A Unified Formalism of the Generalized Lagrangian Mean and the Transformed Eulerian Mean Theories

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Abstract

The generalized Lagrangian mean (GLM) theory and the transformed Eulerian mean (TEM) theory were reconstructed into a unified formalism. Application of a basic postulate that reciprocal translation is possible between Eulerian and Lagrangian information revealed the existence of a dual structure of governing equations having two standard forms which respectively correspond to the Lagrangian and Eulerian approaches. Either of these forms provides a complete equation set for the continuum concept, and can easily be transformed from one to the other. Therefore, whenever a new set of continuum equations is introduced which has either standard form, a new hypothetical continuum can be defined. As in the case of the original equations, the governing equations of the new continuum can also be easily transformed into various forms (other standard forms and the material form). The definition of new dependent variables is essential, since the hypothetical continuum contains particular arbitrariness due to the definition. Additionally the new governing equations are formulated without any physical restrictions such as initial conditions or spatial symmetry.

The GLM theory is reconstructed by selecting a form for new dependent variables and their related additional terms in the Lagrangian mean continuum. In this reconstruction, a new particle representing the original particle group exists at its Lagrangian mean position, and moves with the Lagrangian mean velocity. Similarly, the TEM theory is reconstructed as one choice from the Eulerian mean continuum, and also describes a new particle, being representative of a newly labeled domain moving with the transformed velocity. As a by-product of these reconstructions, the non-acceleration theorem in both theories was also investigated and subsequently extended.

1. Introduction

1.1 Wave, mean-flow interaction theory

The fundamental hypothesis of the non-acceleration theorem, *i.e.* steady, nondissipative waves do not affect the mean flow, was first proposed by Eliassen and Palm (1961) and Charney and Drazin (1961). This theorem is of key importance in the wave, mean-flow interaction theory and has been applied in mechanistic modeling for several drastic changes in global atmospheric circulation. For example, it was used in the quasi biannual oscillation model for the equatorial stratosphere (Lindzen and Holton, 1968; Holton and Lindzen, 1972; Plumb and McEvan, 1978), as well as in the stratospheric sudden warming model for high-latitude upper atmosphere (Matsuno, 1971; Matsuno and Nakamura, 1979).

In parallel with the success of these applications, theoretical studies were also in progress, *e.g.*, atmospheric wave propagation was studied by Bretherton

(1966), Jones (1967), and Dickinson (1968, 1969), and the conservation law of wave action was developed by Whitham (1965, 1970) and Bretherton and Garrett (1968). In addition, wave behavior in the neighborhood of singular points for small amplitude wave disturbances was investigated by Booker and Bretherton (1967), Dickinson (1970), Breeding (1971), and Geisler and Dickinson (1974).

Formulation of the wave, mean-flow interaction was also developed on the basis of the above studies. The transformed Eulerian mean (TEM) description was independently proposed by Boyd (1976) and Andrews and McIntyre (1976). The most important in-common advance was that some dependent variables were transformed into new ones. In the TEM description, the new mean meridional circulation is defined as the sum of the conventional mean meridional circulation and several additional wave disturbance quadratic terms. The virtue of the transformation lies in the fact that the wave effects on the mean flow can be separated from the mean properties. Furthermore, Andrews and

McIntyre (1978c; denoted hereafter as AMc) extended the TEM description to general waves for the zonally-symmetrical mean flow in a compressible atmosphere on the rotating earth. Concurrently, the generalized Lagrangian mean (GLM) theory, which validates the non-acceleration theorem for finite amplitude wave disturbances was developed by Andrews and McIntyre (1978a, b; denoted hereafter as AMa and AMb, respectively). They provided their formulation via a hybrid method, *i.e.*, the dependent variables express Lagrangian mean quantities, whereas the independent variables are in Eulerian coordinates. Thus the non-acceleration theorem was extended and more elegant proof was developed.

1.2 Background and objectives

The GLM theory was extended in an effort to create more practical applications for a atmospheric science. Dunkerton (1983) investigated the theoretical properties of the Lagrangian time-mean equations, and Noda (1988) highly generalized the hybrid method by extending the GLM description to the general coordinate system through the introduction of four-dimensional (time-space) Lagrangian coordinates and an initial hypersurface. In this extended GLM description, Noda did not use the initial nondisturbance postulate as was used in AMa and AMb, which can not be satisfied under actual atmospheric conditions. Furthermore, the four-dimensional symmetrical procedure overcomes difficulties stemming from the true vector Lagrangian mean operation.

Simultaneously, research on the TEM theory progressed toward developing practical atmospheric applications. Plumb (1986) extended the TEM description to Eulerian time-mean equations for a quasi-geostrophic system in order to investigate the expected response to transient eddy forcing. Iwasaki (1989) formulated a new TEM description in the primitive (nongeostrophic) equations system by introducing new vertical coordinates. This new method was developed to approximate Lagrangian mean meridional circulations in finite amplitude nongeostrophic systems.

Since the objectives of the present paper include investigating the theoretical essence of the GLM and TEM theories and identifying the key differences between them, discussions such as those of Noda and Iwasaki are beyond its scope. Consequently, the primary goal here is not to extend the GLM and TEM theories, but instead to develop a unified formalism of both. Through the reconstruction process, similarities and differences between both theories will be clarified, thereby elucidating the understanding of each theory. To this end, only the most basic geometry and mean operations are employed. Moreover, in the mathematical formulations, no physical restrictions such as initial conditions are included;

hence making the approach completely symmetrical in a non-relativistic sense.

1.3 General overview

The continuum is an abstract, macroscopic concept expressing the statistical properties of microscopic molecular motion, while the wave, mean-flow interaction theory is a simplified paradigm for understanding the nonlinear effects of fluid motion through the diminishing degree of freedom via the mean operation.

The time-honored approach to easily understand a complex problem is to mold it into a familiar model, *i.e.*, the averaged equations are regarded as the governing equations of another continuum. This *similarity principle* can only be applied by summarizing the fundamental structure of the original continuum equations.

A basic postulate in the continuum is the possibility of reciprocal translations between Eulerian and Lagrangian information. In Section 2, this postulate is mathematically expressed as an invertible mapping between a particle's label and position. Two standard-form continuum equations in label (Lagrangian) and position (Eulerian) space are defined, and it is shown that each can be derived from the other, *i.e.* they are mathematically equivalent. Such equivalency can be easily confirmed using a formula for exchanging independent variables in differential equations. This formula was derived by using only the invertibility postulate, and is not dependent upon any physical attributes (given in Appendix A). Therefore, whenever a new set of continuum equations of either standard form is introduced, a new continuum can be defined and the other standard form derived.

Two new types of continuum are defined in Section 3. A hypothetical continuum with new positions will be termed the *Lagrangian mean continuum* (LMC), corresponding to the GLM description, while a hypothetical continuum with new labels will be termed the *Eulerian mean continuum* (EMC), corresponding to the TEM description. In both new continuum equations, the dependent variables are transformed not only in the EMC but also in the LMC. It should be noted that the definitions of the new dependent variables are essential; a point which was not stressed in previous studies about the GLM and TEM theories.

In Section 4, the LMC dynamics of compressible atmosphere on the rotating earth are investigated using cartesian geometry. The specific effects of initial mass and entropy disturbances are discussed to clarify the role of each in the non-acceleration problem. Such a separation of these influences has been not presented in previous GLM theory.

Section 5 correspondingly investigates the EMC dynamics of compressible atmosphere on the rotat-

ing earth. The TEM description presented here differs slightly from that of AMc. The various roles of the mean state balance and spatial mean operations, being common postulates in many TEM descriptions, are investigated. It is subsequently shown that only when these postulates are accepted, is the presented TEM theory equivalent to that of AMc. Additionally, the effect of each wave-packet approximation or wave transience/dissipation in the non-acceleration problems is identified. In contrast with the case of AMc, no Lagrangian information such as particle displacement is used in proving relevant theorems. Concluding remarks are presented in Section 6.

2. Fundamental structure of the continuum equations

2.1 Invertible mapping between label and position

A general continuum containing material in non-relativistic time and space is assumed in which an infinitesimal element of continuous material is called a particle. The possibility of reciprocal translation between Eulerian and Lagrangian information in a continuum is guaranteed by the invertibility between the particle's label and position described as

$$\mathbf{x} = \mathbf{x}(t, \mathbf{a}) \quad \text{and} \quad \mathbf{a} = \mathbf{a}(t, \mathbf{x}), \quad (2-1a)$$

where t , $\mathbf{a}$, and $\mathbf{x}$ are respectively time, particle label, and particle position. Through this invertible mapping, arbitrary field variables in the time and position space (hereafter simply called position space) can be translated into field variables in the time and label space (hereafter called label space), and also transformed vice versa, *i.e.*,

$$\phi(t, \mathbf{x}) = \phi(t, \mathbf{x}(t, \mathbf{a})) = \phi^*(t, \mathbf{a}), \quad (2-1b)$$

and

$$\psi(t, \mathbf{a}) = \psi(t, \mathbf{a}(t, \mathbf{x})) = \psi^*(t, \mathbf{x}), \quad (2-1c)$$

where the asterisks in (2-1b, 2-1c) indicate that the functional form differs in the position and label space; being omitted hereafter for notation simplicity. These descriptions are referred to as Eulerian in the position space and Lagrangian in the label space.

The differential form of (2-1a) is

$$dx_\mu = da_\nu \partial x_\mu / \partial a_\nu \quad (\mu, \nu = 0, 1, 2, 3), \quad (2-1c)$$

and

$$da_\nu = dx_\mu \partial a_\nu / \partial x_\mu \quad (\mu, \nu = 0, 1, 2, 3), \quad (2-1d)$$

where four-dimensional summation convection is employed, and $\partial x_0 / \partial a_\nu = \delta_{\nu 0}$, $\partial a_0 / \partial x_\mu = \delta_{\mu 0}$, since $x_0 = a_0 = t$ from non-relativistic geometry considerations. Here, Lagrangian (substantial) temporal differentiation is denoted as $\partial / \partial a_0$ and Eulerian (local) temporal differentiation as $\partial / \partial x_0$. From (2-1c, 2-1d),

$$J_a^x \neq 0, \quad J_x^a \neq 0 \quad \text{and} \quad J_a^x J_x^a = 1. \quad (2-1e)$$

where $J_a^x (\equiv \partial(x_1, x_2, x_3) / \partial(a_1, a_2, a_3))$ and $J_x^a (\equiv \partial(a_1, a_2, a_3) / \partial(x_1, x_2, x_3))$ are the respectively specific volume and its reciprocal.

From only the non-relativistic invertibility between label and position, the following key relation is derived. (see P. Chadwick (1976)) as

$$J_a^x \partial (J_a^x) / \partial a_0 = \text{div} (\partial \mathbf{x} / \partial a_0). \quad (2-1f)$$

2.2 Standard forms in label and position space

In many wave, mean-flow interaction studies, the initial continuum equations are conventionally written in the material form, *i.e.*,

$$d/dt(\rho) + \rho \text{div}(\mathbf{v}) = 0, \quad (2-2a)$$

$$d/dt(\mathbf{v}) = \mathbf{f} / \rho, \quad (2-2b)$$

$$d/dt(s) = q / \rho, \quad (2-2c)$$

where all physical quantities are field variables, $\rho = \rho(t, \mathbf{x})$ is the mass density, $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ the velocity vector, $s = s(t, \mathbf{x})$ the entropy per unit mass, and $\mathbf{f}(t, \mathbf{x}) / \rho$ and $q(t, \mathbf{x}) / \rho$ are the forcing vector and entropy source per unit mass, respectively. Note that the set of (2-2a, 2-2b, 2-2c) is not a closed system since it contains the Lagrangian differentiation ($d/dt \equiv \partial / \partial a_0$) of field variables. In order to make it "closed" such that ρ , $\mathbf{v}$, and s can be solved for as field variables in position space when the RHS of (2-2b, 2-2c) are given, a mechanical relation is needed, *i.e.*,

$$d\mathbf{x}/dt = \mathbf{v}. \quad (2-2d)$$

Until d/dt is expanded to $\partial / \partial t + \mathbf{v} \cdot \nabla$, the set of equations (2-2a, 2-2b, 2-2c) can not be mathematically closed.

On the other hand, no information about the label space can be obtained from (2-2a, 2-2b, 2-2c) alone; and the mechanical relation (2-2d) is needed to translate from position into label space. By backward integrating (2-2d) over time, the label of a particle occupying position $\mathbf{x}$ at the present time can be identified because the label was defined in terms of the initial particle position. However, the material form of equations (2-2a, 2-2b, 2-2c) is not necessarily appropriate for constructing a unified formalism of the TEM and GLM theories due to the poor translation symmetry between Eulerian and Lagrangian information.

2.2.1 Standard form in label space

Using (2-1f), 2-2a, 2-b, 2-c, 2-d) are arranged such that

$$\underline{m}_{\mu,a} = \underline{f}_{\mu} \quad (\mu = 0, 1, 2, 3, 4) \quad (2-3a)$$

and

$$x_{i,a} = \underline{m}_i / \underline{m}_0, \quad (i = 1, 2, 3) \quad (2-3b)$$

where $\underline{m}_0(t, \mathbf{a}) \equiv J_a^x \rho$, $\underline{m}_i(t, \mathbf{a}) \equiv \underline{m}_0 v_i$, $\underline{m}_4(t, \mathbf{a}) \equiv \underline{m}_0 s$, $\underline{f}_0(t, \mathbf{a}) \equiv 0$, $\underline{f}_i(t, \mathbf{a}) \equiv \underline{m}_0 f_i / \rho = J_a^x f_i$, $\underline{f}_4(t, \mathbf{a}) \equiv \underline{m}_0 q / \rho = J_a^x q$ and Lagrangian differentiation ($d/dt \equiv \partial/\partial a_0$) is denoted as $(\cdot)_a$. Each component of (2-3a) has been obtained by multiplying J_a^x by (2-2a), $\underline{m}_0$ by (2-2b), and $\underline{m}_0$ by (2-2c), respectively. The dependent variables are $\underline{m}_\mu$ and $\underline{f}_\mu$, being mass, momentum, entropy densities, and their sources in the label space, respectively, with time and the label being independent variables. Hereafter, Greek indices indicate the values 0, 1, 2, 3, 4 and Latin indices those of 1, 2, 3.

The formalism of (2-3a, 2-3b) is consistent with the general Lagrangian point of view, since all of the dependent variables are densities in the *label space*. Furthermore, (2-3a) is more simple than (2-2a, 2-2b, 2-2c) because the latter set is a closed *linear* system for given source terms. Information in the label space can be obtained from the solution of (2-3a) and can be translated into the position space using the mechanical relation of (2-3b). The equations (2-3a, 2-3b) will be referred to as the *standard* form of continuum equations in the label space because it is a *complete set* containing both Lagrangian and Eulerian information.

2.2.2 Standard form in position space

Using $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$, another arrangement of (2-2a, 2-2b, 2-2c, 2-2d) is also possible, *i.e.*,

$$m_{\mu,t} + (m_\mu m_j / m_0)_{,j} = f_\mu, \quad (2-4a)$$

$$x_{i,a} = m_i / m_0, \quad (2-4b)$$

where $m_0(t, \mathbf{x}) \equiv \rho$, $m_i(t, \mathbf{x}) \equiv \rho v_i$, $m_4(t, \mathbf{x}) \equiv \rho s$, $f_0(t, \mathbf{x}) \equiv 0$, $f_4(t, \mathbf{x}) \equiv q$, and the Eulerian temporal and j -th spatial partial differentiations are denoted as $_{,t}$ and $_{,j}$, respectively. The i -th or fourth component of (2-4a) has been obtained by multiplying m_0 and (2-2b) or (2-2c), respectively. The variables m_μ and f_μ , representing mass, momentum, entropy densities and their sources in the position space, are dependent while time and position are independent variables.

The system of (2-4a, 2-4b) holds to the original Eulerian description because all the dependent variables are densities in *position space* and the set is a closed *non-linear* system for the given source terms. Information in the position space can be obtained from the solution of (2-4a), and translated into the label space using (2-4b). Since the set of equations (2-4a, 2-4b) is also a *complete set* containing both Lagrangian and Eulerian information, it is called the *standard* form of continuum equations in the position space.

2.3 Pair structure of the standard forms

The two defined standard forms have an exceptionally simple relationship with each other: the ratio of the two sets of dependent variables is the specific volume J_a^x , *i.e.*,

$$J_a^x = \underline{m}_\mu(t, \mathbf{a}) / m_\mu(t, \mathbf{x}) = \underline{f}_\mu(t, \mathbf{a}) / f_\mu(t, \mathbf{x}), \quad (2-5a)$$

where $\underline{m}_\mu$, m_μ , $\underline{f}_\mu$ and f_μ have been previously defined in (2-3a, 2-3b)–(2-4a, 2-4b), respectively. Additionally, the specific volume J_a^x can also be obtained from the ratio of the two conservation equations:

$$J_a^x = (\text{Eq. (2-3a)}) / (\text{Eq. (2-4a)}). \quad (2-5b)$$

To obtain the relationship in (2-5b), a formula for exchanging independent variables ((A.6a), Appendix A) has been used, which was derived using just the postulate of differentiability and invertibility between two independent variables types, with non-relativity being the only physical constraint. Furthermore, (2-3b) and (2-4b) are obviously equivalent. This *pair structure* consisting of the standard forms in label and position space is a fundamental continuum equation structure.

Based on the *similarity principle*, if a new set of differential equations which has either kind of standard form is introduced, a new continuum, being a *hypothetical* one, can be assumed because the standard form equation set is complete for a continuum. In addition, the other standard form can be obtained by using the formula (A.6a), and both standard forms can be arranged to create the material form whenever mass conservation holds.

3. Two kinds of hypothetical continua

Two kinds of hypothetical continua can be defined corresponding to the GLM and TEM descriptions. Their formulations are straightforward, with the procedures being reciprocals of each other.

3.1 Hypothetical continuum with new position

First, a new continuum containing particles with original labels and new positions is formulated. The new dependent variables (capital letters) in the label space are defined as

$$\underline{M}_\mu \equiv \{\underline{m}_\mu\} + \underline{M}_\mu^{add}, \quad (3-1a)$$

$$\underline{F}_\mu \equiv \{\underline{f}_\mu\} + \underline{M}_\mu^{add},_a \quad \text{and} \quad (3-1b)$$

$$X_i \equiv a_i + \int_0^t \underline{M}_i / \underline{M}_0 dt, \quad (3-1c)$$

where the $\underline{M}_\mu^{add}$ are accumulated additional arbitrary terms, $\{\psi\}$ denotes the Lagrangian mean of an arbitrary function ψ , and the label “a” remains fixed in the integral of (3-1c). The Lagrangian mean

definition and some of its properties are presented in Appendix B. From the original equation (2-3a), the new dependent variables (3-1a, 3-1b, 3-1c) obviously satisfy the following standard form equations in the label space:

$$\underline{M}_{\mu,a} = \underline{F}_\mu \quad (3-2a)$$

and

$$X_{i,a} = \underline{M}_i / \underline{M}_0 \quad (3-2b)$$

where (3-2a) is also obtained by applying the procedure that after taking the Lagrangian mean of (2-3a), $\underline{M}_\mu^{add}$, is added to both sides of it. This new hypothetical continuum is called the *Lagrangian mean continuum* (LMC).

If $\underline{M}_\mu^{add}$ is defined as

$$\underline{M}_\mu^{add} \equiv -\{\underline{m}_0^l v_\mu^l\}, \quad (3-3a)$$

then $\underline{M}_\mu$, $\underline{F}_\mu$ and X_i can all be uniquely determined¹, i.e.,

$$\begin{aligned} \underline{M}_\mu &= \{\underline{m}_0\} \{v_\mu\}, \quad \underline{F}_\mu = \{\underline{m}_0\} \{f_\mu / m_0\} \\ &\text{and } X_i = \{x_i\}, \end{aligned} \quad (3-3b)$$

where $v_\mu \equiv \underline{m}_\mu / m_0$ and $\psi^l \equiv \psi - \{\psi\}$. Thus, the LMC is completely defined. It is found that the new particle with the label “a” occupies the Lagrangian mean position $\{\mathbf{x}\}$ and moves with the Lagrangian mean velocity $\underline{\mathbf{M}} / \underline{M}_0 = \{\mathbf{v}\}$. In addition, mass conservation clearly holds ($\underline{F}_0 = 0$) in the LMC because $f_0 = 0$ in (3-3b).

As far as the mapping between “a” and $\mathbf{X}$ is invertible, the following standard form in the new position space also holds by using (A.6a) of Appendix A:

$$M_{\mu,T} + (M_\mu M_J / M_0)_{,J} = F_\mu, \quad (3-4a)$$

$$X_{I,a} = M_I / M_0, \quad (3-4b)$$

where $J_X^a \equiv \partial(a_1, a_2, a_3) / \partial(X_1, X_2, X_3)$, and $(M_\mu, F_\mu) \equiv J_X^a (M_\mu, F_\mu)$, and the notations $_{,T}$ and $_{,J}$ indicate temporal and spatial partial differentiations in the new position space $(T, \mathbf{X})$.

By mass conservation ($\underline{M}_{0,a} = 0$), the material form equations are obtained by dividing (3-2a) by $\underline{M}_0$:

$$V_{\mu,a} = \underline{F}_\mu / \underline{M}_0, \quad (3-5a)$$

¹ $\underline{M}_\mu = \{\underline{m}_0 v_\mu\} - \{\underline{m}_0^l v_\mu^l\} = \{\underline{m}_0\} \{v_\mu\}$. $\underline{F}_\mu = \{\underline{f}_\mu\} - \{\underline{m}_0^l v_\mu^l\}_{,a} = \{\underline{f}_\mu\} - \{\underline{m}_0^l (v_\mu^l)_{,a}\} = \{\underline{f}_\mu\} - \{\underline{m}_0^l (v_{\mu,a})^l\} = \{\underline{m}_0 f_\mu / m_0\} - \{\underline{m}_0^l (f_\mu / m_0)^l\} = \{\underline{m}_0\} \{f_\mu / m_0\} = \{\underline{m}_0\} \{f_\mu / m_0\}$, where $\underline{m}_{0,a}^l = (\underline{m}_{0,a})^l = 0$ and $(v_\mu^l)_{,a} = (v_{\mu,a})^l = (\underline{f}_\mu / \underline{m}_0)^l$. Also $X_i = a_i + \int_0^t \{v_i\} dt = a_i + \int_0^t \{\underline{m}_i / \underline{m}_0\} dt = \{a_i + \int_0^t \underline{m}_i / \underline{m}_0 dt\} = \{x_i\}$, where the first relation of (3-3b) and (2-3b) have been used.

where $V_\mu \equiv \underline{M}_\mu / \underline{M}_0$, and (3-5a) is rewritten as $\{v_{\mu,a}\} = \{f_\mu / m_0\}$ which agrees with the Lagrangian mean equation of the original material form motion and entropy equations (2-2b, 2-2c). This results from the specific definition of $\underline{M}_\mu^{add}$, in that if another definition is chosen, such an agreement does not occur.

It should be noted that the postulate of initial nondisturbances, which is a physical restriction, has not been used in this formulation. The rest of the present study is devoted to investigating the properties of the new governing equations.

3.2 Hypothetical continuum with new label

In formulating a new continuum containing particles with original positions and new labels, the new dependent variables (capital letters) in the position space are defined as

$$M_\mu \equiv \langle m_\mu \rangle + M_\mu^{add}, \quad (3-6a)$$

$$F_\mu \equiv \langle f_\mu \rangle + F_\mu^{add}, \quad (3-6b)$$

$$A_i \equiv x_i + \int_t^0 (M_i / M_0) dt, \quad (3-6c)$$

where $F_\mu^{add} \equiv M_\mu^{add}{}_{,t} + (M_\mu M_J / M_0)_{,j} - \langle m_\mu m_j / m_0 \rangle_{,j}$, and $\underline{M}_\mu^{add}$ are accumulated additional arbitrary terms, $\langle \psi \rangle$ denotes the Eulerian mean of an arbitrary function ψ , and label $\mathbf{A}$ remains fixed in the integration of (3-6c). The Eulerian mean definition and properties are shown in Appendix B. The backward time integration defines the label $\mathbf{A}(t, \mathbf{x})$ of the new hypothetical particle existing at position $\mathbf{x}$ at the present time. It is important to realize that the dependent variables (M_μ, F_μ) in (3-6a, 3-6b) are different from those in (3-4a, 3-4b), even though the symbolic representation is the same. From the original equation (2-4a), the dependent variables (3-6a, 3-6b, 3-6c) obviously satisfy the following standard form equations in the position space:

$$M_{\mu,t} + (M_\mu M_J / M_0)_{,j} = F_\mu, \quad (3-7a)$$

$$x_{i,A} = M_i / M_0, \quad (3-7b)$$

where (3-7a) is also obtained by applying the procedure that after taking the Eulerian mean of (2-4a), F_μ^{add} is added to both sides of it. This new hypothetical continuum is called the *Eulerian mean continuum* (EMC).

By using an antisymmetrical tensor $C_{ik} \equiv \langle m_0 \rangle (\langle s \rangle_{,i} \langle s' v'_k \rangle - \langle s \rangle_{,k} \langle s' v'_i \rangle) / |\nabla \langle s \rangle|^2$, if M_μ^{add} is defined as

$$M_0^{add} \equiv 0, \quad M_i^{add} \equiv C_{ik,k}, \quad M_4^{add} \equiv 0, \quad (3-8a)$$

the M_μ , F_μ and A_i can all be uniquely determined², i.e.,

$$M_0 = \langle m_0 \rangle, \quad M_i = (\langle m_i \rangle + C_{ik,k}), \\ M_4 = \langle m_4 \rangle, \quad (3-8b)$$

$$F_0 = \langle f_0 \rangle = 0, \quad (3-8c)$$

$$F_i = \langle f_i \rangle + C_{ik,kt} + (C_{ik,k} M_j / M_0)_{,j} + W_{ij,j}, \quad (3-8d)$$

$$F_4 = \langle f_4 \rangle + W_{4j,j}, \quad (3-8e)$$

and

$$A_i \equiv x_i + \int_t^0 ((\langle m_i \rangle + C_{ik,k}) / \langle m_0 \rangle) dt, \quad (3-8f)$$

where $W_{\mu j} \equiv C_{jk,k} \langle m_\mu \rangle / \langle m_0 \rangle + \langle m_\mu \rangle \langle m_j \rangle / \langle m_0 \rangle - \langle m_\mu m_j / m_0 \rangle$, $v_i \equiv m_i / m_0$, $s \equiv m_4 / m_0$, and $\psi' \equiv \psi - \langle \psi \rangle$. Thus, the EMC is completely defined. The new particle at position $\mathbf{x}$ has the label $\mathbf{A}$ and moves with velocity M_i / M_0 . Mass conservation also holds ($F_0 = 0$) in the EMC because $f_0 = 0$ in (3-8c).

As far as the mapping between $\mathbf{A}$ and $\mathbf{x}$ is invertible, the following standard form in the new label space also holds by using the formula (A.6a) of Appendix A:

$$\underline{M}_{\mu,A} = \underline{F}_\mu, \quad (3-9a)$$

$$x_{i,A} = \underline{M}_i / \underline{M}_0, \quad (3-9b)$$

where $J_A^x \equiv \partial(x_1, x_2, x_3) / \partial(A_1, A_2, A_3)$, $(\underline{M}_\mu, \underline{F}_\mu) \equiv J_A^x(M_\mu, F_\mu)$, and the notation “ $_{,A}$ ” ($\equiv \partial / \partial t + \mathbf{V} \cdot \nabla$, $\mathbf{V} \equiv \mathbf{M} / M_0$) denotes the new Lagrangian differentiation in the new label space $(T, \mathbf{A})$. Again, note that the variables $(\underline{M}_\mu, \underline{F}_\mu)$ in (3-9a, 3-9b) are different from those in (3-2a, 3-2b).

Because mass conservation holds, dividing (3-9a) by $\underline{M}_0$, the material form equations are obtained as

$$V_{i,A} = \underline{F}_i / \underline{M}_0 = F_i / M_0 = \langle f_i \rangle / \langle m_0 \rangle \\ + (C_{ik,k} / \langle m_0 \rangle)_{,A} + W_{ij,j} / \langle m_0 \rangle, \quad (3-10a)$$

and

$$V_{4,A} = \underline{F}_4 / \underline{M}_0 = F_4 / M_0 \\ = \langle f_4 \rangle / \langle m_0 \rangle + W_{4j,j} / \langle m_0 \rangle, \quad (3-10b)$$

² $F_0 = \langle f_0 \rangle + F_0^{add} = M_{j,j} - \langle m_j \rangle_{,j} = C_{jk,kj} = 1/2 (C_{jk} + C_{kj})_{,kj} = 0$, where the antisymmetry of C_{ij} has been used. $F_i = \langle f_i \rangle + M_i^{add,t} + (M_i M_j / M_0)_{,j} - \langle m_i m_j / m_0 \rangle_{,j} = \langle f_i \rangle + M_i^{add,t} + (M_i^{add} M_j / M_0)_{,j} + (\langle m_i \rangle M_j / \langle m_0 \rangle - \langle m_i m_j / m_0 \rangle)_{,j} = \langle f_i \rangle + C_{ik,kt} + (C_{ik,k} M_j / M_0)_{,j} + (C_{jk,k} \langle m_i \rangle / \langle m_0 \rangle + \langle m_i \rangle \langle m_j \rangle / \langle m_0 \rangle - \langle m_i m_j / m_0 \rangle)_{,j}$. F_4 is same as in F_i , except for $M_4^{add} = 0$.

where $V_i \equiv M_i / M_0$. It is found that F_i / M_0 includes not only the wave stress term $W_{ij,j} / \langle m_0 \rangle$, but also the new Lagrangian transient term $(C_{ik,k} / \langle m_0 \rangle)_{,A}$. The spatial symmetry postulate such as zonally-symmetrical mean flow was not used in deriving these formulations. The remaining subject is the investigation of properties of the new governing equations.

4. Lagrangian mean continuum dynamics

For fluid in a constant rotating system, the general mass momentum and entropy source terms are

$$f_0 \equiv 0, \quad f_i \equiv -(2\boldsymbol{\Omega} \times \mathbf{m})_i + \tau_{ij,j} - m_0 \Psi_{,i} \\ \text{and} \quad f_4 \equiv q, \quad (4-1)$$

where τ_{ij} is the stress tensor, $\boldsymbol{\Omega}$ the angular velocity of system rotation, $\Psi(t, \mathbf{x})$ the potential per unit mass for arbitrary potential forces, and q the entropy source density.

4.1 Effects of initial nondisturbance on mass

After substituting (4-1) into the RHS of (3-4a), and applying (A-6b) in Appendix A,

$$F_i = J_X^a \underline{F}_i = J_X^a \underline{M}_0 \{f_i / m_0\} = -(2\boldsymbol{\Omega} \times \mathbf{M})_i \\ + \{\underline{m}_0\} \{T_{ij,j} / \underline{m}_0\} - M_0 \{\Psi_{,i}\}, \quad (4-2)$$

where $\xi_j \equiv x_j - X_j$, and $T_{ij} \equiv J_X^x \tau_{ik} (\delta_{jk} - \xi_{n,K} X_{j,n})$ is the new asymmetrical stress tensor. In derivation of (4-2), $\tau_{ij,j} / m_0 = J_X^x \tau_{ik,k} / (J_X^a \underline{m}_0) = (J_X^x \tau_{ik} X_{j,k})_{,j} / (J_X^a \underline{m}_0) = (J_X^x \tau_{ik} (\delta_{jk} - \xi_{j,k}))_{,j} / (J_X^a \underline{m}_0)$ and $\xi_{j,k} = \xi_{j,k} - \xi_{j,K} + \xi_{j,K} = \delta_{nk} \xi_{j,n} - x_{n,K} \xi_{j,n} + \xi_{j,K} = -\xi_{n,K} \xi_{j,n} + \xi_{n,K} \delta_{jn} = \xi_{n,K} X_{j,n}$ have been used.

According to the postulate of initial nondisturbances on only mass density in the label space, it is found that the M_0 , $\mathbf{M}$ and $\{T_{ij,j}\}$ in (4-2) correspond to the m_0 , $\mathbf{m}$ and $\tau_{ij,j}$ in (4-1). This occurs because from $\underline{m}_{0,a} = 0$, if $\underline{m}_0^l = 0$ at the initial time, then $\underline{m}_0^l = 0$ for any time, i.e., $\underline{m}_0 = \{m_0\}$. Thus, the necessary condition for this similarity is more relaxed than that of AMa, since it is not required to postulate the initial nondisturbance on any other quantities.

4.2 Effects of initial nondisturbance on entropy

The RHS of (3-5a) is arranged as

$$\underline{F}_i / \underline{M}_0 = \{f_i / m_0\} = \{X_{j,I} f_j / m_0\} \\ = \{x_{j,I} f_j / m_0\} - \{\xi_{j,I} f_j / m_0\} \\ = \{x_{j,I} \tau_{jk,k} / m_0\} - \{x_{j,I} \Psi_{,j}\} \\ - \{x_{j,I} (2\boldsymbol{\Omega} \times \mathbf{v})_j\} - \{\xi_{j,I} v_{j,a}\}, \quad (4-3a)$$

where $\boldsymbol{\xi} (\equiv \mathbf{x} - \mathbf{X})$ is a vector describing particle displacement from the Lagrangian mean position $\mathbf{X}$, and (4-1) has been substituted into the f_j in the first term of the second line of (4-3a), while $m_0 v_{j,a}$ has been substituted into the f_j in the corresponding second term.

Each term in the bottom two lines of (4-3a) is expanded in Appendix C. When motion is inviscid (τ_{ij} is diagonal) and all of the mean quantities are independent of the direction X_i , (4-3a) can be reduced as

$$\underline{E}_i/\underline{M}_0 = -(2\Omega \times V)_i - P_{i,a} + \{T_e^l s^l, I\}, \quad (4-3b)$$

where $P_i \equiv \{\xi_{j,I}(v_j^l + (\Omega \times \xi)_j)\}$ and T_e is temperature.

According to the postulate of initial nondisturbance on only entropy per unit mass, it is found that steady waves do not affect the LMC flow. This is true because from $s^l, a = 0$, if $s^l = 0$ at the initial time, then $s^l = 0$ for any time. Consequently, the necessary condition for nonacceleration is more relaxed than that of AMa, since no postulates for initial nondisturbance on any other quantities are required.

4.3 Effects of initial nondisturbance on mass and entropy

For adiabatic inviscid motion, from (4-2) and (4-3b),

$$\begin{aligned} M_0 P_{i,a} + \{m_0\} \{B_{ij,J}/m_0\} - M_0 \{T_e^l s^l, I\} \\ - \{m_0\} (J_X^x p)_{,I}/m_0 - M_0 \{\Psi, i\}, \end{aligned} \quad (4-4a)$$

where $B_{ij} \equiv J_X^x p \xi_{n,I} X_{j,n}$ and p is pressure. Also $\tau_{ik} = -p\delta_{ik}$ have been used. If the two postulates of initial nondisturbance on both mass and entropy are applied, (4-4a) is reduced to

$$(M_0 P_i)_{,T} + (M_0 P_i V_j + \{B_{ij}\})_{,J} = 0, \quad (4-4b)$$

where $M_0 P_{i,a} = (M_0 P_i)_{,T} + (M_0 P_i V_j)_{,J}$ and $\{J_X^x p\}_{,I} = \Psi_{,i} = 0$ have been used. Equation (4-4b) describes the pseudo-momentum conservation. The necessary conditions for this conservation are not differ from those in the case of AMa.

4.4 Comparison with the GLM theory of AMa and AMb

If the postulates of initial nondisturbance on both mass and entropy are accepted, the presented LMC dynamics theory is completely equivalent to the GLM theory of AMa and AMb. For example, the zeroth component of (3-4a), fourth of (3-5a), and (4-4b) are respectively equivalent to Eq. (4.2), (2.21) of AMa, and (2.15) of AMb, because the presented notation $M_0, \rho_a, V_{j,J}, \{f_4/m_0\}, P_i$, and $\{B_{ij}\}$ correspond to their $\bar{\rho}, \bar{D}^L, \nabla \cdot \bar{u}^L, -\bar{Q}^L, A$, and B_j , respectively. Additionally, (4-2) and (4-3b) correspond to Eq. (8.7a) and (3.8) of AMa because the presented notation $V_i, \{T_{ij}\}$ and $\{\Psi, i\}$ correspond to their $\bar{u}_i^L, R_{ij} - (\bar{p}^L)\delta_{ij}$, and $((\bar{\Phi}, i)^S + \bar{\Phi}, i)$, respectively.

However, LMC dynamics can be formulated without such postulates. A more important consideration is that the LMC is an example of a hypothetical continuum with similarity to the original continuum.

5. Eulerian mean continuum dynamics

For fluid on a β plane of the rotating earth, the mass, momentum, and entropy source terms are

$$\begin{aligned} f_0 \equiv 0, \quad f_i \equiv -p_{,i} - (2\Omega \times m)_i + m_0 g_i + d_i \\ \text{and} \quad f_4 \equiv q, \end{aligned} \quad (5-1)$$

where x_1 is the zonal direction, $\Omega (\equiv (0, 0, \Omega_3))$ a vector with the vertical projection of the earth's angular velocity, $g (\equiv (0, 0, -g))$ the gravity acceleration vector, and d_i and q are the dissipation terms.

5.1 Effects of the mean state balance

It is natural to assume that all transient parts of the mean state quantities are small (*i.e.*, $\langle m_0 \rangle_A \sim \langle v_\mu \rangle_A \sim O(a)$, where “ a ” is wave amplitude). This assumption implies that $\langle v_j \rangle_j \sim \langle f_\mu \rangle / \langle m_0 \rangle \leq O(a)$, from (2-2a, 2-2b, 2-2c).

With this assumption, the mean state transient parts are removed from the wave action equation (*i.e.*, the fifth and sixth lines of (D-6) in Appendix D vanishes because $\langle \rangle_A \sim Dt \langle \rangle$). On the other hand, for fluid on the rotating earth, the assumption implies that the zonal mean flow is *uni-directional* (*i.e.*, $\langle v_2 \rangle \sim \langle v_3 \rangle \leq O(a)$, because $\langle f_\mu \rangle / \langle m_0 \rangle \sim 2\Omega_3 \langle v_2 \rangle \leq O(a)$ and $\langle v_j \rangle_j \sim \langle v_2 \rangle_{,2} + \langle v_3 \rangle_{,3} \leq O(a)$, where it has been assumed that mean dissipation terms are less than $O(a)$).

5.2 Effects of the spatial mean operation

If the mean parameter is zonal direction x_1 , all of mean quantities are independent of x_1 . By this homogeneity, $C_{ik,k}$ is reduced to

$$C_{ik,k} = (C_{12,2} + C_{13,3}, C_{23,3}, C_{32,2}). \quad (5-2a)$$

From the antisymmetry of C_{ij} ($C_{32} = -C_{23}$), it is found that C_{23} becomes a stream function of the meridional additional term. Also C_{23} and C_{32} can be reduced to

$$\begin{aligned} C_{ij} = -\langle m_0 \rangle \langle s' v'_i \rangle / \langle s \rangle_{,j} \\ + \kappa \langle s \rangle_{,i} / \langle s \rangle_{,j}, \quad (i, j = 2, 3) \end{aligned} \quad (5-2b)$$

where $\kappa \equiv \langle m_0 \rangle \langle s' v' \rangle \cdot \nabla \langle s \rangle / |\nabla \langle s \rangle|^2$. In derivation of (5-2b), $C_{23} = \langle m_0 \rangle \langle s \rangle_{,2} \langle s \rangle_{,3} \langle s' v'_3 \rangle - \langle s \rangle_{,3} \langle s \rangle_{,2} \langle s' v'_2 \rangle / (\langle s \rangle_{,3} |\nabla \langle s \rangle|^2) = \langle m_0 \rangle (-|\nabla \langle s \rangle|^2 \langle s' v'_2 \rangle + \langle s \rangle_{,2} \langle s \rangle_{,3} \langle s' v'_3 \rangle + \langle s \rangle_{,2} \langle s \rangle_{,2} \langle s' v'_2 \rangle) / (\langle s \rangle_{,3} |\nabla \langle s \rangle|^2)$ has been used. And C_{32} is similar except for exchanging 2 and 3. Furthermore $W_{\mu j,j}$ is also reduced to

$$\begin{aligned} W_{\mu j,j} = (C_{jk,k} \langle v_\mu \rangle - \langle m_0 \rangle \langle v'_\mu v'_j \rangle) + O(a^3) \\ = (-C_{23} \langle v_\mu \rangle_{,3} - \langle m_0 \rangle \langle v'_\mu v'_2 \rangle)_{,2} \\ + (-C_{32} \langle v_\mu \rangle_{,2} - \langle m_0 \rangle \langle v'_\mu v'_3 \rangle)_{,3} + O(a^3) \\ = (D_{23}^\mu - \kappa \langle v_\mu \rangle_{,3} \langle s \rangle_{,2} / \langle s \rangle_{,3})_{,2} \\ + (D_{32}^\mu - \kappa \langle v_\mu \rangle_{,2} \langle s \rangle_{,3} / \langle s \rangle_{,2})_{,3} + O(a^3), \end{aligned} \quad (5-2c)$$

where $D_{ij}^\mu = \langle m_0 \rangle (\langle s'v_i' \rangle \langle v_\mu \rangle_{,j} / \langle s \rangle_{,j} - \langle v_\mu' \rangle v_i' \rangle$, and (5-2b), $C_{23} + C_{32} = 0$, and $\langle m_\mu \rangle \langle m_j \rangle / \langle m_0 \rangle - \langle m_\mu m_j / m_0 \rangle = -\langle m_0 \rangle \langle v_\mu' v_j' \rangle + O(a^3)$ have been used. Thus $W_{4j,j}$ is now a diffusion type dissipation term, *i.e.*,

$$W_{4j,j} = -\text{div}(\kappa \nabla \langle s \rangle), \quad (5-2d)$$

where $D_{ij}^4 = 0$ has been used.

5.3 Non-acceleration theorem for an acoustic wave packet

For a nondissipative acoustic wave packet, Ω , g_i , d_i and q of (5-1) vanish. From the mean state balance, $\langle p \rangle_{,i} \leq O(a)$. If a three-dimensional wave packet is assumed, then using (E-3) in Appendix E, $W_{\mu j,j}$ is reduced to

$$W_{ij,j} = -k_i (\langle p'v_j' \rangle / \underline{\omega})_{,j} + O(a^3) \\ \text{and } W_{4j,j} = O(a^3), \quad (5-3a)$$

where $\underline{\omega} \equiv \omega - \mathbf{k} \cdot \langle \mathbf{w} \rangle$, ω is wave frequency and $\mathbf{k}$ is wave number. After averaging (D-6) in Appendix D, and substituting (E-3) into it, the conservation of wave action for acoustic waves is obtained as

$$\langle m_0 \rangle (\langle E \rangle / \underline{\omega})_{,A} + \text{div}(\langle p'v' \rangle / \underline{\omega}) = O(a^3), \quad (5-3b)$$

where $\langle E \rangle \equiv \langle |v'|^2 \rangle / 2 + C_v \langle p'^2 \rangle / (2C_p \langle p \rangle \langle m_0 \rangle)$, $\langle E \rangle$ is the sum of kinetic and compressible wave energy per unit mass, and C_v and C_p are respectively specific heats for constant volume and pressure.

Substituting (5-3a, 5-3b) into (3-10a, 3-10b), the EMC material form equations are reduced to

$$V_{i,A} = -\langle p \rangle_{,i} / \langle m_0 \rangle + \chi_{i,A} \\ \text{and } V_{4,A} = O(a^3), \quad (5-4)$$

where $\chi_i \equiv C_{ik,k} / \langle m_0 \rangle + k_i \langle E \rangle / \underline{\omega}$. Thus, the following three-dimensional non-acceleration theorem holds: In a steady nondissipative acoustic wave packet, all wave effects vanish.

5.4 Non-acceleration theorem for a Rossby-gravity-acoustic wave packet

For a nondissipative Rossby-Gravity-Acoustic wave packet, d_i and q of (5-1) vanish. From the assumption of mean state balance with uni-directional mean flow, the thermal wind balance is as follows:

$$2\Omega_3 \langle v_1 \rangle_{,3} = [\langle p \rangle_{,2} \langle m_0 \rangle_{,3} \\ - \langle p \rangle_{,3} \langle m_0 \rangle_{,2}] / \langle m_0 \rangle^2. \quad (5-5)$$

Consider a sinusoidal wave packet in the x_1 direction, which may have an arbitrary structure in the x_2 and x_3 directions. Using (E-5a, b, c) of Appendix E, $W_{1j,j}$ is reduced to

$$W_{1j,j} = -k_1 (\langle p'v_j' \rangle / \underline{\omega})_{,j} \\ + (2\Omega_3 \langle m_0 \rangle \langle s'v_2' \rangle / \langle s \rangle_{,2})_{,3} + O(a^3), \quad (5-6a)$$

where $\underline{\omega} = \omega - k_1 \langle v_1 \rangle$. After averaging (D-6) of Appendix D, and substituting (E-5a, b, c) into it, the wave action conservation for Rossby-Gravity-Acoustic waves is obtained ³ as

$$\langle m_0 \rangle (\langle E \rangle / \underline{\omega})_{,A} + (\langle p'v_j' \rangle / \underline{\omega})_{,j} = O(a^3). \quad (5-6b)$$

Using (5-6a, 5-6b), the EMC material form equations are reduced ⁴ to

$$V_{1,A} = -2(\Omega \times \mathbf{V})_1 + \chi_{1,A} \quad \text{and} \quad V_{4,A} = O(a^3), \quad (5-7)$$

where $\chi_1 = C_{1k,k} / \langle m_0 \rangle + k_1 \langle E \rangle / \underline{\omega}$. Hence, the following one-dimensional non-acceleration theorem holds: In a steady nondissipative Rossby-gravity-acoustic wave packet, the wave effects vanish in the x_1 direction.

5.5 Effects of wave transiency and dissipation

Assuming the mean state balance is the same as in the previous subsection, consider arbitrary Rossby-gravity-acoustic waves with dissipation. The following symbols, which express the departure from steady nondissipative waves, are defined as $Dc(v_1') \equiv (\partial/\partial t + c\partial/\partial x_1)v_1' - (d_1/m_0)'$ and $Dc(s') \equiv (\partial/\partial t + c\partial/\partial x_1)s' - (q/m_0)'$, where c is a constant. From these definitions, for steady nondissipative propagating waves with a constant phase velocity c in the x_1 direction, $Dc(v_1')$ and $Dc(s')$ vanish. Substituting (E-7a, b) of Appendix E into (5-2c), results in

$$W_{1j,j} = -k_1 (\langle p'v_j' \rangle / \underline{\omega})_{,j} - J(\langle s \rangle, T_v) \\ + J(\langle u \rangle^\dagger, T_{vs}) \\ + J(\langle u \rangle^\dagger, \kappa \nabla \langle s \rangle \cdot \nabla \langle u \rangle^\dagger / J(\langle u \rangle^\dagger, \langle s \rangle)) \\ - 2\Omega_3 C_{32,3} + O(a^3), \quad (5-8a)$$

and

$$W_{4j,j} = -(\kappa \langle s \rangle_{,j})_{,j}, \quad (5-8b)$$

where $\underline{\omega} = k_1 c - k_1 \langle v_1 \rangle$, $\langle u \rangle^\dagger \equiv \langle v_1 \rangle - \int^x 2\Omega_3 dx_2$, $T_v \equiv \langle m_0 \rangle \langle v^\dagger Dc(v_1') \rangle / J(\langle u \rangle^\dagger, \langle s \rangle)$, $T_{vs} \equiv \langle m_0 \rangle \langle v^\dagger Dc(s') \rangle / J(\langle u \rangle^\dagger, \langle s \rangle)$, T_s

³Substituting (E-5a, b) into the LHS of mean of (D-6), $\langle m_0 \rangle (\langle E \rangle / \underline{\omega})_{,A} + \text{div}(\langle p'v' \rangle / \underline{\omega}) + \langle m_0 \rangle \langle v_1 \rangle_{,2} \langle v_1 \rangle_{,3} \langle s'v' \rangle \cdot \nabla \langle s \rangle / (\underline{\omega} \langle s \rangle_{,2} \langle s \rangle_{,3}) - 2\Omega_3 \langle m_0 \rangle \langle v_1 \rangle_{,3} \langle s'v_3' \rangle / (\underline{\omega} \langle s \rangle_{,2}) + \langle s'v' \rangle \cdot \nabla \langle p \rangle / (C_p \underline{\omega}) = O(a^3)$, assuming a unidirectional mean flow. The third term vanishes by substituting (E-5c). Substituting (5-5) into the fourth term, and adding the fifth term, the sum is reduced to $\langle p \rangle_{,2} / (C_p \underline{\omega} \langle s \rangle_{,2}) \langle s'v' \rangle \cdot \nabla \langle s \rangle$. When substituting (E-5c), this term also vanishes.

⁴Substituting (5-6a) into (3-10a), $V_{1,A} = -(2\Omega \times \mathbf{V})_1 + (2\Omega \times M^{add}/M_0)_1 + (C_{1k,k} / \langle m_0 \rangle)_{,A} - k_1 / \langle m_0 \rangle (\langle p'v_j' \rangle / \underline{\omega})_{,j} + 2\Omega_3 / \langle m_0 \rangle [\langle m_0 \rangle \langle s'v_2' \rangle / \langle s \rangle_{,2}]_{,3} + O(a^3)$. Using (5-6b), the fourth term of the RHS is rewritten as $(k_1 \langle E \rangle / \underline{\omega})_{,A}$. Substituting the definition of M^{add} into the second term, $(2\Omega \times M^{add}/M_0)_1 = -2\Omega_3 C_{23,3} / \langle m_0 \rangle = +2\Omega_3 [\langle m_0 \rangle \langle s'v_2' \rangle / \langle s \rangle_{,2}]_{,3}$, where (5-2b) and (E-5c) have been used. Therefore, using (E-5c) the second term is canceled by the fifth term.

$\equiv \langle m_0 \rangle \langle s' Dc(s') \rangle / J(\langle u \rangle^\dagger, \langle s \rangle)$, $v^\dagger \equiv v'_1 - k_1 p' / (\langle m_0 \rangle \omega)$, and $J(\phi, \psi) \equiv \partial(\phi, \psi) / \partial(x_2, x_3)$. Additionally,

$$\begin{aligned} -(2\boldsymbol{\Omega} \times \langle \mathbf{m} \rangle)_1 &= -(2\boldsymbol{\Omega} \times \mathbf{M})_1 + (2\boldsymbol{\Omega} \times \mathbf{M}^{add})_1 \\ &= -(2\boldsymbol{\Omega} \times \mathbf{M})_1 - 2\Omega_3 C_{23,3}. \end{aligned} \quad (5-8c)$$

Substituting (5-8a, 5-8b, 5-8c) into (3-10a, 3-10b),

$$\begin{aligned} V_{1,A} &= -(2\boldsymbol{\Omega} \times \mathbf{V})_1 + \langle d_1 \rangle / \langle m_0 \rangle \\ &\quad + (C_{1k,k} / \langle m_0 \rangle)_{,A} \\ &\quad + 1 / \langle m_0 \rangle [-k_1 (\langle p' v'_j \rangle / \omega)_{,j} - J(\langle s \rangle, T_v) \\ &\quad + J(\langle u \rangle^\dagger, T_r)] + O(a^3) \end{aligned} \quad (5-9a)$$

and

$$V_{4,A} = \langle q \rangle / \langle m_0 \rangle - (\kappa \langle s \rangle_{,j})_{,j} / \langle m_0 \rangle, \quad (5-9b)$$

where $T_r \equiv T_{vs} + \kappa \nabla \langle s \rangle \cdot \nabla \langle u \rangle^\dagger / J(\langle v \rangle^\dagger, \langle s \rangle)$ and $C_{23} + C_{32} = 0$ has been used. In (5-9a), there exists not only the wave action flux divergence term but also two forcing terms which are effective as vector products, since if $\phi_{,1} = \psi_{,1} = 0$, then $\nabla \phi \times \nabla \psi \equiv (J(\phi, \psi), 0, 0)$. One forcing term is the vector product of $\nabla \langle s \rangle$ and ∇T_v , while the other is the vector product of $\nabla \langle u \rangle^\dagger$ and ∇T_r .

5.6 Comparison with the TEM theory of AMc

The TEM theory of AMc is based on the assumptions of both zonal mean operation and the uni-directional mean flow. The presented EMC dynamics theory, however, is constructed without such restrictions.

The additional mean momentum density terms of AMc are written as follows using the presented notation:

$$M_1^{add} = 0, \quad M_2^{add} = \Phi_{,3} \quad \text{and} \quad M_2^{add} = -\Phi_{,2}, \quad (5-10a)$$

where

$$\Phi \equiv (\nabla \langle s \rangle \times \langle s' (m_0 \mathbf{v})' \rangle)_1 / |\nabla \langle s \rangle|^2. \quad (5-10b)$$

Using the assumption of uni-directional mean flow ($\langle v \rangle = (\langle v_1 \rangle, 0, 0) + O(a)$), (5-10b) is rewritten as

$$\Phi = \langle m_0 \rangle (\nabla \langle s \rangle \times \langle s' \mathbf{v}' \rangle)_1 / |\nabla \langle s \rangle|^2. \quad (5-10c)$$

This equation agrees with the (2, 3) component of the current additional term tensor (C_{ij}). Thus, it is found that when applying the assumptions of zonal mean operation and uni-directional mean flow, Eqs. (3-10a, 3-10b) are mathematically equivalent to (6.12a, b, c, d) of AMc, except for the fact that the latter are written in cylindrical coordinates.

In this sense, the present Eulerian mean continuum (EMC) dynamics theory includes the TEM theory of AMc, although the EMC wave stress and entropy source term ($W_{\mu j, j}$) do not explicitly include

the disturbance of mass density (m'_0). On the other hand, AMc used particle displacement in proving the non-acceleration theorem, while, the presented theory does not use such Lagrangian quantities.

6. Conclusions

The GLM and TEM theories have been reconstructed in a unified formalism which respectively corresponds to the presented LMC and EMC dynamics. The significant aspects of this formalism are as follows:

- (1) The concept of a new hypothetical continuum particle can be defined which is representative of the mean state. As in the case of the original equations, the governing equations of the new continuum can be easily transformed into various forms (two standard forms and a material form), using a transformation formula for independent variables.
- (2) The definition of new dependent variables is essential to construct the hypothetical continuum, which has an arbitrariness due to this definition. The new governing equations are formulated without physical restrictions such as initial conditions or spatial symmetry.

The main properties of the LMC dynamics are as follows:

- (1) The GLM theory can be reconstructed using the definition (3-3a) (among other possible choices), where a new particle, being a representative of the group of original particles, exists at the Lagrangian mean position of the original particles group and moves with the Lagrangian mean velocity.
- (2) The GLM equations are mathematically equivalent to the result obtained when applying the Lagrangian mean operation to the material form of the original equations. The hybrid structure in AMa and AMb (the independent variables are Eulerian and the dependent variables are Lagrangian mean quantities) is a reflection of the continuum's fundamental property with various form governing equations.
- (3) The postulate of initial nondisturbances is separated into a postulate on mass and one on entropy, and the effects of each were investigated. For the similarity of the stress tensor, the postulate on mass is required. For the non-acceleration theorem, the postulate on entropy is required. For the conservation of Pseudo momentum, both postulates are required. For practical applications to the atmosphere, the extension to curvilinear coordinates may be required. In Noda (1988), the postulate of initial nondisturbance was relaxed, by introducing the initial hypersurface in space-time. Here, however, the relaxation results from the transformation of the dependent variables.

The main properties of the EMC dynamics are as follows:

- (1) The TEM theory can be reconstructed using the

definition in (3-8a) (among other possible choices), where a new particle is defined as representative of the domain with a new label and moves with the transformed velocity.

(2) The formulation as a hypothetical continuum is exact, although the non-acceleration theorem is approximate.

(3) As a result of effects of the uni-directional mean flow and the spatial mean operation, an additional term to Eulerian mean meridional momentum circulation is represented by a stream function, and the wave entropy source term becomes a diffusion term.

(4) For nondissipative wave packet disturbances, the non-acceleration theorem holds. The effects of wave transience and dissipation produce the forcing terms which are effective as vector products. This result may be important under largely nonhomogeneous conditions in which wave packet approximation can not be used.

(5) For practical applications to the atmosphere, an extension to curvilinear coordinates is required. Iwasaki (1989) formulated a new TEM description for the primitive system in order to approximate Lagrangian mean meridional circulation. However, regardless of whether the TEM description is an approximation of the GLM description, the essential TEM framework should be used in practical studies.

The underlying dual structure of both the GLM and TEM theories has been revealed, with the main difference between them being that one is based on the Lagrangian point of view while the other is based on the Eulerian point of view. Since each theory has advantages and disadvantages, the theory most beneficial/applicable should be employed in any given study.

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Appendix

A. Transformation rule of independent variables

Let $z_{\alpha\beta}$ and $\bar{z}_{\alpha\beta}$ be the (α, β) element and its cofactor of 4×4 matrix Z , respectively. These must satisfy the next relation;

$$z_{\alpha\beta}\bar{z}_{\alpha\gamma} = \delta_{\beta\gamma}\det Z = \bar{z}_{\gamma\alpha}z_{\beta\alpha}, \quad (\text{A-1})$$

where $\det Z \equiv 1/4! \varepsilon_{\alpha\mu\lambda} \varepsilon_{\beta\nu\kappa\sigma} z_{\alpha\beta} z_{\mu\nu} z_{\lambda\kappa} z_{\sigma\lambda}$, $\bar{z}_{\alpha\beta} \equiv 1/3! \varepsilon_{\alpha\mu\lambda} \varepsilon_{\beta\nu\kappa\sigma} z_{\mu\nu} z_{\lambda\kappa} z_{\sigma\lambda}$, and $\varepsilon_{\alpha\mu\lambda}$ is the four dimensional Edington's ε ($\alpha, \beta, \gamma, \mu, \nu, \lambda, \kappa, \lambda, \sigma = 0, 1, 2, 3$).

If $\mathbf{a} = (a_0, a_1, a_2, a_3)$ and $\mathbf{b} = (b_0, b_1, b_2, b_3)$ are four dimensional vectors, and the mapping between “ $\mathbf{a}$ ” and “ $\mathbf{b}$ ” is invertible, then the following relation must be satisfied;

$$\partial a_\alpha / \partial b_\beta \partial b_\gamma / \partial a_\alpha = \delta_{\beta\gamma} = \partial b_\alpha / \partial a_\gamma \partial a_\beta / \partial b_\alpha. \quad (\text{A-2})$$

Defining $z_{\alpha\beta}$ as $\partial a_\alpha / \partial b_\beta$, from comparing (A-2) and (A-1), it can be seen that J_b^a and $J_b^a \partial b_\gamma / \partial a_\alpha$ correspond to $\det Z$ and $\bar{z}_{\alpha\gamma}$, respectively, where $J_b^a \equiv \partial(a_0, a_1, a_2, a_3) / \partial(b_0, b_1, b_2, b_3)$. In addition,

$$J_a^b \neq 0, J_b^a \neq 0 \quad \text{and} \quad J_a^b J_b^a = 1. \quad (\text{A-3})$$

Also, an identical relation can be derived as

$$\partial / \partial b_\beta (J_b^a \partial b_\beta / \partial a_\alpha) = \partial / \partial b_\beta (\bar{z}_{\alpha\beta}) = 0, \quad (\text{A-4})$$

in which the definition of $\bar{z}_{\alpha\beta}$ has been used.

If ψ is a function of (b_0, b_1, b_2, b_3) , then $J_a^b \psi$ is a function of (a_0, a_1, a_2, a_3) . The following identical relation is therefore obtained as

$$\begin{aligned} J_b^a \partial (J_a^b \psi) / \partial a_\alpha &= J_b^a \partial b_\beta / \partial a_\alpha \partial (J_a^b \psi) / \partial b_\beta \\ &= \partial / \partial b_\beta (J_b^a \partial b_\beta / \partial a_\alpha \psi) \\ &= \partial / \partial b_\beta (\psi \partial b_\beta / \partial a_\alpha), \end{aligned} \quad (\text{A-5})$$

where (A-4) has been used.

When a_0 and b_0 are nonrelativistic time (*i.e.*, $a_0 = b_0$), (A-5) is reduced to

$$J_b^a \partial (J_a^b \psi) / \partial a_0 = \partial \psi / \partial b_0 + \partial / \partial b_j (\psi \partial b_j / \partial a_0) \quad (j = 1, 2, 3) \quad (\text{A-6a})$$

and

$$J_b^a \partial (J_a^b \psi) / \partial a_i = \partial / \partial b_j (\psi \partial b_j / \partial a_i), \quad (i, j = 1, 2, 3) \quad (\text{A-6b})$$

where $\partial b_0 / \partial a_0 = 1$ and $\partial b_0 / \partial a_i = 0$. Specifically, if “ $\mathbf{a}$ ” is the particle label and “ $\mathbf{b}$ ” is its position, (A-6a) provides a rule by which the Lagrangian temporal differentiation is transformed to the Eulerian differentiation. For the case when $\psi = 1$, (A-6a) contains the key relation (2-1f) described in Section 2. In addition, if “ $\mathbf{a}$ ” presents the particle position and “ $\mathbf{b}$ ” the Lagrangian mean particle position, then (A-6b) plays an important role as shown in Section 4.

B. The mean operation and its relatives

If the mean value of ψ with respect to c_μ is denoted as $\ll \psi \gg^\mu$, the mean operation is defined such as

$$\ll \psi \gg^\mu \equiv \frac{1}{2L} \int_{c-L}^{c+L} \psi(c_0, c_1, c_2, c_3) dc_\mu, \quad (\mu = 0, 1, 2, 3) \quad (\text{B-1})$$

where c_0 is time, ψ an arbitrary function of (c_0, c_1, c_2, c_3) , and the departure from the mean value of ψ is cyclic with respect to c_μ in the range of $2L$. If the mean parameter is c_1 for example, the integration in (B-1) is performed such that c_0, c_2 , and c_3 are fixed. If (c_1, c_2, c_3) is a fluid particle label “ $\mathbf{a}$ ”, (B-1) becomes the definition of the Lagrangian mean operation and the mean value is denoted as $\{\psi\}^\mu$. On the other hand, if (c_1, c_2, c_3) is a fluid particle position “ $\mathbf{x}$ ”, (B-1) describes the Eulerian mean operation and the mean value is denoted as $\langle\psi\rangle^\mu$. Hereafter, the superscript μ , which indicates the mean parameter, is omitted except when needed for clarity.

From the Lagrangian mean definition,

$$\{t\} = t, \{\mathbf{a}\} = \mathbf{a}, d\mathbf{a}/dt = 0, \{d\psi/dt\} = d\{\psi\}/dt, \\ \{\{\psi\}\} = \{\psi\}, \text{ and } \{\psi - \{\psi\}\} = 0. \quad (\text{B-2})$$

The mapping between $(t, \mathbf{a})$ and $(t, \mathbf{X})$ is given by (3-1c). For example, when the mean parameter is a_1 , $X_i = a_i + \int_0^t (\underline{M}_i/\underline{M}_0)dt$, where the second term of the RHS is independent of a_1 , i.e., $\partial X_1/\partial a_1 = 1$. Therefore, $J_a^X = \partial(X_1(a_0, a_1, a_2, a_3), X_2(a_0, a_2, a_3), X_3(a_0, a_2, a_3))/\partial(a_1, a_2, a_3) = \partial(X_2, X_3)/\partial(a_2, a_3)$, $\{\psi\}^1 = 1/(2L) \int \psi(a_0, a_1, a_2, a_3) da_1 = 1/(2L) \int \psi(a_0, a_1, (a_0, X_1, a_2, a_3), a_2, a_3) dX_1$, and $J_X^x = \det(\partial x_i/\partial X_j) = \det(\partial x_i/\partial a_k \partial a_k/\partial X_j) = \det(\partial x_i/\partial a_k) \det(\partial a_k/\partial X_j) = J_a^x J_X^a$.

Thus, regardless of whether the mean parameter is a_μ or X_μ , the Lagrangian mean of ψ is denoted as $\{\psi\}$ and

$$\{\partial\psi/\partial X_\mu\} = \partial\{\psi\}/\partial X_\mu, \{J_a^X\} = J_a^X \\ \text{and } J_X^x = J_a^x J_X^a. \quad (\text{B-3})$$

On the other hand, from the Eulerian mean definition,

$$\langle t \rangle = t, \langle \mathbf{x} \rangle = \mathbf{x}, \partial \mathbf{x} / \partial t = 0, \\ \langle \partial \psi / \partial x_j \rangle = \partial \langle \psi \rangle / \partial x_j, \langle \langle \psi \rangle \rangle = \langle \psi \rangle, \\ \text{and } \langle \psi - \langle \psi \rangle \rangle = 0. \quad (\text{B-4})$$

The mapping between $(t, \mathbf{x})$ and $(t, \mathbf{A})$ is given by (3-6c). For example, when the mean parameter is x_1 , $x_i = A_i + \int_0^t (M_i/M_0)dt$, where the second term of the RHS is independent of x_1 , i.e., $\partial A_1/\partial x_1 = 1$. Therefore, $J_A^x = \partial(A_1(x_0, x_1, x_2, x_3), A_2(x_0, x_2, x_3), A_3(x_0, x_2, x_3))/\partial(x_1, x_2, x_3) = \partial(A_2, A_3)/\partial(x_2, x_3)$, $\langle \psi \rangle^1 = 1/(2L) \int \psi(x_0, x_1, x_2, x_3) dx_1 = 1/(2L) \int \psi(x_0, x_1(x_0, A_1, x_2, x_3), x_2, x_3) dA_1$, and $J_A^a = \det(\partial a_i/\partial A_j) = \det(\partial a_i/\partial x_k \partial x_k/\partial A_j) = \det(\partial a_i/\partial x_k) \det(\partial x_k/\partial A_j) = J_x^a J_A^x$. Thus, regardless of whether the mean parameter is x_μ or A_μ , the Eulerian mean of ψ is denoted as $\langle \psi \rangle$ and

$$\langle \partial \psi / \partial A_\mu \rangle = \partial \langle \psi \rangle / \partial A_\mu, \langle J_A^x \rangle = J_A^x \\ \text{and } J_A^a = J_x^a J_A^x. \quad (\text{B-5})$$

C. Wave energetics in the LMC

The third line of (4-3a) are

$$-\{h\}_{,I} + \{T_e\}\{s\}_{,I} + \{T_e^l s^l\}_{,I} \\ + \{x_{j,I} \tau_{jk,k}^d/m_0\} - \{\Psi\}_{,I}, \quad (\text{C-1})$$

where $\tau_{ik}^d \equiv \tau_{ik} + p\delta_{ik}$; and p , h , and T_e are pressure, enthalpy, and temperature, respectively. Also, $dp/m_0 = dh - T_e ds$ and $x_{j,I} \partial/\partial x_j = \partial/\partial X_I$ have been used.

The first term of the bottom line of (4-3a)

$$= -2\{(X_j + \xi_j)_{,I} (\boldsymbol{\Omega} \times \mathbf{v})_j\} \\ = -2\{(\boldsymbol{\Omega} \times \mathbf{v})_i\} + \{\xi_{j,I} (\boldsymbol{\Omega} \times (\mathbf{v} + \mathbf{v}^l))_j\} \\ = -2\{(\boldsymbol{\Omega} \times \mathbf{v})_i\} + \{\xi_{j,I} (\boldsymbol{\Omega} \times \boldsymbol{\xi}_{,a})_j\} \\ = -(2\boldsymbol{\Omega} \times \mathbf{v})_i - \{\xi_{j,I} (\boldsymbol{\Omega} \times \boldsymbol{\xi})_j\}_{,a} \\ + \{v_j^l (\boldsymbol{\Omega} \times \boldsymbol{\xi})_j\}_{,I} - \{v_k\}_{,I} \{\xi_{j,K} (\boldsymbol{\Omega} \times \boldsymbol{\xi})_j\}, \quad (\text{C-2})$$

where $\{\boldsymbol{\Omega}\} = \boldsymbol{\Omega}$, $\{\xi_j\} = 0$ and $\mathbf{v}^l = \boldsymbol{\xi}_{,a}$. In addition, the equivalencies $2\{\xi_{j,I} (\boldsymbol{\Omega} \times \boldsymbol{\xi}_{,a})_j\} = \varepsilon_{jmn} \Omega_m \{\xi_{j,I} \xi_{n,a} - \xi_{n,I} \xi_{j,a}\}$ and $\xi_{n,I} \xi_{j,a} = (\xi_n \xi_{j,a})_{,I} - \xi_n (\xi_{j,a})_{,I} = (\xi_n \xi_{j,a})_{,I} - \xi_n (\xi_{j,I})_{,a} - \{v_k\}_{,I} \xi_n \xi_{j,K}$ have been employed.

The second term of the bottom line of (4-3a)

$$= -\{\xi_{j,I} v_j^l\}_{,a} \\ = -\{(\{v_j^l \xi_{j,I}\}_{,a} - v_j^l (\xi_{j,I})_{,a})\} \\ = -\{v_j^l \xi_{j,I}\}_{,a} + 1/2 \{|v_j^l|^2\}_{,I} \\ - \{v_k\}_{,I} \{v_j^l \xi_{j,K}\}, \quad (\text{C-3})$$

where $v^l = \xi_{,a}$. Furthermore, $\{v_j^l (\xi_{j,I})_{,a}\} = \{v_j^l (\xi_{j,a})_{,I}\} - \{v_k\}_{,I} \{v_j^l \xi_{j,K}\} = 1/2 \{|v_j^l|^2\}_{,I} - \{v_k\}_{,I} \{v_j^l \xi_{j,K}\}$ has been used.

D. Wave action equation in the EMC

The material form governing equations are rewritten as follows:

$$m_{0,a} + m_0 \text{div}(\mathbf{v}) = 0, \quad v_{\mu,a} = f_\mu/m_0, \quad (\text{D-1})$$

where $\mu=0$ is a dummy argument. Eulerian mean equations of (D-1) are

$$Dt(\langle m_0 \rangle) + \langle m_0 \rangle \text{div} \langle \mathbf{v} \rangle = O(a^2), \quad (\text{D-2a})$$

$$Dt(\langle v_\mu \rangle) = \langle f_\mu/m_0 \rangle + O(a^2), \quad (\text{D-2b})$$

where $Dt(\cdot) \equiv (\cdot)_{,t} + \langle \mathbf{v} \rangle \cdot \nabla(\cdot)$. The deviation equations of (D-1) are

$$Dt(m'_0) + \text{div}(\langle m_0 \rangle \mathbf{v}') + m'_0 \text{div} \langle \mathbf{v} \rangle = O(a^2), \quad (\text{D-3a})$$

$$Dt(v'_\mu) + \mathbf{v}' \cdot \nabla \langle v_\mu \rangle = (f_\mu/m_0)' + O(a^2). \quad (\text{D-3b})$$

Taking the inner product of $\langle m_0 \rangle \mathbf{v}'$ and the first, second, and third components of (D-3b),

$$K_{,t} + \text{div}(K \langle \mathbf{v} \rangle) + \langle m_0 \rangle v'_i \mathbf{v}' \cdot \nabla \langle v_i \rangle \\ = \langle m_0 \rangle \mathbf{v}' \cdot (\mathbf{f}/m_0)' + O(a^3), \quad (\text{D-4a})$$

where $K(\equiv \langle m_0 \rangle |\mathbf{v}'|^2/2)$ is kinetic energy density. Taking the product of p'/C_p and the fourth component of (D-3b) using the substitution $v'_4 = C_v p'/\langle p \rangle - C_p m'_0/\langle m_0 \rangle + O(a^2)$,

$$U_{,t} + \text{div}(U \langle \mathbf{v} \rangle + p' \mathbf{v}') \\ + C_v/C_p p' \mathbf{v}' \cdot \nabla \langle p \rangle / \langle p \rangle \\ - U/\langle T_e \rangle Dt(\langle T_e \rangle) \\ = \mathbf{v}' \cdot \nabla p' + p'/C_p (f_4/m_0)' + O(a^3), \quad (\text{D-4b})$$

where C_p, C_v, T_e and $U(\equiv C_v p'^2/(2C_p \langle p \rangle))$ are the specific heat in constant pressure, specific heat in constant volume, temperature, and compressible energy density, respectively. Also (D-2a) and (D-3a) have been employed.

The sum of (D-4a) and (D-4b) is

$$(\langle m_0 \rangle E)_{,t} + \text{div}(\langle m_0 \rangle E \langle \mathbf{v} \rangle + p' \mathbf{v}') \\ + (\langle m_0 \rangle v'_i \mathbf{v}' \cdot \nabla \langle v_i \rangle) \\ + s' \mathbf{v}' \cdot \nabla \langle p \rangle / C_p - U/\langle T_e \rangle Dt(\langle T_e \rangle) \\ = \langle m_0 \rangle \mathbf{v}' \cdot (\mathbf{d}/m_0)' \\ + p'/C_p (q/m_0)' + O(a^3), \quad (\text{D-5})$$

where $E \equiv (K+U)/\langle m_0 \rangle$, $s' \equiv v'_4$, and Eq. (5-1) has been used. Assuming that ω, k_1, k_2 , and k_3 are arbitrary constant parameters, (D-5) can be expressed in another form as

$$(\langle m_0 \rangle E/\underline{\omega})_{,t} + \text{div}(\langle m_0 \rangle E/\underline{\omega} \langle \mathbf{v} \rangle + p' \mathbf{v}'/\underline{\omega}) \\ + 1/\underline{\omega} (\langle m_0 \rangle v'_i \mathbf{v}' - k_i p' \mathbf{v}'/\underline{\omega}) \cdot \nabla \langle v_i \rangle \\ + s' \mathbf{v}' \cdot \nabla \langle p \rangle / (\underline{\omega} C_p) \\ - [\langle m_0 \rangle \mathbf{v}' \cdot (\mathbf{d}/m_0)' + p'/C_p (q/m_0)']/\underline{\omega} \\ + \langle m_0 \rangle E/\underline{\omega}^2 k_i Dt(\langle v_i \rangle) \\ - U/\langle T_e \rangle Dt(\langle T_e \rangle) = O(a^3), \quad (\text{D-6})$$

where $\underline{\omega} \equiv \omega - k_1 \langle v_1 \rangle - k_2 \langle v_2 \rangle - k_3 \langle v_3 \rangle$. If ω and k_i are respectively the wave frequency and wave number, the mean equation of (D-6) is regarded as the wave action equation. The constant parameters ω and k_i ($i=1, 2, 3$) imply a three-dimensional slowly-varying wave packet. However, if uni-directional mean flow is assumed, such a restriction can be relaxed to an arbitrary structure on the meridional plane, since all terms containing k_2 and k_3 in (D-6) vanish.

E. Wave energetics in the EMC

E.1 Acoustic wave packet

Substituting the nondissipative wave packet solution $(v'_i, s', p') \sim \exp(j(\omega t - \mathbf{k} \cdot \mathbf{x}))$ into (D-3b) of

Appendix D,

$$j\underline{\omega} \begin{pmatrix} v_{0i} \\ s_0 \end{pmatrix} + \begin{pmatrix} \langle v_i \rangle_{,2} & \langle v_i \rangle_{,3} \\ \langle s \rangle_{,2} & \langle s \rangle_{,3} \end{pmatrix} \begin{pmatrix} v_{02} \\ v_{03} \end{pmatrix} \\ - j k_i / \langle m_0 \rangle \begin{pmatrix} p_0 \\ 0 \end{pmatrix} = O(a^2) \quad (\text{E-1a})$$

where $\underline{\omega} = \omega - \mathbf{k} \cdot \langle \mathbf{v} \rangle$, v_{0i}, p_0 and s_0 are amplitudes of v'_i, p' and s' , respectively. Also, $\langle f_i \rangle \leq O(a)$, “ j ” is an imaginary unit, and the differentiation of deviations has been performed to only the phase function, keeping in mind the short wavelength asymptotics. From $\text{Re}((E-1b)s_0^*)$,

$$\langle s \rangle_{,2} \text{Re}(s_0^* v_{02}) + \langle s \rangle_{,3} \text{Re}(s_0^* v_{03}) = O(a^3), \quad (\text{E-2a})$$

where ψ^* is the complex conjugate of ψ . Solving (E-1a, b) about v_{02} and v_{03} , from $\text{Im}(v_{02} v_{02}^*)$,

$$\langle s \rangle_{,3} \text{Re}(v_{0i} v_{02}^*) - \langle v_i \rangle_{,3} \text{Re}(s_0 v_{02}^*) \\ - k_i \langle s \rangle_{,3} \text{Re}(p_0 v_{02}^*) / (\underline{\omega} \langle m_0 \rangle) = O(a^3), \quad (\text{E-2b})$$

and from $\text{Im}(v_{03} v_{03}^*)$,

$$\langle s \rangle_{,2} \text{Re}(v_{0i} v_{03}^*) - \langle v_i \rangle_{,2} \text{Re}(s_0 v_{03}^*) \\ - k_i \langle s \rangle_{,2} \text{Re}(p_0 v_{03}^*) / (\underline{\omega} \langle m_0 \rangle) = O(a^3). \quad (\text{E-2c})$$

Using the relationship of $\langle \phi' \psi' \rangle = \langle \text{Re}(\phi') \text{Re}(\psi') \rangle = \text{Re}(\phi_0 \psi_0^*)/2$, from (E-2b, c, a), the following is obtained:

$$D_{23}^i = -k_i \langle p' v'_2 \rangle / \underline{\omega} + O(a^3), \\ D_{32}^i = -k_i \langle p' v'_3 \rangle / \underline{\omega} + O(a^3) \\ \text{and } \kappa = O(a^3). \quad (\text{E-3})$$

E.2 Rossby-gravity-acoustic wave packet

Assume a nondissipative wave packet with arbitrary structures in the x_2 and x_3 directions. Substituting $(v'_i, s', p') \sim \exp[j(\omega t - k_1 x_1)]$ into (D-3b),

$$j\underline{\omega} \begin{pmatrix} v_{01} \\ s_0 \end{pmatrix} + \begin{pmatrix} \langle v_1 \rangle_{,2} - 2\Omega_3 & \langle v_1 \rangle_{,3} \\ \langle s \rangle_{,2} & \langle s \rangle_{,3} \end{pmatrix} \begin{pmatrix} v_{02} \\ v_{03} \end{pmatrix} \\ - j k_1 / \langle m_0 \rangle \begin{pmatrix} p_0 \\ 0 \end{pmatrix} = O(a^2), \quad (\text{E-4a})$$

where $\underline{\omega} = \omega - k_1 \langle v_1 \rangle$. These equations are same as in the case of acoustic-gravity waves, except for exchanging of $\langle v_i \rangle_{,2}$ and $\langle v_1 \rangle_{,2} - 2\Omega_3$. Similarly as the case of acoustic wave packet,

$$D_{23}^1 = -k_1 \langle p' v'_2 \rangle / \underline{\omega} + O(a^3), \quad (\text{E-5a})$$

$$D_{32}^1 - 2\Omega_3 \langle m_0 \rangle \langle s' v'_3 \rangle / \langle s \rangle_{,2} \\ = -k_1 \langle p' v'_3 \rangle / \underline{\omega} + O(a^3) \quad (\text{E-5b})$$

$$\text{and } \kappa = O(a^3). \quad (\text{E-5c})$$

On the other hand, for transient, dissipative waves using the notation of $Dc(v'_1)$ and $Dc(s')$, (D-3b) are rewritten as

$$\begin{pmatrix} Dc(v'_1) \\ Dc(s') \end{pmatrix} + \begin{pmatrix} \langle u \rangle^{\dagger,2} & \langle u \rangle^{\dagger,3} \\ \langle s \rangle^{\dagger,2} & \langle s \rangle^{\dagger,3} \end{pmatrix} \begin{pmatrix} v'_2 \\ v'_3 \end{pmatrix} \\ = \omega/k_1 \begin{pmatrix} v^{\dagger,1}_{s',1} \end{pmatrix} + O(a^2), \quad \begin{matrix} \text{(E-6a)} \\ \text{(E-6b)} \end{matrix}$$

where $\langle u \rangle^{\dagger} \equiv \langle v_1 \rangle - \int^x 2\Omega_3 dx_2$, $v^{\dagger} \equiv v'_1 - k_1 p' / (\langle m_0 \rangle \omega)$, $\omega \equiv k_1 c - k_1 \langle v_1 \rangle$, and Eq. (5-1) has been used.

After solving (E-6a, E-6b) about (v'_2, v'_3) , from $\langle (\langle s \rangle^{\dagger,3} v^{\dagger} - \langle u \rangle^{\dagger,3} s') v'_2 \rangle$, $\langle (\langle s \rangle^{\dagger,2} v^{\dagger} - \langle u \rangle^{\dagger,2} s') v'_3 \rangle$ and $\langle s' Dc(s') \rangle$, the following relations are respectively obtained;

$$D_{23}^1 = -k_1 \langle p' v'_2 \rangle / \omega + T_v \langle s \rangle^{\dagger,3} \\ - T_{vs} \langle u \rangle^{\dagger,3} + T_s (\langle u \rangle^{\dagger,3})^2 / \langle s \rangle^{\dagger,3} + O(a^3) \quad \text{(E-7a)}$$

$$D_{32}^1 = -k_1 \langle p' v'_3 \rangle / \omega - T_v \langle s \rangle^{\dagger,2} \\ + T_{vs} \langle u \rangle^{\dagger,2} - T_s (\langle u \rangle^{\dagger,2})^2 / \langle s \rangle^{\dagger,2} \\ + 2\Omega_3 \langle m_0 \rangle \langle s' v'_3 \rangle / \langle s \rangle^{\dagger,2} + O(a^3), \quad \text{(E-7b)}$$

$$\kappa = -\langle m_0 \rangle \langle s' Dc(s') \rangle / |\nabla \langle s \rangle^{\dagger}|^2 + O(a^3), \quad \text{(E-7c)}$$

where $T_v \equiv \langle m_0 \rangle \langle v^{\dagger} Dc(v'_1) \rangle / J(\langle u \rangle^{\dagger}, \langle s \rangle^{\dagger})$, $T_s \equiv \langle m_0 \rangle \langle s' Dc(s') \rangle / J(\langle u \rangle^{\dagger}, \langle s \rangle^{\dagger})$, $T_{vs} \equiv \langle m_0 \rangle \langle v^{\dagger} Dc(s') + s' Dc(v'_1) \rangle / J(\langle u \rangle^{\dagger}, \langle s \rangle^{\dagger})$, and $J(\phi, \psi) \equiv \partial(\phi, \psi) / \partial(x_2, x_3)$.

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一般化されたラグランジュ平均（GLM）理論と 変換されたオイラー平均（TEM）理論の統一形式

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一般化されたラグランジュ平均（GLM）理論と変換されたオイラー平均（TEM）理論が統一形式において再構築された。

オイラー的情報とラグランジュ的情報の相互の翻訳が可能であるという基本的な前提条件を応用して、2種類の標準型を持つ支配方程式系の対構造が明らかにされた、そして、それらは各々ラグランジュ的視点とオイラー的視点に対応している。これら2種の標準型のいずれか一方だけでも連続体概念の完全な方程式系を形づくり、そして一方から他方へ簡単に変換することができる。

従って、いずれか一方の標準型を持つ連続体方程式の新しいセットを導入しさえすればいつでも、新しい仮想的連続体が定義される。そして、この新しい連続体の支配方程式系もまた、元の方程式の場合と同様に、簡単に様々な型（他方の標準型やマテリアル型）に変換され得る。新しい従属変数の定義が本質的である、なぜならこの仮想的連続体はこれらの定義に起因する任意性を持っているからである。また、この新しい支配方程式系は初期条件とか空間的対称性のようないかなる物理的前提条件も無しに定式化される。

ラグランジュ平均連続体において新しい従属変数の形とそれらの付加項を選択することによって、GLM理論が再構築される。そこでは元の粒子群を代表する新しい粒子が、元の粒子群のラグランジュ平均位置に存在し、ラグランジュ平均速度で移動する。

同様に、オイラー平均連続体におけるひとつの選択肢として、TEM理論が再構築され、そこではある領域の代表としての新しい粒子が、新しいラベルを持ち、ある変換された速度で移動する。

これらの再構築の副産物として、両理論における非加速定理が調べられ、若干の拡張が得られた。