A Study of Stationary Wave-Wave Interactions in a Low Order Stratospheric Model

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Abstract

Interactions among stationary planetary waves in a low-order spherical stratospheric model are studied both analytically and numerically. A triad of interacting waves is examined using a steady-state hemispheric quasi-geostrophic model with a mean zonal wind in solid body rotation. Due to the structure of the model basic state, dissipation is required for the waves to interact. It is found that the nonlinear structure of the two gravest forced modes is independent of their relative position only if the third mode is unforced at the lower boundary. In this case the degree to which the nonlinearities act to amplify or phase shift the linear waves is shown to be dependent upon the vertical propagation characteristics of the waves. Large horizontal scale waves and weak zonal westerlies are found to be conditions which can result in significant amplitude changes due to the wave-wave interaction. As the scale of the wave is reduced and/or the speed of the mean zonal westerlies is increased, the predominant changes occur in the wave phases. Numerical solutions using realistic boundary forcing amplitudes and realistic dissipation are found to be only weakly nonlinear, despite the fact that the planetary waves distort considerably the polar vortex.

1. Introduction

Starting with the pioneering work of Charney and Drazin (1961), the majority of forced stationary planetary wave studies of the winter stratosphere have employed linearized equations to calculate the wave structure. The justification of using linear models lay in the fact that the solutions resembled qualitatively the observed stationary wave structure, as in the work of Matsuno (1970), for example. Remarkably, however, the resulting large amplitude waves distorted considerably the zonal flow at high elevations, which would apparently contradict the underlying linear assumption of small amplitude disturbances, thereby implying that nonlinear interactions among the waves be important.

Egger (1976) and Ashe (1979) employed nonlinear steady-state models with two levels in the vertical to calculate the structure of forced stationary waves in the troposphere. Their topographically-forced nonlinear solutions did not differ drastically from the linear ones, indicating that wave-wave interactions were relatively unimportant.

Using monthly mean data from the upper troposphere and lower stratosphere Derome (1984) showed that a linear relationship existed between the zonally-averaged quasi-geostrophic potential

vorticity and streamfunction. The consequence of such a flow structure is that a linear, inviscid solution of the steady-state quasi-geostrophic potential vorticity equation is a finite-amplitude solution. Thus, large amplitude stationary wave disturbances, forced by topography for example, can exist without interacting nonlinearly provided dissipation and other nonconservative forcing terms are absent.

Robinson (1986), using a highly truncated quasi-geostrophic model with Cartesian geometry, examined the nonlinear stratospheric stationary wave response to tropospheric forcing. The structure of his zonal basic state was such that wave-wave interactions were a consequence solely of dissipation in the model. Despite the relatively weak dissipation (at mid-stratospheric levels the ratio of the mean flow advective and the dissipative time-scales was less than 0.1), the nonlinear solutions differed by up to 25 % from the corresponding linear structure. In addition Robinson found that the solutions were highly dependent on the phase of the largest scale waves at the lower boundary.

In this paper stationary dissipation-induced wave-wave interactions are examined in the context of a spherical model. As will be seen in the subsequent sections, several significant differences were found between our results and those of Robinson.

2. The model equation

Since stationary stratospheric planetary waves are large-scale extratropical disturbances, quasi-geostrophic theory should adequately describe their first-order dynamics (see, for example, Matsuno, 1970 or Schoeberl and Geller, 1977). The particular model that we have chosen keeps the Coriolis parameter constant except where it appears in the term involving the advection of the zonal potential vorticity and so is a spherical equivalent of the quasi-geostrophic β -plane model.

In order to isolate the stationary wave-wave interactions, the forcing of the time-averaged flow by transient disturbances is neglected. The wave equation is then derived by separating the flow into a zonal mean part and a perturbation, denoted by an overbar and a prime, respectively. As did Robinson (1986), Ashe (1979) and Egger (1976) in their non-linear stationary wave studies, we will specify the mean zonal flow in our search for the wave structure. Our model (and theirs, too, for that matter) can thus be viewed as applying in the limit of strong zonal flow forcing which drives the basic state to some prescribed condition.

Due to the insensitivity of quasi-geostrophic models to vertical variations of the basic state temperature profile (Schoeberl and Geller, 1977), an isothermal basic state is assumed, implying that the Brunt-Väisälä frequency N is constant. Rayleigh friction and Newtonian cooling, whose coefficients are denoted by γ and α , are the dissipative mechanisms included in this model. The resulting wave potential vorticity equation in spherical geometry and log pressure coordinates is therefore

$$\bar{\omega} \frac{\partial q'}{\partial \lambda} + \bar{\beta} \frac{\partial \psi'}{\partial \lambda} + \gamma \nabla^2 \psi' + \frac{\varepsilon_o^2}{\rho_o} \frac{\partial}{\partial z} \left[\rho_o \alpha \frac{\partial \psi'}{\partial z} \right] = -J(\psi', q') + \overline{J(\psi', q')} \quad (1)$$

where

$$q' = \nabla^2 \psi' + \frac{\varepsilon_o^2}{\rho_o} \frac{\partial}{\partial z} \left[\rho_o \frac{\partial \psi'}{\partial z} \right]$$

$$\bar{\omega} = \frac{\bar{U}}{a \cos \theta}$$

$$\bar{\beta} = \frac{1}{a^2 \cos \theta} \left\{ 2\Omega \cos \theta - \frac{1}{a} \frac{\partial}{\partial \theta} \left[\frac{1}{\cos \theta} \frac{\partial}{\partial \theta} (\bar{U} \cos \theta) \right] - \frac{\varepsilon_o^2 a}{\rho_o} \frac{\partial}{\partial z} \left[\rho_o \frac{\partial \bar{U}}{\partial z} \right] \right\}$$

and $\varepsilon_o^2 = \frac{f_o^2}{N^2}$.

In Eq. (1), $\bar{U}$ is the speed of the mean zonal flow and ψ' and q' are the disturbance streamfunction and potential vorticity, respectively. The longitude

and latitude are denoted by λ and θ , respectively. Ω is the earth's angular velocity, a the earth's radius and f_o the Coriolis parameter evaluated at some specified latitude. The differential operators J and ∇^2 denote the spherical geometry Jacobian and two-dimensional Laplacian, respectively. The reference density is given by $\rho_o = \rho_s \exp(-z/H)$ and the log-pressure vertical coordinate by $z = -H \ln(p/p_s)$ with the subscript s referring to the lower boundary and with H the scale height.

Ignoring specific forcing mechanisms such as topography and heating, the wave structure is forced at the model lower boundary by specifying the streamfunction. The height of the lower boundary is taken to be some lower stratospheric elevation. Boundary conditions of zero wave amplitude at both the equator and pole are assumed.

Different vertical profiles of the mean zonal wind will be examined, but in each case the horizontal structure will be that of solid body rotation. The implications of using such a basic state will now be discussed.

3. Theoretical results

An examination of the linearized potential vorticity Eq. (1) indicates that if the ratio of $\bar{\beta}$ and $\bar{\omega}$ is independent of latitude then, in the absence of both dissipative and internal forcing mechanisms, $q' = -(\bar{\beta}/\bar{\omega})\psi'$.

Consequently, a linear solution is also a finite-amplitude solution since $J(\psi', q') = 0$ and so all wave-wave interactions vanish (Derome, 1984).

For our basic state zonal wind in solid body rotation the zonally averaged wind $\bar{U}$ is proportional to $\cos \theta$ and so both $\bar{\omega}$ and $\bar{\beta}$ are independent of latitude. Since no internal forcing mechanisms (such as heating) are present in the model, the wave-wave interactions arise solely as a result of dissipation.

The layout of this section is as follows. First, to obtain a maximum simplification of the nonlinear wave equation, the model is truncated to three waves. Since the forcing of stratospheric planetary waves occurs on the longest horizontal scales this seems to be a reasonable assumption. Then the asymptotic behavior of the wave triad is examined in the limit of both weak nonlinearity and dissipation. There it will be demonstrated how the weakly nonlinear solutions are dependent on the propagation characteristics of the basic state as well as on the number of waves forced at the lower boundary.

3.1 Weakly nonlinear wave triad equations

To derive the triad equations we first separate the horizontal and vertical dependence of the solution by expanding ψ' and q' in terms of spherical harmonics. Thus

$$\begin{Bmatrix} \psi' \\ q' \end{Bmatrix} = \sum_m \sum_n \begin{Bmatrix} \Psi_{m,n}(z) \\ Q_{m,n}(z) \end{Bmatrix} e^{z/2H} P_{m,n}(\theta) e^{im\lambda} \quad (2)$$

where $\Psi_{m,n} = \Psi_{m,n}^*$ (the asterisk denoting the complex conjugate) and the $P_{m,n}$'s are orthonormal over the horizontal domain. Substituting Eq. (2) into Eq. (1), multiplying by $P_{q,r} e^{-iq\lambda}/(4\pi \cos \theta)$, integrating over θ and λ and dropping the overbars from the basic state quantities yields

$$(1 - i\delta_m) Q_{m,n} + \frac{\beta}{\omega} \Psi_{m,n} = -\frac{i\varepsilon}{m\omega} J_{m,n} \quad (3)$$

where $J_{m,n}$ represents the projection of the nonlinear terms onto wave (m, n) and δ_m and ε are the small nondimensional parameters of dissipation (*i.e.*, $\delta_m \equiv \gamma/(m\omega)$) and nonlinearity, respectively. For the purpose of demonstration a constant value of potential vorticity damping γ has been used, implying equal and constant rates of Rayleigh friction and Newtonian cooling. (Note that the assumption of weak interactions has enabled the Jacobian terms in Eq. (1) to be scaled by ε . In the sequel it will be assumed that $O(\delta_m) < O(\varepsilon)$.) The term $Q_{m,n}$ designates the potential vorticity for a particular mode and is given by

$$Q_{m,n} = \left\{ \varepsilon_o^2 \left[\frac{d^2}{dz^2} - \frac{1}{4H^2} \right] - \frac{n(n+1)}{a^2} \right\} \Psi_{m,n} \quad (4)$$

For a single triad of waves, composed of the largest horizontal scales that satisfy the equatorial boundary condition of zero wave amplitude, Eq. (3) reduces to a system of three coupled nonlinear ordinary differential equations. The associated nonlinear terms are given by

$$\begin{aligned} J_{1,2} &= i\mu e^{z/2H} [\Psi_{1,4}^* Q_{2,3} - \Psi_{2,3} Q_{1,4}^*] \\ J_{2,3} &= i\mu e^{z/2H} [\Psi_{1,4} Q_{1,2} - \Psi_{1,2} Q_{1,4}] \\ J_{1,4} &= i\mu e^{z/2H} [\Psi_{2,3} Q_{1,2}^* - \Psi_{1,2}^* Q_{2,3}] \end{aligned} \quad (5)$$

where $\mu = (35)^{1/2}/a^2$. By expanding $\Psi_{m,n}$ and $Q_{m,n}$ in powers of ε (*i.e.*, $\Psi = \Psi^{(0)} + \varepsilon \Psi^{(1)} + \dots$) the nonlinear system (3) yields a set of forced linear equations. The nonlinear terms up to first order in ε are

$$\begin{aligned} J_{1,2} &\cong \left\{ G \Psi_{2,3}^{(0)} \Psi_{1,4}^{(0)*} \right\} + \varepsilon \left\{ -H \left\| \Psi_{2,3}^{(0)} \right\|^2 \Psi_{1,2}^{(0)} \right\} \\ J_{2,3} &\cong \varepsilon \left\{ H \left[\left\| \Psi_{1,2}^{(0)} \right\|^2 + \left\| \Psi_{1,4}^{(0)} \right\|^2 \right] \Psi_{2,3}^{(0)} \right\} \\ J_{1,4} &\cong \left\{ -G \Psi_{2,3}^{(0)} \Psi_{1,2}^{(0)*} \right\} + \varepsilon \left\{ -H \left\| \Psi_{2,3}^{(0)} \right\|^2 \Psi_{1,4}^{(0)} \right\} \end{aligned} \quad (6)$$

where $G(z) = (3\mu\beta\delta_1)/(2\omega)e^{z/2H}$, $H(z) = \mu e^{z/2H} G(z)/\omega$ and $\| \cdot \|$ denotes the absolute value. In deriving Eq. (6), second-order terms in the dissipation parameter have been neglected, which explains the absence of dissipation-induced terms such as $\Psi_{m,n}^{(1)}$. Similarly, the potential vorticity terms appearing in Eq. (5) were eliminated by relating their zeroth-order values to the zeroth-order streamfunctions.

Note that in the absence of dissipation, the nonlinear forcing terms vanish.

3.2 Examination of the leading-order forced response

From the structure of the forcing terms in Eq. (6) several qualitative statements can be made about the weakly nonlinear solutions. For the case to be discussed in the subsequent sections in which wave (1, 4) is not forced at the lower boundary, the nonlinear forcing for waves (1, 2) and (2, 3) is independent of the relative position of the linear waves since the nonlinear coupling involves only the moduli of the zeroth order waves. We see for example from Eq. (6) that $J_{1,2}$ is 180 degrees out of phase with $\Psi_{1,2}^{(0)}$ and that $J_{2,3}$ is in phase with $\Psi_{2,3}^{(0)}$, regardless of the position of $\Psi_{1,2}^{(0)}$ and $\Psi_{2,3}^{(0)}$. Hence, the weakly nonlinear solutions for waves (1, 2) and (2, 3) do not depend upon the relative position of the linear waves. With the inclusion of linear wave (1, 4), nonlinear wave (2, 3) remains as before independent of the relative phase of the linear waves but wave (1, 2), however, does not. Its forcing and consequently its nonlinear streamfunction are dependent on the phase of linear waves (2, 3) and (1, 4).

Let us now examine in a qualitative fashion the leading-order response to the forcing terms given in Eq. (6) for the case in which wave (1, 4) is not forced at the lower boundary. This enables the position of the nonlinear correction to be determined and so reveals the degree to which the nonlinearity amplifies and phase shifts the linear waves. As did Robinson (1985), who performed a similar analysis for a β -plane model, we will proceed as follows.

Let us first relate the wave potential vorticity and streamfunction in the following manner

$$Q_{m,n} = -(\alpha_R + i\alpha_I) \Psi_{m,n} \quad (7)$$

where α_R and α_I are z -dependent real quantities. Substituting Eq. (7) into Eq. (3) and expressing $\Psi_{m,n}$ and $J_{m,n}$ in polar form, with ϕ_ψ and ϕ_J the respective phases, yields

$$(a^2 + b^2)^{1/2} \|\Psi\| e^{i(\phi_\psi - \phi)} = \frac{\|J\|}{m\omega\alpha_R} e^{i\phi_J} \quad (8)$$

where $\phi = \tan^{-1} b/a$

$$\begin{aligned} a &= \alpha_I/\alpha_R - \delta_m & \text{and} \\ b &= 1 - \beta/(\omega\alpha_R) + \delta_m\alpha_I/\phi a_R. \end{aligned}$$

In the discussion that follows we will focus our attention on the phase response ϕ or equivalently the longitudinal difference $\Delta\lambda$ between the position of the forcing $J_{m,n}$ wave and that of the streamfunction response wave. Thus, from Eq. (8) we get

$$\Delta\lambda = (\phi_\psi - \phi_J)/m = \phi/m = m^{-1} \tan^{-1}(b/a). \quad (9)$$

The quantities a and b which appear in Eq. (9) depend on α_I and α_R and so can only be determined by solving the potential vorticity equation. In certain limiting cases, however, the phase response as given by Eq. (9) can be deduced. We shall assume that the term containing the vertical derivative in the expression for the wave potential vorticity, *i.e.*, Eq. (4), is small in comparison to the other terms. This will be true provided that the nonlinear correction streamfunction response, and therefore its corresponding forcing, be smoothly varying in the vertical. In this case the potential vorticity and the streamfunction of the nonlinear correction will be nearly 180 degrees out of phase, implying that $\alpha_R \gg \alpha_I$. If, in addition, the damping is sufficiently weak (*i.e.*, small δ_m), then $a \ll 0(1)$ and $b \cong 1 - \beta/(\omega\alpha_R)$.

For the remainder of this section we will examine two limiting cases, namely (i) $b \rightarrow 1$, *i.e.*, $\omega\alpha_R \gg \beta$ and (ii) $b \rightarrow 0$, *i.e.*, $\omega\alpha_R \approx \beta$. Provided that β remains close to the value of $2\Omega/a^2$ (refer to Eq. (1) for the expression for β), which is true for $\omega \ll \Omega$ and weak vertical shear, the above limits are equivalent to a respective increase or decrease in the magnitude of $\omega\alpha_R$.

The first case (*i.e.*, $b \rightarrow 1$ implying $\omega\alpha_R \gg \beta$) applies to the situation where either the horizontal scale of the wave is small (large n and hence large α_R) or the mean zonal wind is westerly and strong. Since these are conditions which prevent the vertical propagation of wave energy, this case can therefore be viewed as corresponding to the situation of vertically trapped waves. Referring to Eqs. (8) and (9) and recalling that $|a|$ was assumed small, the angle ϕ consequently tends to $\pi/2$ and so the streamfunction response is located one quarter of a wavelength to the west of the forcing $J_{m,n}$ (*i.e.*, $\Delta\lambda = \pi/(2m)$).

Now let us apply this result to our weakly nonlinear wave triad model for the situation in which linear waves (1, 2) and (2, 3) are trapped and wave (1, 4) is not forced at the lower boundary. Recall from Eq. (6) that in the absence of linear wave (1, 4), $J_{1,2}$ and $\Psi_{1,2}^{(0)}$ are out of phase while $J_{2,3}$ and $\Psi_{2,3}^{(0)}$ are in phase. By combining the previous results pertaining to the response of a trapped wave to an internal forcing term with the above results concerning the phase relationship between the forcing and the linear waves, we conclude that the nonlinear correction is located one quarter of a wavelength to the east of the linear wave for mode (1, 2) and one quarter of a wavelength to the west for mode (2, 3). Remembering also that the nonlinear corrections are of small amplitude, the weakly nonlinear waves (*i.e.*, linear plus correction) undergo primarily a change in phase which shifts linear wave (1, 2) eastwards and linear wave (2, 3) westwards. It should be pointed out that, because the nonlinear correction and the linear wave are separated by

1/4 of a wavelength, changes in amplitude are secondary. This can be seen by considering the sum of two waves $\cos(\lambda)$ and $\Delta \sin(\lambda)$ which to second order in the small parameter Δ yields the composite wave $(1 + \Delta^2/4) \cos(\lambda - \Delta)$.

The second case (*i.e.*, $b \rightarrow 0$ or equivalently $\omega\alpha_R \rightarrow \beta$) corresponds to weaker westerlies or larger horizontal scales and so typifies conditions for vertically propagating waves. Since both a and b are small, nothing definite can be said about the response since, depending upon the relative magnitudes of these terms, ϕ can vary anywhere from $+\pi$ to $-\pi$. Unlike trapped waves, the streamfunction response of a propagating wave can be located to either side of the forcing with the distance separating the two being, in general, different from 1/4 of a wavelength.

Envisage now a situation where linear waves (1, 2) and (2, 3) are propagating and linear wave (1, 4) is absent as before. By following the same line of reasoning as with the trapped waves, we find that the nonlinear corrections are no longer restricted to being separated from the linear waves by 1/4 of a wavelength. Hence the weakly nonlinear waves can undergo changes in amplitude of the same order as the changes in phase. Since wave (1, 2) is the more vertically propagating it can be expected to undergo larger relative amplitude changes than would wave (2, 3). The amplification can be seen if we again consider the sum of the two waves $\cos \lambda$ and $\Delta \sin(\lambda + \phi)$, where ϕ is now some arbitrary phase angle (in the trapped wave case the 1/4 wavelength phase shift between the linear and nonlinear correction meant ϕ was zero). Since to first order in Δ , the sum of the two is $(1 + \Delta \sin \phi) \cos(\lambda - \Delta \cos \phi)$, order Δ changes in both the phase and the amplitude can occur.

This ends the theoretical section concerning the weakly nonlinear behaviour of a triad of forced planetary waves, interacting as a result of dissipation. In the following section the nonlinear wave triad equations will be solved using two different vertical zonal wind profiles. The results of this past section regarding the structure of the lower boundary forcing as well as the effect of the vertical structure of the waves (*i.e.*, their propagation characteristics) will be put to use in explaining the nonlinear wave structure.

4. Numerical results

In this section we will calculate numerically the nonlinear wave triad solutions using two different vertical profiles of the zonal wind specified by

$$\bar{U} = \{U_1 + U_2 \operatorname{sech}[\alpha(z - z_o)]\} \cos \theta \quad (10)$$

where $\alpha = 0.4 \times 10^{-4} \text{ m}^{-1}$, $z_o = 60 \text{ km}$ and U_1 and U_2 are constants. In the first case U_1 and U_2 are given

Fig. 1. Vertical profile of solid body rotation zonal wind (ms^{-1}) evaluated at the equator: profile A (solid); profile B (dashed).

values of 0 and 120 ms^{-1} , respectively. The resulting zonal wind structure is denoted profile A. For the second basic state configuration, called profile B, values of 40 and 70 ms^{-1} are used for U_1 and U_2 , respectively. The corresponding vertical zonal wind profiles evaluated at the equator are illustrated in Fig. 1.

The square of the refractive index (r^2) for the waves (not shown) illustrates the differences in the propagation characteristics of the two basic states. For zonal wind profile A, r^2 is positive for both waves (1, 2) and (2, 3) throughout the lower part of the domain, indicating a region of wave propagation. With profile B wave (1, 2) remains propagating up to about 55 km, although the values of r^2 are considerably reduced at lower levels. Wave (2, 3), on the other hand, is now trapped throughout the domain.

Values of $2 \times 10^{-2} \text{ s}^{-1}$ for the Brunt-Väisälä frequency and 7 km for the scale height H are assumed, implying a basic state temperature of approximately 239°K . The Coriolis parameter f_o is evaluated at 60°N . A constant rate of $5 \times 10^{-7} \text{ s}^{-1}$ is used for both the Rayleigh friction and Newtonian cooling coefficients.

The vertical structure equations for the same triad of waves are determined, as before, by substituting Eq. (2) into Eq. (1), multiplying by $P_q e^{-iq\lambda}/(4\pi \cos \theta)$ and integrating over the horizontal domain. Second-order centered finite differences are used to evaluate the vertical derivatives. Solutions are calculated using a 2 km grid spacing and a vertical domain extending from 10 to 100 km. At the lower boundary the streamfunction of the three waves is specified. At the upper boundary the condition of vanishing wave amplitude is imposed.

The nonlinear solutions are calculated by iteration. Initially the nonlinear terms are neglected and the linear solutions are calculated using a method

devised by Lindzen and Kuo (1969). The linear solutions are then used to evaluate the nonlinear terms which are subsequently used as forcing to determine the solution at the second iteration. This procedure is repeated until the desired level of convergence is attained.

In the majority of the experiments, relatively weak lower boundary forcing and dissipation are used. To aid in the analysis of the nonlinear wave structure, we will resort frequently to the qualitative theoretical results of Section 3. An examination of the phase response to the nonlinear forcing will reveal the relationship between the nonlinear wave structure and the propagation characteristics of the linear waves. In another experiment, the importance of the relative position of the linear waves will be demonstrated by forcing wave (1, 4) at the lower boundary. We will conclude the numerical studies by examining the effect of larger boundary forcing and stronger dissipation.

4.1 Numerical results using zonal wind profile A

In Fig. 2 are shown the amplitude and phase of linear waves (1, 2) and (2, 3). For presentation purposes, we have defined $\psi(z, \lambda) = A(z)(g/f_o) \exp(z/2H) \cos(m\lambda + \phi)$ for a given mode, so that the amplitude A has units of length and is proportional to the square root of the energy density. Note that the phase ϕ of zonal wave m is defined such that an increase in ϕ indicates a westward phase tilt. The westward phase shift obtained for wave (1, 2) in Fig. 2 is characteristic of a propagating wave. The amplitude minimum and sudden phase shift of the wave in the vicinity of 15 km indicates the presence of an internal mode associated with the internal wave reflection in the upper part of the model where r^2 is negative. The linear solutions are not near points of resonance as was verified by setting the dissipation to zero. In addition, the upper boundary was found to be sufficiently high since any further increase altered little the solutions.

(a) Weakly nonlinear solutions with wave (1, 4) not forced.

In this first case wave (1, 4) is not forced at the lower boundary and so it appears only through interactions with waves (1, 2) and (2, 3). To parallel the approach used in the weakly nonlinear analysis presented in Section 3, only three iterations are made. Since linear wave (1, 4) is absent, the first nonlinear corrections to waves (1, 2) and (2, 3) do not occur until the third iteration.

The difference in the amplitude and phase between the third and first (linear) iterations, denoted ΔA and $\Delta \phi$ respectively, is shown in Fig. 3. For comparison, ΔA and $\Delta \phi$ using the fully nonlinear solutions (20 iterations) are also provided. The weakly nonlinear solutions show a tendency for an amplification of wave (1, 2) and a reduction of wave

Fig. 2. Amplitude (m) and phase (deg) of the linear solution waves (1, 2) and (2, 3) using zonal wind profile A. An increase in phase indicates a westward phase tilt.

(2, 3), with both waves undergoing an increase in phase and hence a westward phase shift. The degree to which amplitude changes or phase changes dominate can be seen by examining the nondimensional ratio $R^* = (\Delta A/A_{lin})/\Delta\phi_{rad}$ where ΔA has been scaled by the linear amplitude and $\Delta\phi$ has been expressed in radians. Referring to Table 1 we see that wave (1, 2), with the largest values of R^* , undergoes the largest relative amplitude changes.

The proportionately larger amplitude changes experienced by wave (1, 2) can be understood in light of the discussion in Section 3.2 regarding the phase lag response of vertically-trapped and propagating waves to their respective nonlinear forcing. In Table 1 the phase lag $\Delta\lambda^*$, expressed as a fraction of a wavelength, for the two primary waves indicates that vertically propagating wave (1, 2) has its response greater than $1/4$ of a wavelength from its forcing in contrast to trapped wave (2, 3). Consequently, the nonlinearities tend to both amplify and shift wave (1, 2) while mainly shifting wave (2, 3).

(b) *Weakly nonlinear solutions with wave (1, 4) forced.*

The previous numerical results pertained to the case where linear wave (1, 4) was absent. In Section 3.1 its inclusion was found to render weakly nonlinear wave (1, 2) dependent on the relative position of the waves at the lower boundary. Nonlinear wave (2, 3), on the other hand, was unaffected by the phase of the linear waves. Since Eq. (6) is valid for arbitrary vertical shear, the same behaviour can be expected here. In addition, the increase in the amplitude of $J_{2,3}$ arising from the presence of linear wave (1, 4) can be expected to further diminish the amplitude of nonlinear wave (2, 3), since as we have already

Fig. 3. The solid and dashed lines represent the difference in amplitude (units of 10^{-1} m) and phase (deg) between the third iteration and linear solutions using zonal wind profile A, for waves (1, 2) and (2, 3). The dotted curves were obtained in the same manner but using the 20th iteration solutions. The linear solutions are shown in Fig. 2.

seen in Fig. 3 the nonlinearities associated with $J_{2,3}$ have reduced the amplitude of wave (2, 3).

To illustrate these points in the context of zonal wind profile A, wave (1, 4) is forced at the lower boundary with an amplitude of 25 m and a phase of 45° . The lower boundary amplitudes of waves (1, 2) and (2, 3) are kept at their previous values of 50 m and the phase of wave (2, 3) is changed by increments of 30 degrees from the value shown in Fig. 2. The results, not shown, indicate that the change in amplitude and phase from the linear values varied from about -0.5 m to 1.8 m and from -0.1° to 1.2° , respectively, for wave (1, 2) while remaining nearly independent of the relative position of the two linear waves in the case of wave (2, 3).

(c) *Solutions using increased lower boundary forcing and dissipation.*

In this last subsection we will examine the effect of increased lower boundary forcing and larger, more realistic dissipation. Here the amplitudes of waves (1, 2) and (2, 3) are doubled and wave (1, 4) is once again set to zero. The Newtonian cooling and Rayleigh friction coefficients are now z -dependent and no longer equal to one another. The new Newtonian cooling profile is taken from Holton and Mass (1976) and is given by

$$\alpha(z) = (1.5 + \tanh((z - 35)/7)) \times 10^{-6} \text{ s}^{-1} \quad (11)$$

where z is in kilometers. The Rayleigh friction coefficient is increased from its background value of

Table 1. Nondimensional ratio $R_{(m,n)}^* = (\Delta A/A_{lin})/\Delta\phi_{rad}$ and longitudinal difference $\Delta\lambda_{(m,n)}^*$ between the nonlinear forcing at the third iteration and the corresponding streamfunction response for waves (1,2) and (2,3) using both zonal wind profiles A and B. $\Delta\lambda^*$ has been expressed as a fraction of a wavelength so that a value of $\Delta\lambda^* = 0.25$, for example, means the response wave is 1/4 of a wavelength to the west of the forcing. Refer to Figs. 2, 3, 6 and 7 for the corresponding linear solutions and for ΔA and $\Delta\phi$.

Z(km)	Wind profile A				Wind profile B			
	$R_{(1,2)}^*$	$R_{(2,3)}^*$	$\Delta\lambda_{(1,2)}^*$	$\Delta\lambda_{(1,2)}^*$	$R_{(1,2)}^*$	$R_{(2,3)}^*$	$\Delta\lambda_{(1,2)}^*$	$\Delta\lambda_{(2,3)}^*$
20	0.19	0.28	-0.24	0.29	0.28	0.10	0.31	0.26
30	0.62	0.21	-0.32	0.28	0.22	0.09	0.30	0.26
40	0.81	0.16	-0.34	0.27	0.19	0.07	0.29	0.26
50	0.92	0.14	-0.35	0.27	0.17	0.06	0.29	0.26
60	0.99	0.12	-0.36	0.27	0.16	0.06	0.29	0.26
70	1.05	0.11	-0.37	0.26	0.15	0.05	0.28	0.26
80	1.11	0.11	-0.37	0.26	0.15	0.06	0.28	0.25

$5 \times 10^{-7} \text{ s}^{-1}$ above 50 km according to the relation

$$\gamma(z) = (5.5 + 5 \tanh[10(z - 70)]) \times 10^{-6} \text{ s}^{-1}. \quad (12)$$

Thirty iterations are used to calculate the nonlinear solutions which are accurate to at least 5 significant figures. Both the linear and nonlinear solutions are qualitatively similar to those using the weaker forcing and dissipation in Figs. 2 and 3 and so will not be shown here. As before, the nonlinearities have amplified wave (1, 2), damped wave (2, 3) and shifted both waves westwards. Due to the larger boundary forcing and damping, the relative changes to the linear waves have increased. For example, at 36 km $\Delta A/A_{lin}$ for wave (1, 2) has increased by nearly a factor of 10 from that in Figs. 2 and 3.

To examine the changes occurring in the horizontal wave structure, the total streamfunction at a given elevation is calculated by integrating Eq. (10) with respect to latitude and adding to it the wave triad given by Eq. (2). Thus,

$$\psi_T(\theta, \lambda, z) = \frac{RT}{f_o H} z - aU(z) \sin \theta + \psi'(\theta, \lambda, z) \quad (13)$$

where R is the gas constant, T the basic state temperature and $U(z)$ is given by the curly brackets in Eq. (10). The z -dependent “constant” of integration was evaluated using the fact that the basic state is isothermal.

In conjunction with ψ_T , the total potential vorticity q_T will also be presented and is calculated as follows

$$q_T(\theta, \lambda, z) = \nabla^2 \psi_T + \frac{\varepsilon_o^2}{\rho_o} \frac{\partial}{\partial z} \left[\rho_o \frac{\partial \psi}{\partial z} T \right] + 2\Omega \sin \theta. \quad (14)$$

In Fig. 4a are shown both nonlinear ψ_T and q_T on a latitude-longitude grid at 30 km. Zonal waves 1 and 2 are visible as seen by the presence of the ψ_T ridges

Fig. 4. (a) Total nonlinear streamfunction ψ_T (solid contours) in units of 10^3 m and the corresponding potential vorticity q_T (dashed contours) in units of 10^{-5} s^{-1} , at 30 km; (b) the difference between the nonlinear and linear streamfunctions (dam).

at 150° and 330°. The fact that the nonlinearities are weak is most clearly illustrated by the near alignment of the ψ_T and q_T contours, indicating that $J(\psi_T, q_T) \cong 0$. The change between the nonlinear and linear streamfunctions, $\Delta\psi_T$, at 30 km is given in Fig. 4b and indicates that the large ridge in Fig. 4a has been displaced slightly westwards while the smaller ridge at 330° has been reduced in ampli-

Fig. 5. As in Fig. 4 but at 60 km.

tude. The presence of wave (1, 4), which has arisen through interactions of the two primary waves, is also seen by the latitudinal tilt of the $\Delta\psi_T$ contours.

Plots of ψ_T , q_T and $\Delta\psi_T$ at 60 km are shown in Figs. 5a and 5b. At this elevation zonal wave 1 is predominant, with its streamfunction ridge being positioned further westwards than at 30 km. Since the ψ_T and q_T contours are less well aligned than in Fig. 4a the wave-wave interactions play a more important role at this height. South of 50°N the nonlinearities have again shifted the ridge westwards, while in the northern portion they have deepened and moved eastwards the trough at 330°.

It is informative to compare these results with those of Robinson (1986). In our case the linear and nonlinear solutions shown in Figs. 4 and 5 differ by only about 5% in contrast to the 25% changes in Robinson's model. In addition, our nonlinear solutions are dependent upon the phase of the waves at the lower boundary only if wave (1, 4) is forced. Robinson's finite-amplitude results, on the other hand, were highly dependent on the wave positioning in the case where wave (1, 4) was not forced at the lower boundary. Thus, even though the dissipation in the two models is of comparable strength, our model with large boundary forcing behaves only weakly nonlinearly. The reason for these differences is most likely due to Robinson's 40° wide β -plane channel (whose Jacobian y -derivatives are twice as large as those in our hemispheric model) and to the stronger meridional curvature of his zonal wind (which results in larger potential vorticity for his waves).

Fig. 6. Amplitude (m) and phase (deg) of the linear solution waves (1,2) and (2,3) using zonal wind profile B. Compare with Fig. 2 for zonal wind profile A.

4.2 Numerical results using zonal wind profile B

The linear wave structure of waves (1, 2) and (2, 3) using zonal wind profile B and a constant value of $5 \times 10^{-7} \text{ s}^{-1}$ for γ and α is shown in Fig. 6. By comparison with Fig. 2, which applies to zonal wind profile A, it is evident that the vertical propagation of both waves has been reduced. While wave (1, 2) still has the structure of an internal mode, wave (2, 3) is now clearly trapped.

Proceeding as before, three iterations are made, with wave (1, 4) appearing only through wave-wave interactions. The differences between the third iteration and linear amplitudes ΔA and phases $\Delta\phi$ are illustrated in Fig. 7. The changes in amplitude are similar to those obtained with profile A, with wave (1, 2) being amplified and wave (2, 3) diminished. The modifications to the phase of wave (1, 2), however, are different from those in Fig. 3, since the weakly nonlinear wave is now shifted slightly eastward of its linear counterpart. In addition, phase changes are seen to be of greater relative importance as compared to the results using zonal wind model A as is indicated in Table 1 by both the nondimensional ratio R^* and phase lag $\Delta\lambda^*$. The stronger westerlies in profile B have reduced the vertical propagation of the waves, which in turn tends to make the nonlinear response lag its forcing by 1/4 of a wavelength, resulting in the reduced wave amplification.

5. Conclusions

In the context of a steady-state hemispheric quasi-geostrophic model, we have examined how the structure of forced planetary waves in the winter stratosphere is affected by wave-wave interactions. In order that the essential features be presented in as concise a manner as possible, a severely truncated

Fig. 7. The difference in amplitude (units of 10^{-2} m) and phase (0.1 deg) between the third iteration and linear solutions using zonal wind profile B, for waves (1, 2) and (2, 3). The linear solutions are shown in Fig. 6. Compare with Fig. 3 for zonal wind profile A.

model with a zonal wind in solid body rotation with realistic vertical shear was chosen. The choice of such a basic state not only simplified the mathematics but also resulted in wave-wave interactions that were due entirely to the presence of the model dissipation.

Since observations of the winter extratropical stratosphere indicate the predominance of large scale waves, the model was truncated to include only the three gravest horizontal modes. The wave triad problem was then examined both analytically and numerically. By considering small amplitude nearly-inviscid disturbances, the weakly nonlinear asymptotic behaviour of the wave triad equations was derived. From the structure of the leading-order forcing terms appearing in the expansion, several important features were deduced about the weakly nonlinear structure of waves (1, 2) and (2, 3). These general features were demonstrated in the presentation of the numerical results.

Firstly, the nonlinear wave structure was found to depend crucially upon the number of waves forced at the lower boundary. In particular, if wave (1, 4) was not forced, the only coupling between waves (1, 2) and (2, 3) was through the amplitudes of the linear waves. Consequently, the relative position of the waves at the lower boundary had no effect whatsoever on the weakly nonlinear waves. By forcing wave (1, 4), nonlinear wave (2, 3) remained as before unaffected but the forcing and therefore the nonlinear structure for wave (1, 2) became dependent on the phase of the zeroth order waves.

Secondly, by examining the streamfunction response to the leading-order nonlinear forcing terms,

the weakly nonlinear wave structure was shown to depend upon the vertical structure of the linear waves. It was demonstrated that the position of the response with respect to the forcing was determined to a large extent by the vertical propagation characteristics of the zeroth-order waves. In the case of a vertically trapped wave, the leading-order nonlinear correction lagged its forcing by $1/4$ of a wavelength. The behaviour for propagating waves, however, was different. There the streamfunction response could either lag or lead its forcing, with the distance separating them no longer restricted to $1/4$ of a wavelength.

The phase response to the nonlinear forcing positioned the nonlinear correction, and so revealed the extent of amplification or phase shift of the linear waves. For instance when wave (1, 4) was not forced, it was shown that if linear waves (1, 2) and (2, 3) were trapped then the predominant nonlinear effect was a phase shift of the linear waves, resulting in wave (1, 2) moving eastward and wave (2, 3) westward. If, however, the linear waves were propagating, the wave-wave interactions resulted in both modifications of the amplitude and phase. The numerical calculations showed that wave (1, 2), being the largest scale wave and hence the most able to propagate vertically, underwent the largest amplitude changes. Depending upon the strength of the zonal wind, it could experience either a westward or eastward phase shift.

The close resemblance shared by the linear and nonlinear solutions presented here may provide some insight into why linear models have been fairly successful in reproducing the gross features of observed stationary planetary waves in the winter stratosphere. The fact that our model results were only weakly nonlinear in spite of relatively large wave amplitudes was a consequence of the fact that the dissipation, which induced the wave-wave interactions, was itself relatively weak.

Our model results depend on the structure of the solid body rotation atmosphere in which a linear relationship exists between the meridional gradient of the zonal potential vorticity and the zonal wind. Thus, the extent to which dissipation-induced wave-wave interactions are relevant in the stratosphere depends upon the structure of the zonal basic state, since an arbitrary latitudinal shear could break this necessary linear relationship. As Derome (1984) has shown, however, the lower stratosphere mean zonal flow closely approximates a linear relationship between the zonal wind and the meridional gradient of the quasi-geostrophic potential vorticity, which suggests that wave-wave interactions may not be very strong in the absence of wave transience, as in the present study. In a later study, using a model with higher resolution and a realistic basic state and boundary forcing obtained from observations, the

nonlinear response to the steady forcing will be computed to determine the importance of the nonlinearities. While the model used in the present study is highly simplified, it should provide a useful basis for the interpretation of the results of the more complex model.

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低次成層圏モデルにおける定常波動の Wave-Wave Interaction の研究

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低次球面成層圏モデルにおける定常波動間の interaction を解析的及び数値的に研究した。Solid body rotation の平均帯状流を持つ定常状態の半球準地衡風モデルにより波動の triad interaction を調べた。モデル基本場は特殊な構造を持つので波動が interact する為には dissipation が必要である。

第一及び第二強制モードの非線型的構造は第三モードが下部境界で強制されない場合に限り相対的な位置によらない事が見い出された。この場合には非線型効果による線型波の振巾の増大や位相の変化の程度は波動の鉛直伝播特性によることが示される。

波動の水平スケールが大きい事と西風が弱い事が wave-wave interaction により振巾が顕著に変化する為の条件であることも見い出された。波動のスケールが減少したり、平均帯状流の速度が増大すると、波動の位相に顕著な変化が起きる。

現実的な境界振巾強制と現実的な dissipation を与えた数値解は planetary wave が極渦をかなりゆがめるにもかかわらず弱い非線型にとどまる。