of inertia-gravity waves on the equatorial β -plane using ray-tracing techniques.

Fig. 1. Evolution of the initial disturbance with perturbation in the height field for $\epsilon = 1000$. The initial disturbance is symmetric about the equator and occurs in mid-latitudes. Time is measured by $a/\sqrt{gH}$ ($= 2.2 \times 10^5$ s). a. Contours of deviation height superposed on the velocity field. Solid and dotted lines show negative and positive heights, respectively.

Fig. 1b. Contours of divergence superimposed on the velocity field. Dotted and solid lines show convergence (upward motion) and divergence (downward motion), respectively.

Fig. 2. Amplitude of normal modes (waves) of $s=1$ which compose the symmetric disturbance occurring in mid-latitudes, when $\varepsilon=1000$. A dotted, solid and broken line show the case with initial perturbation in the height, divergence and vorticity field, respectively.

waves from the region where the disturbance is initiated. In fact, the solution obtained (not shown) indicates that the initial perturbation given in the divergence field is not retained; the perturbation seen in the height and velocity field is rapidly dispersed. The feature of the dispersion is similar to that depicted in Fig. 1b: first, inertia-gravity waves propagate equatorward and are subsequently dispersed in a longitudinal direction. In this case also, the disturbance cannot propagate poleward.

Figure 2 shows that the initial disturbance with vorticity is composed of both Rossby and inertia-gravity waves; this is because the both waves have vorticity. This composition suggests that the subsequent evolution of this initial state is intermediate between those of the above two initial states. In practice, the solution in this case (not shown) indicates that the initial vorticity is not completely retained, but a height field geostrophically balanced with the initial vorticity is formed; the evolution of the height field in this case is, to some extent, similar to the superposition of those in the two cases described above. The propagation and dispersion of the inertia-gravity waves made in this case is also similar to the above two cases.

b. solutions for $\varepsilon=1$ or 10

Figure 3 shows the evolution of the initial disturbance having only vorticity for $\varepsilon=1$. Note that time is measured by $a/\sqrt{gH}=6.9 \times 10^3$ s for this value of ε . The initial vorticity is substantially retained throughout the evolution of the disturbance; further the height field to be balanced with its vorticity is gradually formed until $t=1.0$. Fig. 3b shows that in this process inertia-gravity waves are emitted from the disturbance and its emission is almost isotropic. It is noted that by $t=1.0$ an emission of inertia-gravity waves ceases; a geostrophic adjustment is completed by this time. From $t=1.0$ the emitted inertia-gravity waves propagate isotropically without dispersion. It should be noted that, for small ε , i.e., when the deformation radius is larger than the wavelength of the excited inertia-gravity waves, the inertia-gravity waves are close to being pure gravity waves, so that the dispersion and the anisotropic propagation do not occur. Fig. 3b shows that at $t=4.0$ the gravity waves are concentrated at the vicinity of antipodean points of the locations of the initial disturbances; at $t=8.0$ the gravity waves expand from the antipodean points. In short, the excited gravity waves oscillate between the locations of the initial disturbances and the vicinity of their antipodean points.

Fig. 3. As in Fig. 1 but for the initial disturbance with vorticity and $\epsilon = 1$. The unit of time is 6.9×10^3 s.
a. Contours of deviation height.

Fig. 3b. Contours of divergence.

Further, a close inspection of Fig. 3a shows that from $t=1.0$ the geostrophic disturbance propagates westward very slowly. This propagation is a consequence of the disturbance being a superposition of Rossby waves. It may be seen that the shape of this disturbance begins to break down at $t=8.0$ (i.e. 5.5×10^4 s), while for $\varepsilon=1000$, as seen in Fig. 1, the shape of the disturbance is not so modified at $t=0.8$ (i.e. 1.8×10^6 s). This is because for small ε the phase velocity of Rossby waves more strongly depends on the wavenumber than for large ε ; as a result Rossby waves are more dispersive for small ε .

The solutions for initial states with perturbation in the height or divergence field are also obtained (not shown). In these cases the initial disturbances are not retained at all; the disturbance spreads over the whole sphere as gravity waves. This fact can be understood from a consideration that these initial states do not involve Rossby waves. When ε is small, i.e., the deformation radius is large in comparison with the spatial scale of the initial disturbance, Rossby waves are non-divergent. Hence, these initial states with perturbation in the height or divergence field cannot involve Rossby waves.

c. discussion

First, we discuss the validity of some simple models, which are referred to in the introduction, in the light of the results obtained above. As long as we are concerned with the equilibrium states established for the given disturbance, the results in our model agree with what the current concept of "geostrophic adjustment" on an f -plane predicts: for large ε (small deformation radius), the velocity (vorticity) field is adjusted to the given height field, while for small ε (large deformation radius) the height field is adjusted to the given velocity (vorticity) field. These equilibrium states have been obtained in a one-dimensional model on an f -plane also (e.g. Rossby (1938) and Gill (1982)), as described in the introduction. However, it should be emphasized that the geostrophic adjustment by gravity waves is made in a different way between the one-dimensional infinite models and our two-dimensional model. In the former models inertia-gravity waves transport a surplus momentum or mass from the initial disturbance to regions infinitely distant from it. Hence, it is important for the former model to be infinitely extended without a wall. On the other hand in our model on the finite sphere the adjustment is established and held by that inertia-gravity waves are dispersed (for large ε) or exist almost always (for small ε) in a wide region outside the disturbance. This is possible because the area occupied by the initial disturbance is sufficiently small in comparison with that on the whole sphere.

Second, we discuss the validity of two-dimensional models on an f -plane or a β -plane. Our results in

this section indicate that, if we wish to examine the behaviour of inertia-gravity waves excited by a non-geostrophic state, treatment on a sphere is necessary. In fact, for large ε the inertia-gravity waves are shown to propagate equatorward and to be restricted to latitudes equatorward of the initial disturbance. This equatorward propagation of inertia-gravity waves is evidently due to a dependence of the Coriolis parameter on latitude, so that this behaviour cannot be reproduced on an f -plane, though this may be done on a β -plane. Further, the restriction of these inertia-gravity waves to the equatorial region cannot be reproduced on a mid-latitude β -plane. For small ε , it is shown that a packet of the excited gravity waves oscillates between the location of the initial disturbance and its antipodean point. Evidently such motion cannot be reproduced by a model on an infinite plane, i.e., an f -plane or a β -plane. Thus, treatment on a sphere is indispensable even for a disturbance in mid-latitudes in a geostrophic adjustment problem, if we wish to reproduce adequately the excited inertia-gravity waves.

4. Evolution of disturbances in the tropics

In this section we examine the evolution of initial disturbances created in the tropics for $\varepsilon=10$ (or 1) and 1000. Table 1 indicates that the spatial scale of the initial disturbance assumed here (i.e., the zonal scale is $36^\circ - 60^\circ$) is smaller than (or comparable to) the equatorial radius of deformation for $\varepsilon=10$ (or 1), but is larger than that for $\varepsilon=1000$.

a. solutions for $\varepsilon=1000$

First, we examine the evolution of the disturbance with perturbation symmetric about the equator in the height field of $\varepsilon=1000$. We present the evolution of the height and velocity fields in Fig. 4a and the divergence and velocity fields in Fig. 4b. The initial height field is seen in Fig. 4a at $t=0.0$. At $t=0.1$ there exists an almost isotropic inflow toward the centre of the disturbance, though the directions of the winds are slightly deviated from its centre by the Coriolis force. This deviation is enlarged at $t=0.2$ and 0.5 .⁴ The height and velocity fields at $t=0.5$ are reminiscent of the steady solution for tropical localized heating symmetric about the equator. (See Fig. 1 for $\varepsilon=1000$ in Matsuda and Kato (1987) and Fig. 1 in Gill (1980).) The divergence field in Fig. 4b at $t=0.5$ is not similar to that of the steady solution. The similarity in the

⁴It is supposed that this evolution from $t=0.0$ through $t=0.5$ is not much different from the response of the atmosphere to a sudden switch-on of tropical localized heating. This response has been calculated, e.g. by Heckley and Gill (1984) based on "long-wave approximation" on the equatorial β -plane. However, it should be emphasized that, as understood from our result from $t=0.1$ through $t=0.5$, a sudden heating mainly excites gravity waves of high frequency in its initial stage; hence a "long-wave approximation" cannot be applied to the initial stage of a problem of this kind.

Fig. 4. As in Fig. 1 except that the initial disturbance is located in the tropics. The unit of time is 2.2×10^5 s. a. Contours of deviation height.

Fig. 5. As in Fig. 2 except that the initial disturbance is located in the tropics.

height and velocity field between the present and the steady solution indicates that the disturbance treated here is also mainly made up of contributions from Kelvin and Rossby waves, as in the steady solution. In fact, Fig. 5 shows that this disturbance is mainly composed of Kelvin and Rossby waves. The subsequent evolution of the disturbance is predicted to be a split into the eastward propagating Kelvin and westward propagating Rossby waves, as seen in practice, in the figures at $t = 1.0$ and 2.0 . The characteristics of the Kelvin wave are clearly seen in the phase relation between height deviation, zonal velocity and divergence, which are depicted in Fig. 4a and 4b at $t = 2.0$. On the other hand, the westward-propagating disturbance is a superposition of some Rossby waves with different latitudinal wavenumbers. Under the "long-wave approximation" on the equatorial β -plane, the phase velocity of Kelvin waves is larger than that of the fastest Rossby waves by a factor of three (*e.g.*, Gill, 1980). This agrees with our result shown in Fig. 4: the velocity of the eastward propagating disturbance is much faster than that of westward propagating one. Further, it should be noted that the westward propagating disturbance shows a northeast-southwest (southeast-northwest) tilt in the Northern

(Southern) Hemisphere. This is because, as pointed out above, the phase velocity of Rossby waves decreases with latitude.⁵

Thus it is found that, when the height field is perturbed in the tropics for $\epsilon = 1000$, the disturbance behaves as equatorial waves except the initial stage; the disturbance in the height field is not retained at all, although the deformation radius is smaller than the spatial scale of the initial disturbance in this case. The geostrophic relation is established only for the westward-propagating disturbance except the region just adjacent to the equator.

Second, we discuss the evolution of an initial disturbance antisymmetric about the equator in the height field. According to the results obtained (not shown), the initial height field is not so drastically changed with time as in the symmetric case, and a velocity field geostrophically balanced with the

⁵The evolution of the tropical disturbance was examined by Yamagata (1987) with the linear shallow water equations supplemented by the simplified moisture equation. Like ours, his results show the excitation and the subsequent split of Kelvin and Rossby waves. However, both the waves, especially the Kelvin wave, obtained in his study are longitudinally much elongated in comparison with ours. This elongation may be caused by the excitation of the waves due to a latent heat release.

given height field is formed. This geostrophic disturbance propagates westward with a velocity slower than that in the symmetric case. This disturbance is mainly composed of Rossby and mixed Rossby-gravity waves. Under a "long-wave approximation" on the equatorial β -plane the phase velocities of these antisymmetric Rossby waves are equal to or smaller than $\sqrt{gH}/5$, so that this disturbance propagates westward more slowly than that in the above symmetric case. In the antisymmetric case, as an eastward-propagating wave, we have only inertia-gravity waves. It is seen that these inertia-gravity waves are excited and propagating eastward. In addition, a cross-equatorial flow from the initial disturbance on one hemisphere to that on another one appears. In short, in comparison with the symmetric case the evolution of the antisymmetric disturbance is similar to that in mid-latitudes. This can be understood from the fact that the deviation height vanishes at the equator in the antisymmetric case, so that the main part of this disturbance is some distance from the equator.

In the rest of this subsection we discuss the evolution of the initial disturbance with vorticity or divergence. In these cases the evolutions are complicated by the excitation of inertia-gravity waves. Fig. 5 shows that for the initial disturbance with vorticity symmetric about the equator amplitudes of Rossby and Kelvin waves are dominant, but those of the inertia-gravity waves (especially westward ones) are not negligible. A superposition of these waves composes the complicated fields. For the initial disturbance with antisymmetric vorticity, *i.e.*, one positive vorticity having its centre at the equator, the subsequent evolution of the fields is comparatively simple (not shown). This initial state strongly excites eastward inertia-gravity, Rossby and mixed Rossby-gravity waves. As a result, the evolution of this case is similar to that of the antisymmetric disturbance in the height field except that the fields in this case are complicated by a superposition of the inertia-gravity waves.

As shown in Fig. 5, for the disturbance with a perturbation in the divergence field, the excited Rossby waves have a negligible amplitude compared to inertia-gravity waves. (In the antisymmetric case an amplitude of mixed Rossby-gravity waves is comparable to those of inertia-gravity waves.) Hence, the evolution of this case (not shown) is only such that the excited inertia-gravity waves are dispersed over and restricted to the tropics.

The results obtained in this subsection show that, when ε is large, all kinds of the excited waves are restricted to the equatorial region. Thus we may conclude that for large ε the evolution of a disturbance can be treated on an equatorial β -plane.

b. solutions for $\varepsilon = 1$ or 10

In Fig. 6 we present the evolution of a disturbance with vorticity symmetric about the equator for $\varepsilon = 1$. Fig. 6a shows that the initial vorticity is more or less retained in the original location through the evolution and the height field geostrophically balanced with the given vorticity is gradually formed from $t = 0.1$ through $t = 1.0$. From $t = 1.0$ the height and velocity fields are not much changed with time. Fig. 6b shows that gravity waves are excited in the disturbance and propagate in all directions. It should be noted that for $\varepsilon = 1$ an inertia-gravity wave is close to a pure non-dispersive gravity wave. At $t = 3.0$ (not shown) the excited gravity waves are concentrated at the antipodean point of the location of the given disturbance without dispersion; at $t = 5.0$ the gravity waves expand from the antipodean point. The gravity waves will again concentrate at the original location of the disturbance. Evidently, in this case gravity waves continue to oscillate between the two antipodean points on the sphere.

In the case that the height or divergence field is initially perturbed, the initial fields are composed almost only of gravity waves, because Rossby waves are almost non-divergent for small ε . Hence, the evolution of the fields is merely the excitation of gravity waves and their isotropic propagation; the disturbance disappears from its original location. Thus, in all cases, when ε is small, the initial disturbance created in the tropics evolves in the similar way to in mid-latitudes. The reason for this result may be understood as follows. When ε is small, all waves are distinctly classified into pure gravity waves and non-divergent Rossby waves; "equatorial waves" do not exist (see, *e.g.* Longuet-Higgins, 1968). Hence, a disturbance given even in the tropics is represented by a superposition of pure gravity and non-divergent Rossby waves, as in mid-latitudes. As a result the geostrophic adjustment analogous to that in mid-latitudes is made in the tropics also. It is evident that such evolution of tropical disturbances for small ε cannot be treated by a model on an equatorial β -plane; hence, treatment on a sphere is indispensable in this case.

5. Summary and comments

We can summarize the results obtained in the present study as follows.

(1) When a non-geostrophic disturbance is in mid-latitudes on the sphere, the geostrophic equilibrium state of the given disturbance agrees with what is anticipated from the geostrophic adjustment on an f -plane: when the deformation radius is smaller (larger) than the spatial scale of the given disturbance, then the vorticity (mass) field comes to be adjusted to the mass (vorticity) field.

(2) When ε is large, the excited inertia-gravity

Fig. 6. As in Fig. 1 except that the initial disturbance with vorticity is located in the tropics and $\varepsilon = 1$. The unit of time is 6.9×10^3 s. a. Contours of deviation height.

Fig. 6b. Contours of divergence.

waves propagate equatorward and are restricted to latitudes equatorward of the disturbance. Hence, the inertia-gravity waves do not perturb the geostrophically balanced state. When ε is small, the excited almost pure gravity waves oscillate between the location of the disturbance and its antipodean point. Hence, the behaviour of the excited inertia-gravity waves in the non-geostrophic disturbance in mid-latitudes cannot be treated by a model on an f -plane or on a β -plane.

(3) After the geostrophic balance is established, the disturbance in mid-latitudes evolves as Rossby wave groups. When ε is large, this geostrophic disturbance shows a northeast-southwest (southeast-northwest) tilt on the Northern (Southern) Hemisphere. When ε is small, this geostrophic disturbance shows a tendency to break down at $t=8.0$ because of the dispersive nature of Rossby wave.

(4) When ε is large, the disturbance given in the tropics evolves as equatorial waves. For example, the perturbation symmetric about the equator in the height field evolves to split out into Kelvin and equatorial Rossby waves. For large ε we can treat the problem on an equatorial β -plane.

(5) When ε is small, a geostrophic adjustment similar to that in mid-latitudes is made even for a disturbance in the equatorial region. Hence, in this case, we cannot treat the evolution of a tropical disturbance on an equatorial β -plane.

In this study we calculated the evolutions of disturbances which are respectively symmetric and antisymmetric about the equator. As shown in section 3, a disturbance created in mid-latitudes evolves independently of a disturbance in another hemisphere; hence the evolution of the isolated disturbance created in one hemisphere almost agrees with that of the disturbance in one hemisphere in the symmetric or antisymmetric case. On the other hand, when an *asymmetric* isolated disturbance occurs in the tropics, it is necessary to combine the symmetric and antisymmetric fields in order to predict the evolution of this disturbance. This examination will be made in a subsequent work.

In this study the evolution of disturbances is calculated in the horizontal fields using the shallow water equations for some values of ε . In order to follow the evolution of disturbances in the three-dimensional atmosphere, it is necessary to combine the solutions of the shallow water equations for various values of ε . This effort will be also made in a work in the future.

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球面上における擾乱の時間発展と地衡風調節

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f 面上における地衡風調節の概念が球面上の大気に適用出来るか調べる為、中緯度及び低緯度に与えられた局所的な非地衡風擾乱の時間発展を組織的に研究した。色々な初期条件に対する初期値問題として球面上の線型化した浅水方程式を数値的に解いた。この計算に際して、球面上の浅水方程式の固有函数 (Hough 函数) を用いた。

初期擾乱が中緯度に与えられた場合、 $\varepsilon = \{ (\text{惑星の回転速度}) / (\text{重力波の速度}) \}^2$ が大きい時 (変形半径が擾乱の空間スケールよりも小さい時)、地衡風調節の過程で擾乱から出た慣性重力波は赤道方向に伝播し、低緯度地方で拡散した。 ε が小さい時 (変形半径が擾乱の空間スケールより大きい時) は、励起された重力波は分散することなしに、擾乱の位置とその対蹠点の間を振動した。

初期擾乱が低緯度地方に与えられかつ ε が大きい時は、擾乱の時間発展は赤道波の時間発展として理解出来る。初期擾乱が赤道に対して対称な場合は、擾乱はケルビン波とロスビー波に分裂する。 ε が小さい時は、低緯度地方に与えられた擾乱の時間発展は中緯度と同様で、地衡風調節が行われた。

地衡風の関係が成立してからの擾乱の時間発展も議論した。

¹現在：広島地方気象台