Methods for Obtaining Stationary Solutions of the Primitive Equation Model and Stationary and Periodic Solutions of the Quasi-Geostrophic Model

By Hisanori Itoh

*Department of Earth Sciences, Wakayama University, Wakayama 640, Japan
(Manuscript received 25 August 1992, in revised form 22 December 1992)*

Abstract

Methods for obtaining stationary solutions of the primitive equation model and stationary and periodic solutions of the quasi-geostrophic model are proposed. The key is to construct the spectral models based on the spectral coefficient method, in which non-linear terms are calculated by the interaction coefficient method. This is required to obtain explicitly Jacobian matrices which are necessary for solving the non-linear equation system by Newton's method. Thus, solutions can be obtained for the models with a resolution which can simulate the real atmosphere. Further, a linear stability analysis for these solutions can be made. In this course, several devices for computations are necessary, particularly since the both models have diagnostic equations and the calculations of non-linear terms is complicated. Applications to various problems and future developments are also discussed.

1. Introduction

Large-scale motions in the extra-tropics are highly chaotic, because baroclinic waves, whose time and spatial scales are relatively small, are dominant. However, *order* is also formed in this chaotic motion. Here *order* means the phenomena of blockings, teleconnection patterns, intraseasonal oscillations, and so on. Their time scales are large, and their spatial structures are widely coherent. These phenomena have been paid attention and extensively studied by many authors since their discoveries, but the mechanisms have not yet been clarified.

One promising approach toward explaining these phenomena is non-linear dynamics. From this approach, several studies about blockings have been made. The first research was done by Charney and DeVore (1979). Using a low-order spectral model on a barotropic β -plane, they concluded that two stable equilibria exist for a certain range of parameters; one is the solution in which the zonal flow is dominant, and the other is the solution in which the meridional flow is dominant, *i.e.*, the blocking solution. Further, Charney and Straus (1980) extended this model to a two-level baroclinic one. However, since there exist baroclinic waves in the mid-latitude atmosphere, these solutions must be unstable when the truncation is loosened. Indeed, Reinhold and

Pierrehumbert (1982, 1985) found that there is no stable stationary solution in their model including baroclinic waves. Instead, they proposed the idea of the weather régimes in which distribution density is high in restricted volumes in a phase space for time-integrated data, and the blocking is dominant in one of the régimes. However, this was also proved to be false by the study of Cehelsky and Tung (1987) in which higher modes were included. The lesson obtained from a series of these studies is that it is dangerous to draw a general conclusion from the results of low-order models.

On the other hand, Legras and Ghil (1985) fundamentally elucidated the importance of unstable stationary solutions in dynamical systems. In dissipative systems, *i.e.*, volume contracting systems, examining the linear stability of those solutions, the number of stable eigenvalues is much more than that of unstable eigenvalues. Trajectories near an unstable equilibrium solution are therefore attracted from almost all directions to this solution, and are repelled in only a few directions. This is the reason why unstable stationary solutions are important. In fact, they found that such motions really occur in their quasi-geostrophic, barotropic model on the sphere. Mukougawa (1988) further clarified that non-stationary local minimum points near which motions are very slow play an important rôle, as well as unstable stationary solutions. Very re-

cently, he conducted a similar research into periodic solutions, and pointed out the importance of unstable periodic solutions in dynamical systems (personal communication). Moreover, he found that pseudo-periodic solutions, which are not strictly periodic, but almost repeat periodic motions, also play an important rôle.

These studies suggest that, as well as the blocking which may be considered as the motion near unstable stationary solutions and/or non-stationary local minimum points, the intra-seasonal oscillation and teleconnection patterns can also be approached from non-linear dynamics. Namely, the intra-seasonal oscillation may be understood as the motion near unstable periodic solutions and/or pseudo-periodic solutions (*cf.* Jin and Ghil, 1990; Ghil and Mo, 1991). The teleconnection patterns may also connect to one or both of them (Kimoto, 1979).

In order to examine these possibilities, the first thing to do is to construct a time integration model which can simulate these phenomena. Then, by obtaining stationary points (including non-stationary minimum points) and periodic solutions (including pseudo-periodic solutions) of the same model, it must be examined whether or not the simulated phenomena coincide with these solutions.

One difficulty in this course is to obtain stationary and periodic solutions, because the model will not be simple but very complicated. Therefore, in order to proceed on this course, methods for obtaining these solutions must be developed. This is the purpose of this paper. In contrast to previous models in which the resolution was coarse and idealized topography was used, the present model must have the following properties: spherical geometry, non-coarse horizontal resolution, multi levels in the vertical, and realistic topography. In order to do so, several devices are necessary, which will be described in this paper.

The future goal is to know the global geometrical structure of meteorological models. We must further know homoclinic orbits, heteroclinic orbits, separatrices, and so on (for these notions, refer to, for example, Abraham and Shaw, 1992; Thompson and Stewart, 1986), in addition to stationary and periodic solutions. When these things can be successfully done, the statistical behavior of the models will be predictable without time integration. This implies that a different approach will be possible to modeling research, in which the ordinary method at the present time is to extract statistical properties from time-integrated data. This study is one step in that direction.

2. General plan

Two models will be constructed. One is the primitive equation model, which can realistically simulate the atmospheric motion. Stationary solutions can be obtained for this model, but it is difficult to

get periodic solutions, because too much computer memory is required (see Subsection 4b). We therefore introduce the quasi-geostrophic model in order to obtain periodic solutions as well as stationary solutions. This model is less accurate, especially in the tropics, than the primitive equation model, but there is no major flaw when we carefully inspect the results of this model, compared to the results of the primitive equation model. Also, its simpler form can be a valuable aid to understanding.

Since it is a primary requirement that the number of dependent variables is as small as possible, spectral models should be adopted. Grid models have a larger number of dependent variables under the same resolution. In spectral models, the spherical harmonic expansion is made in the horizontal while the finite difference is taken in the vertical. Dependent variables are therefore spectral coefficients on each vertical grid, which will be expressed as $x_{\alpha,k}$, where α and k mean spherical harmonic mode and vertical grid level, respectively. They are the unknowns when stationary solutions are sought, while they are functions of time (t) for periodic solutions. In the latter case, by discretizing $x_{\alpha,k}(t)$ with respect to time, $x_{\alpha,k,\tau}$'s become the unknowns in this system, where τ indicates the time level. Then, a simultaneous non-linear equation system with a large number of unknowns is yielded for both cases. This system will be numerically solved, and stationary or periodic solutions will be obtained.

For the calculation of non-linear terms, the interaction coefficient method is utilized. This is required to obtain explicitly Jacobian matrices which are necessary for solving the non-linear equation system by Newton's method. Although approximate Jacobian matrices can also be obtained by the finite difference with respect to the dependent variables, this method is not appropriate in this kind of large system in terms of accuracy and especially computation time.

Since it is important to see how stationary or periodic solutions are changed with parameters (parameters in this kind of problem are often called *bifurcation parameters*), the continuation method based on Newton's method is adopted. Namely, suppose one stationary solution has been obtained for a certain value of a parameter, say p_0 . Then, setting this solution as an initial guess or making an initial guess from this solution, a stationary solution for $p_0 + \Delta p$ is sought by Newton's method, where Δp is a small amount of the parameter. Then, we can successively get the solutions with the change of the parameter. As the starting point of the continuation method, one stationary solution must be obtained by a different method. The easiest way to get it may be as the result of time integration using a certain parameter value for which there exists a stable stationary solution. Another alternative is to start from

the trivial solution in which all components have no amplitudes, if such exists. Other methods have also been proposed (*e.g.*, Keller, 1978).

Linear stability analyses for both stationary and periodic solutions should be made, because it is essential to determine whether these solutions are stable or not. These analyses also give information about the bifurcation of the solutions. For the property of bifurcation in non-linear dynamics, refer to, for example, Guckenheimer and Holmes (1983). These calculations on stability can also be made by utilizing Jacobian matrices.

Next, we will explain how to obtain Jacobian matrices. Both models can be formally written as

$$dx/dt = f(x, y), \quad (1)$$

$$y = g(x). \quad (2)$$

For instance, the equation for $\dot{\sigma}$ for the primitive equation model and the equation of the thermal wind balance for the quasi-geostrophic model are classified as (2), where $\dot{\sigma}$ is the vertical σ -velocity. There are three possible methods for obtaining stationary solutions.

- A. Eliminating y by the insertion of (2) into (1), only x is treated as the dependent variable.
- B. y as well as x is treated as the dependent variable.
- C. The dependent variable is essentially x only, but Jacobian matrices are derived as $\partial f/\partial x + \partial f/\partial y \cdot \partial g/\partial x$.

In practice, Method A is too complicated to obtain Jacobian matrices. Either Method B or C should be selected. On the other hand, for obtaining periodic solutions, Method B is not suitable, because (2) has no time derivative. Therefore, Method C must be adopted.

As noted in the Introduction, it is a serious matter as to how the resolution should be determined. We *a priori* have no principle for determining the number of vertical grids. In this study, since the aim is to show that stationary and periodic solutions can be obtained, simple models are constructed. The number of vertical grids is four for the primitive equation model, and two for the quasi-geostrophic model. Concerning the horizontal resolution, the following principle is adopted: Generally speaking, results change as the resolution increases. After the truncation reaches a certain wavenumber, however, finer resolution will give almost the same result as this truncation. This truncation will be a reasonable one.

For the present, it seems enough that only the Northern Hemisphere is treated, because the interaction between both hemispheres does not seem to be essential. This treatment is equivalent to the

model in which the stream function is antisymmetric about the equator, while the velocity potential, temperature and pressure are symmetric. Triangular truncation is adopted, although rhomboidal truncation is also applicable. Hereafter truncation will be expressed as, for example, T15, which means triangular truncation at wavenumber 15.

Non-linear terms are calculated as follows. There are three types of non-linear terms; $A \cdot B$, $\nabla A \cdot \nabla B$ and $J(A, B)$, where A and B show two variables, ∇ nabla operator and J Jacobian operator. These spectral forms for mode α produced by the interaction of modes β and γ can be written as $C_{\alpha\beta\gamma}A_\beta B_\gamma$, $I_{\alpha\beta\gamma}A_\beta B_\gamma$, and $iJ_{\alpha\beta\gamma}A_\beta B_\gamma$, respectively, where $\alpha \equiv (m_\alpha, n_\alpha)$, m_α is zonal wavenumber, and n_α total wavenumber, *i.e.*, $n_\alpha - m_\alpha$ the number of nodes in the meridional direction. β and γ have the same meaning as α . C , I and J are called the interaction coefficients, and are defined as

$$C_{\alpha\beta\gamma} = \frac{1}{4\pi_0} \int \int Y_\alpha^* Y_\beta Y_\gamma \cos \phi \, d\phi \, d\lambda,$$

$$I_{\alpha\beta\gamma} = \frac{1}{4\pi_0} \int \int Y_\alpha^* \nabla Y_\beta \cdot \nabla Y_\gamma \cos \phi \, d\phi \, d\lambda,$$

and

$$J_{\alpha\beta\gamma} = \frac{1}{i4\pi_0} \int \int Y_\alpha^* J(Y_\beta, Y_\gamma) \cos \phi \, d\phi \, d\lambda,$$

where Y represents the spherical harmonic function, and the superscript $*$ stands for the complex conjugate. Further, π_0 , ϕ and λ indicate the circular constant, latitude and longitude, respectively. These coefficients are calculated in advance, and will be substituted in corresponding places in the calculation of non-linear terms. The method for obtaining these coefficients is described in, *e.g.*, Silberman (1954), Ellsaesser (1966) and Kikuchi (1972).

The number of the interaction coefficients becomes large, as the resolution increases in the order of M^6 , where M means the truncation wavenumber. However, most of these interaction coefficients are zero, so only non-zero coefficients should be retained. As a result, when M is less than 21, the number of non-zero coefficients is estimated as $O(M^4)$. This number is much smaller than the estimation based on the selection rule, *i.e.*, $O(M^5)$ (Orszag, 1970).

Further explanations about the calculation of non-linear terms will be given in Appendix A.

In the following, detailed methods and results for the primitive equation model and the quasi-geostrophic model will be offered.

3. Stationary solutions of the primitive equation model

a. Model

The primitive equation model used in this study is a general one. We seek stationary solutions under tropical heating, in order to see the model work well in the tropics as well as the extra-tropics. The model is described in Appendix B, together with the explanation of notations. In this sub-section, we start from the spectral forms of the equations, where length and time are non-dimensionalized by a (the radius of the earth) and Ω^{-1} (Ω : the angular velocity of the earth's rotation).

The spectral forms of the prognostic equations (B1)–(B4) can be expressed as

$$\begin{aligned} -D_\alpha \frac{\partial \psi_\alpha}{\partial t} = & i \sum_{\beta, \gamma} D_\gamma J_{\alpha\beta\gamma} \psi_\beta \psi_\gamma \\ & + \sum_{\beta, \gamma} D_\gamma I_{\alpha\beta\gamma} \chi_{\beta\gamma} \psi_\gamma - \sum_{\beta, \gamma} D_\beta D_\gamma C_{\alpha\beta\gamma} \chi_{\beta\gamma} \psi_\gamma \\ & - i 2 m_\alpha \psi_\alpha + 2 (a_\alpha \chi_{\alpha-\epsilon} + a_{\alpha+\epsilon} \chi_{\alpha+\epsilon}) \\ & + \sum_{\beta, \gamma} D_\gamma C_{\alpha\beta\gamma} \dot{\sigma}_\beta \frac{\partial \psi_\gamma}{\partial \sigma} - i \sum_{\beta, \gamma} J_{\alpha\beta\gamma} \dot{\sigma}_\beta \frac{\partial \chi_\gamma}{\partial \sigma} \\ & - \sum_{\beta, \gamma} I_{\alpha\beta\gamma} \dot{\sigma}_\beta \frac{\partial \psi_\gamma}{\partial \sigma} - i R \sum_{\beta, \gamma} J_{\alpha\beta\gamma} T_\beta \pi_\gamma \\ & + \nu_H (D_\alpha^2 - 4) D_\alpha \psi_\alpha + F_\psi, \end{aligned} \quad (3)$$

$$\begin{aligned} -D_\alpha \frac{\partial \chi_\alpha}{\partial t} = & \frac{D_\alpha}{2} \sum_{\beta, \gamma} I_{\alpha\beta\gamma} \psi_\beta \psi_\gamma + \frac{D_\alpha}{2} \sum_{\beta, \gamma} I_{\alpha\beta\gamma} \chi_{\beta\gamma} \chi_\gamma \\ & + i D_\alpha \sum_{\beta, \gamma} J_{\alpha\beta\gamma} \psi_\beta \chi_\gamma - \sum_{\beta, \gamma} D_\gamma I_{\alpha\beta\gamma} \psi_\beta \psi_\gamma \\ & + i \sum_{\beta, \gamma} D_\gamma J_{\alpha\beta\gamma} \chi_{\beta\gamma} \psi_\gamma + \sum_{\beta, \gamma} D_\beta D_\gamma C_{\alpha\beta\gamma} \psi_\beta \psi_\gamma \\ & - i 2 m_\alpha \chi_\alpha - 2 (a_\alpha \psi_{\alpha-\epsilon} + a_{\alpha+\epsilon} \psi_{\alpha+\epsilon}) \\ & + \sum_{\beta, \gamma} D_\gamma C_{\alpha\beta\gamma} \dot{\sigma}_\beta \frac{\partial \chi_\gamma}{\partial \sigma} + i \sum_{\beta, \gamma} J_{\alpha\beta\gamma} \dot{\sigma}_\beta \frac{\partial \psi_\gamma}{\partial \sigma} \\ & - \sum_{\beta, \gamma} I_{\alpha\beta\gamma} \dot{\sigma}_\beta \frac{\partial \chi_\gamma}{\partial \sigma} + R \sum_{\beta, \gamma} D_\gamma C_{\alpha\beta\gamma} T_\beta \pi_\gamma \\ & - R \sum_{\beta, \gamma} I_{\alpha\beta\gamma} T_\beta \pi_\gamma + D_\alpha \Phi_\alpha \\ & + \nu_H (D_\alpha^2 - 4) D_\alpha \chi_\alpha + F_\chi, \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{\partial T_\alpha}{\partial t} = & -i \sum_{\beta, \gamma} J_{\alpha\beta\gamma} \psi_\beta T_\gamma - \sum_{\beta, \gamma} I_{\alpha\beta\gamma} \chi_{\beta\gamma} T_\gamma \\ & - \sum_{\beta, \gamma} C_{\alpha\beta\gamma} \dot{\sigma}_\beta \frac{\partial T_\gamma}{\partial \sigma} + \frac{\kappa}{\sigma} \sum_{\beta, \gamma} C_{\alpha\beta\gamma} T_\beta \dot{\sigma}_\gamma \\ & - \kappa \sum_{\beta, \gamma} C_{\alpha\beta\gamma} T_\beta \frac{\partial \dot{\sigma}_\gamma}{\partial \sigma} + \kappa \sum_{\beta, \gamma} D_\gamma C_{\alpha\beta\gamma} T_\beta \chi_\gamma \\ & + \frac{\dot{Q}_\alpha}{C_p} - \nu_H D_\alpha^2 T_\alpha - \mu (T_\alpha - T_{ea}), \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{\partial \pi_\alpha}{\partial t} = & \sum_{k=1}^K \left[-i \sum_{\beta, \gamma} J_{\alpha\beta\gamma} \psi_{\beta,k} \pi_\gamma \right. \\ & \left. - \sum_{\beta, \gamma} I_{\alpha\beta\gamma} \chi_{\beta,k} \pi_\gamma + D_\alpha \chi_{\alpha,k} \right] \Delta \sigma_k, \end{aligned} \quad (6)$$

respectively, where $D_\alpha = n_\alpha(n_\alpha + 1)$, ϵ stands for mode (0,1), and

$$a_\alpha = (n_\alpha^2 - 1) \sqrt{\frac{n_\alpha^2 - m_\alpha^2}{4n_\alpha^2 - 1}}.$$

Subscript k in (6) represents the level index, and K is the maximum of the level index (=4 in the present model). Level indices are omitted in (3)–(5), since these equations are applied to each level. The vertical finite difference forms of $\dot{\sigma}(\partial A / \partial \sigma)$ and $A(\partial \dot{\sigma} / \partial \sigma)$ are

$$\begin{aligned} & \frac{1}{2\Delta\sigma_k} \left[\dot{\sigma}_{k-\frac{1}{2}} (A_{k-1} - A_k) + \dot{\sigma}_{k+\frac{1}{2}} (A_k - A_{k+1}) \right] \\ & \text{and } A_k \frac{\dot{\sigma}_{k-\frac{1}{2}} - \dot{\sigma}_{k+\frac{1}{2}}}{\Delta\sigma_k}, \end{aligned}$$

respectively. The vertical diffusion terms are linear, because p_s and ρ in (B5) are treated as constants. The spectral forms are not special, so the explicit forms are not expressed. The spectral forms of the diagnostic equations (B6)–(B9) are

$$\Phi_{\alpha,1} = \Phi_{\alpha,s} - RT_{\alpha,1} \ln \sigma_1, \quad (7)$$

$$\begin{aligned} \Phi_{\alpha,k} = & \Phi_{\alpha,s} - RT_{\alpha,1} \ln \sigma_1 \\ & + \sum_{l=2}^k (\alpha_l T_{\alpha,l} + \beta_{l-1} T_{\alpha,l-1}) \quad (\text{for } k \geq 2), \end{aligned} \quad (8)$$

$$\begin{aligned} \dot{\sigma}_{\alpha,k-\frac{1}{2}} = & -\sigma_{k-\frac{1}{2}} \sum_{l=1}^K \left[-i \sum_{\beta, \gamma} J_{\alpha\beta\gamma} \psi_{\beta,l} \pi_\gamma \right. \\ & \left. - \sum_{\beta, \gamma} I_{\alpha\beta\gamma} \chi_{\beta,l} \pi_\gamma + D_\alpha \chi_{\alpha,l} \right] \Delta \sigma_l \\ & - \sum_{l=K}^k \left[i \sum_{\beta, \gamma} J_{\alpha\beta\gamma} \psi_{\beta,l} \pi_\gamma \right. \\ & \left. + \sum_{\beta, \gamma} I_{\alpha\beta\gamma} \chi_{\beta,l} \pi_\gamma - D_\alpha \chi_{\alpha,l} \right] \Delta \sigma_l, \end{aligned} \quad (9)$$

$$\begin{aligned} \dot{\sigma}_{\alpha,k-\frac{1}{2}} = & \dot{\sigma}_{\alpha,k+\frac{1}{2}} - \left[\frac{\partial \pi_\alpha}{\partial t} + i \sum_{\beta, \gamma} J_{\alpha\beta\gamma} \psi_{\beta,k} \pi_\gamma \right. \\ & \left. + \sum_{\beta, \gamma} I_{\alpha\beta\gamma} \chi_{\beta,k} \pi_\gamma - D_\alpha \chi_{\alpha,k} \right] \Delta \sigma_k, \end{aligned} \quad (10)$$

respectively, where $\alpha_k = \frac{C_p}{2} [(\frac{\sigma_{k-1}}{\sigma_k})^\kappa - 1]$, and $\beta_k = \frac{C_p}{2} [1 - (\frac{\sigma_{k+1}}{\sigma_k})^\kappa]$.

Table 1. Number of unknowns for the primitive equation model as a function of truncation wavenumber and vertical grid number. They can be counted as the sum of $\frac{M(M+1)}{2}K$ for ψ , $\frac{(M+1)(M+2)}{2}K - K$ for χ , $\frac{(M+1)(M+2)}{2}K$ for T , $\frac{(M+1)(M+2)}{2} - 1$ for π , and $\frac{(M+1)(M+2)}{2}(K-1)$ for $\dot{\sigma}$, where M and K represent truncation wavenumber and number of vertical grids, respectively. In this count, $\chi_{0,0}$ and $\pi_{0,0}$ are excluded.

no. of grids	T9	T11	T13	T15	T17	T19	T21
4	835	1195	1619	2107	2659	3275	3955
6	1253	1793	2429	3161	3989	4913	5933
8	1671	2391	3239	4215	5319	6551	7911

Surface wind speed is assumed to be constant at 5 m/sec in the computation of surface drag. This is because the spectral expansion is impossible if the model surface wind is used. We will mention a possible method for the formulation of the surface flux in Section 6, although we do not think that there are significant differences in practice. Further, ψ and χ at the surface are assumed to be 0.5 times the values at $\sigma=0.875$. Finally, the other constants values used in this model are as follows; $\nu_H=5 \times 10^{17}$ m²/sec, $\nu_V=10$ m²/sec, $C_D=0.0025$, and $\mu=1/(8 \text{ days})$.

b. Method for obtaining stationary solutions

The dependent variables are ψ , χ , T and π in Method C, while $\dot{\sigma}$ is also the dependent variable in Method B, in addition to these variables. There is another method in which geopotential Φ is added to the dependent variable (Schneider, 1989). In that method, since the variables are only coupled to nearest-neighbors, except for the surface pressure, computer memory is saved and a higher-resolution model can be constructed. In this paper, however, that method will not be mentioned further.

Table 1 shows how the number of the unknowns, N , changes with truncation wavenumber and number of vertical grids, when Method B is adopted. Although (3)–(10) are expressed in complex numbers, N is counted as the number of the real unknowns, *i.e.*, the sum of the number of the zonal modes and two times the number of the wave modes.

Both Methods B and C have merits and demerits. The number of unknowns is smaller in Method C than in Method B. However, (9) must be used as the equation governing $\dot{\sigma}$ in Method C, while either (9) or (10) can be used in Method B. As understood from the comparison of (9) and (10), the Jacobian matrix for (9) will be highly complicated. That is, the programming for (9) is much more difficult than for (10). The computation time was almost the same for the two methods. We can choose one of them, according to our taste.

When Method B is adopted, though, we encounter one difficulty as to how the linear stability analysis should be made, because (2) is a diagnostic equation (strictly speaking, (B9) or (10) is used, so $\partial\pi/\partial t$

appears in multiple equations). This problem, however, can be overcome by the following operation: The matrix for calculating the linear stability can be written in a symbolic form as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \lambda \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, \quad (11)$$

where λ is an eigenvalue, A , B , C , D , and F are sub-matrices, and the equation, $C\mathbf{x} + D\mathbf{y} = 0$, represents the diagnostic equation. Then, the matrix should be operated to convert sub-matrix B into a zero matrix so that the part relating to $\mathbf{y}$ is decoupled. For example, in order to make the (i, j) component of B zero, the k -th row of C and D is multiplied by $b_{i,j}/d_{k,j}$ ($d_{k,j} \neq 0$) and subtracted from the i -th row of A and B , and so on. The equation for computing eigenvalues eventually becomes

$$A'\mathbf{x} = \lambda F'\mathbf{x}. \quad (12)$$

Since F' is generally regular, eigenvalues can be obtained from this form.

In order to get stationary solutions by Newton's method, the treatment of mode $\pi_{0,0}$ is one problem. This mode is a very peculiar one; it is non-zero, and changes with time in time integration models, but never affects other modes. This implies that the elements of the row associated with $\pi_{0,0}$ in the Jacobian matrix are all zero, which means that the matrix is singular. Therefore, without excluding $\pi_{0,0}$ from the Jacobian matrix, Newton's method cannot work. Further, since $\chi_{0,0}$ is always zero, it should be eliminated also. After all, we will solve the simultaneous equations except for these two modes. In the linear stability analysis, on the other hand, the equation governing the $\pi_{0,0}$ mode must be added, because $\partial\pi_{0,0}/\partial t$ is not zero.

Although there are several continuation methods for stationary solutions, we adopt the *pseudo-arclength method* (Keller, 1977, 1978; Legras and Ghil, 1985) in this paper. This method can continuously trace stationary solutions by adding one parameter as one unknown, although ordinary continuation methods cannot trace solutions at, for example, limit points, because Jacobian matrices are

singular at those points. The parameter changing the meridional contrast of the radiative equilibrium temperature, γ , will be treated as one of the unknowns (see Eq. (B10)).

Although exact periodic solutions cannot be obtained with current computer facilities, the properties of periodic solutions near Hopf bifurcation points can be examined by using the formulae of Hassard *et al.* (1981). Their formulae describe periods, amplitudes, and the stability of periodic solutions near Hopf bifurcation points by the means of the MacLaurin expansions of a bifurcation parameter. We adopt this method also.

c. Results

First of all, we will show how results change with truncation wavenumber. Figure 1 illustrates the dependence of the $\psi_{0,5}$ mode at $\sigma=0.875$ on γ from T9 to T17. This mode seems to show large differences from one truncation to another, compared with other modes of small wavenumbers. Features of T9 or T11 differ from those of the others, but it is clear that the difference is small for both the solution curves and the bifurcation points among T13, T15, and T17. Truncation at about T15 will give satisfactory results for the present model and parameter values.

Next, the stationary solution of T15 will be examined in detail. Figure 2 shows the amplitudes of the $\pi_{0,2}$ mode as a function of γ . This solution branch is the only branch connecting from the trivial solution under $\gamma=0$ and no tropical heating. Linear stability can also be understood from this figure; S expresses stable solutions and numerals indicate [the number of unstable modes]/2, which means that it is a Hopf bifurcation point where numerals increase. The number of unstable modes is small, at most 20, compared with the number of stable modes. From this fact, these stationary solutions may play an important role in the time evolution.

The solution reveals complicated features at $1.01 < \gamma < 1.02$, which is shown in Fig. 3. Yet even such complicated bending can be traced by the pseudo-arclength method.

We examined the property of the first Hopf bifurcation point by using the method of Hassard *et al.* (1981). The result shows that the bifurcation is supercritical. This implies that stable periodic solutions connect from this point to the right hand side of this diagram.

Finally, the horizontal structure of the solution at $\gamma=1.0$ will be shown. Figure 4 illustrates the horizontal structure at 375 hPa. We can see clearly that the wavetrains radiating from the heat source in the tropics go through the mid-latitudes, again returning to the tropics.

Fig. 1. Amplitude of the $\psi_{0,5}$ mode at $\sigma=0.875$ as a function of γ from T9 to T17 in the primitive equation model. T9 (—), T11 (---), T13 (---), T15 (— · —), T17 (—○—). Letter B represents bifurcation points. Since all components of the external heating have non-zero amplitudes in this model, symmetry breaking cannot occur. Therefore, bifurcation is always Hopf bifurcation. Only the first four bifurcation points are labelled, because the structure of bifurcation is so complicated that periodic solutions coalesce in stationary solutions, and so on.

4. Stationary and periodic solutions of the quasi-geostrophic model

a. Model

The quasi-geostrophic model treated here is called the linear balance model (Lorenz, 1960; Kikuchi,

Fig. 2. Amplitude of the $\pi_{0,2}$ mode as a function of γ . See further in the text.

Fig. 3. Solution curve of the $\pi_{0,2}$ mode between $1.00 < \gamma < 1.03$.

1969). This model is adopted because it is an intermediate model among a large variety of the quasi-geostrophic models. For more complicated models than this model (*e.g.*, Simmons and Hoskins, 1976), almost the same procedure as the primitive equation model is needed. For simpler models which do not include the effect of advection by divergent winds, all the equations are prognostic ones. No device to treat diagnostic equations is necessary.

The basic equations and those spectral forms are shown in detail in Kikuchi (1969). The equation system is composed of the vorticity equation, the thermodynamic equation and the equation of the thermal wind balance. These can be written as

$$\frac{\partial}{\partial t} \nabla^2 \psi = -J(\psi, \nabla^2 \psi) - J(\psi, f) + \nabla \cdot \left(f \nabla \frac{\partial X}{\partial p} \right) - \nu_H \left(\nabla^4 - \frac{4}{a^4} \right) \nabla^2 \psi + F_\psi, \quad (13)$$

$$\frac{\partial T}{\partial t} = -J(\psi, T) + \frac{p}{R} S \nabla^2 X + \frac{\dot{Q}}{C_p} - \nu_H \nabla^4 T - \mu(T - T_e), \quad (14)$$

$$\nabla^2 T = -\frac{p}{R} \nabla \cdot \left(f \nabla \frac{\partial \psi}{\partial p} \right), \quad (15)$$

respectively, where S shows the static stability and X is derived from the vertical p -velocity, ω , as $\omega = \nabla^2 X$. The other notations are the same as in the primitive equation model.

The number of vertical levels is two. The upper and lower boundaries are 200 hPa and 1000 hPa, respectively. Equation (13) is applied to 400 and 800 hPa, while (14) is applied to 600 hPa. These levels are named level $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, and $2\frac{1}{2}$ from the ground to the top, as in the primitive equation model. The boundary conditions are

$$X = 0 \quad \text{at } p = 200 \text{ hPa}, \quad (16)$$

$$\nabla \cdot (f \nabla X) \approx f_0 \rho_s J(\Phi_s, \psi_1) \quad \text{at } p = 1000 \text{ hPa}, \quad (17)$$

where f_0 represents the Coriolis parameter at a standard latitude.

In order to obtain solutions, it is simple to make a diagnostic equation for X , which will be called the X -equation, instead of the thermodynamic equation. After all, the basic equations are composed

Fig. 4. Horizontal structure of the stationary solution at 375 hPa when $\gamma=1$. Height contours and wind vectors are drawn in the Northern Hemisphere, while vorticity contours and wind vectors are illustrated in the Southern Hemisphere. Contour intervals for the height and vorticity are 20 m and $10^{-6}/\text{sec}$, respectively. Areas with positive values are shaded. The scale for winds is shown in the right-lower part in the figure. The zonal mean fields are subtracted. Note that the maximum of the localized heating is at 90°E over the equator (see (B11)).

of the vorticity equation at levels 1 and 2, the X -equation and the equation of the thermal wind balance at level $1\frac{1}{2}$.

The spectral forms of these equations are as follows;

$$\frac{d\psi_{\alpha,1}}{dt} = r(\alpha, 1) - d_{\alpha}X_{\alpha-\epsilon} - e_{\alpha}X_{\alpha+\epsilon}, \quad (18)$$

$$\frac{d\psi_{\alpha,2}}{dt} = q(\alpha, 2) + d_{\alpha}X_{\alpha-\epsilon} + e_{\alpha}X_{\alpha+\epsilon}, \quad (19)$$

$$\begin{aligned} & (SD_{\alpha} + 2d_{\alpha}e_{\alpha-\epsilon} + 2e_{\alpha}d_{\alpha+\epsilon})X_{\alpha} \\ & + 2d_{\alpha}d_{\alpha-\epsilon}X_{\alpha-2\epsilon} + 2e_{\alpha}e_{\alpha+\epsilon}X_{\alpha+2\epsilon} \\ & = \frac{R}{p_{1\frac{1}{2}}}s(\alpha) + d_{\alpha}\{r(\alpha-\epsilon, 1) - q(\alpha-\epsilon, 2)\} \\ & + e_{\alpha}\{r(\alpha+\epsilon, 1) - q(\alpha+\epsilon, 2)\}, \end{aligned} \quad (20)$$

$$\begin{aligned} T_{\alpha} = \frac{p_{1\frac{1}{2}}}{R} \{ & d_{\alpha}(\psi_{\alpha-\epsilon,2} - \psi_{\alpha-\epsilon,1}) \\ & + e_{\alpha}(\psi_{\alpha+\epsilon,2} - \psi_{\alpha+\epsilon,1}) \}, \end{aligned} \quad (21)$$

where length, time and pressure scales are non-dimensionalized by a , Ω^{-1} and 800 hPa, respectively. Level indices are omitted for X and T , because they are always $1\frac{1}{2}$. Further,

$$\begin{aligned} q(\alpha, k) = \left[-i \sum_{\beta, \gamma} D_{\gamma} J_{\alpha\beta\gamma} \psi_{\beta,k} \psi_{\gamma,k} + i 2m_{\alpha} \psi_{\alpha,k} \right] \\ / D_{\alpha} - \nu_H (D_{\alpha}^2 - 4) \psi_{\alpha,k} + F_{V\alpha,k}, \end{aligned}$$

$$r(\alpha, 1) = q(\alpha, 1) - i \sum_{\gamma} \tau_{\alpha\gamma} \psi_{\gamma,1},$$

$$\begin{aligned} s(\alpha) = -i \sum_{\beta, \gamma} J_{\alpha\beta\gamma} \frac{\psi_{\beta,1} + \psi_{\beta,2}}{2} T_{\gamma} \\ + \frac{\dot{Q}_{\alpha}}{C_p} - \nu_H D_{\alpha}^2 T_{\alpha} - \mu (T_{\alpha} - T_{e\alpha}), \end{aligned}$$

$$d_{\alpha} = \frac{2a_{\alpha}}{D_{\alpha}\Delta p}, \quad e_{\alpha} = \frac{2a_{\alpha+\epsilon}}{D_{\alpha}\Delta p},$$

$$\tau_{\alpha\gamma} = \frac{f_0 \rho_s}{D_{\alpha}\Delta p} \sum_{\beta} \Phi_{s\beta} J_{\alpha\beta\gamma},$$

where $\Delta p = 400$ hPa in the dimensional value. $T_{0,0}$, $X_{0,0}$, $T_{M,M}$, and $X_{M,M}$ cannot be determined from these equations, where M represents the largest wavenumber. However, we do not need the equations for determining these values, because $T_{0,0}$ is allowed to be arbitrary, $X_{0,0}$ is zero, and the $T_{M,M}$ and $X_{M,M}$ modes never affect other modes in the triangular truncation.

The vertical diffusion terms can be expressed as

$$\begin{aligned} F_{V\alpha,1} = & -\frac{g}{\Delta p} \left\{ \rho_s C_D V \left(1 + \frac{p_{\frac{1}{2}} - p_1}{\Delta p} \right) + \frac{\rho_{1\frac{1}{2}}^2 g K_V}{\Delta p} \right\} \psi_{\alpha,1} \\ & + \frac{g}{\Delta p} \left\{ \rho_s C_D V \frac{p_{\frac{1}{2}} - p_1}{\Delta p} + \frac{\rho_{1\frac{1}{2}}^2 g K_V}{\Delta p} \right\} \psi_{\alpha,2}, \end{aligned} \quad (22)$$

Fig. 5. Topography used in this study. Contour intervals are 500 m.

and

$$F_{V\alpha,2} = \left(\frac{\rho_{1\frac{1}{2}} g}{\Delta p} \right)^2 K_V (\psi_{\alpha,1} - \psi_{\alpha,2}). \quad (23)$$

In this formulation, ψ at the surface is determined by the extrapolation from ψ_1 and ψ_2 .

It is assumed that there is no external heating, *i.e.*, $\dot{Q}=0$, which is different from the primitive equation model, because the quasi-geostrophic model does not work well in the tropics. Instead, the topography which is shown in Fig. 5 is introduced. The formulation of the radiative equilibrium temperature is the same as (B10). $T_{e0,0}$ is not necessary in this equation system, and $T_{e0,2} = -20$ K. The other modes have no amplitudes. Constants S and f_0 are set as $3 \times 10^{-6} \text{ m}^4 \text{ sec}^2 / \text{kg}$, and $2\Omega \sin 45^\circ$, respectively. The others are the same as in the primitive equation model.

b. Methods for obtaining stationary and periodic solutions

Method *C* must be adopted, as was seen above, because it is difficult to obtain periodic solutions by Method *B*. The dependent variables are, therefore, ψ only. Rewriting (18)–(21) symbolically as

$$\frac{\partial \psi}{\partial t} = F(\psi, X),$$

$$AX = G(\psi, T),$$

and

$$T = H(\psi),$$

Table 2. Number of unknowns for the two-level quasi-geostrophic model as a function of truncation wavenumber. The dependent variables are ψ only.

T9	T11	T13	T15	T17	T19	T21
90	132	182	240	306	380	462

the Jacobian matrix in this system can be calculated as

$$\frac{\partial F}{\partial \psi} + \frac{\partial F}{\partial X} A^{-1} \left(\frac{\partial G}{\partial \psi} + \frac{\partial G}{\partial T} \frac{\partial H}{\partial \psi} \right). \quad (24)$$

Table 2 shows the dependence of the number of the unknowns (not complex numbers but real numbers) on truncation wavenumber.

The method for obtaining stationary solutions is the same as in the primitive equation model.

Several software packages to obtain periodic solutions have been designed. Generally speaking, there are two algorithms. One is called the finite difference method, and the other is the shooting method, which is further divided into two algorithms, the single shooting method and the multiple shooting method. Among software packages based on the finite difference, single shooting, and multiple shooting methods, for example, AUTO (Doedel, 1986), DERPER (Kubíček and Marek, 1983), and PERIOD (Deuffhard, 1984), respectively, are well known. Of these methods, the finite difference method easily yields periodic solutions, even when they are strongly unstable, but needs much memory. On the other hand, the shooting method needs less memory, but in the single shooting method, computations frequently diverge when the instability is strong. The multiple shooting method needs a lot of computation time.

Here we adopt AUTO. Its details are described in Doedel (1986), therefore we will only briefly explain the algorithm. Take an autonomous system of N dimensions,

$$dx/dt = f(x, \lambda), \quad (25)$$

where λ is a bifurcation parameter. Scaling this by the period T , we obtain

$$dx/dz = T \cdot f(x, \lambda) \quad 0 \leq z \leq 1, \quad (26)$$

$$x(0) = x(1). \quad (27)$$

Note that T is also one of the unknowns (strictly speaking, λ is also one of the unknowns, but it will not be described for simplicity). Since the continuation method is also used here, let us consider that we proceed to a new solution (x_k, T_k) at $\lambda = \lambda_k$ from a solution (x_{k-1}, T_{k-1}) at $\lambda = \lambda_{k-1}$, which has already

been obtained. However, Eq. (26) with the boundary condition (27) does not uniquely specify x and T , since $x(z)$ can be translated freely in time. Conditions for removing this indeterminacy are called *phase conditions*. Although there can be various *phase conditions*, AUTO adopts the following equation;

$$\int_0^1 (x_k(z) - x_{k-1}(z))^T \frac{dx_k(z)}{dz} dz = 0, \quad (28)$$

where superscript T expresses the transpose. This equation means that for the two solution points (x_k, T_k) and (x_{k-1}, T_{k-1}) the distance between x_k and x_{k-1} is minimized with respect to translation. The equation system (26) and (28), subject to the boundary condition (27), forms a proper two-point boundary value problem, with $(N+1)$ unknowns and $(N+1)$ equations. Then, we make the finite difference of x with respect to z , obtaining a large number of simultaneous equations in which the unknowns are $x_{k,\tau}$ ($\tau=1$ to τ_{max}) and T , where subscript τ is time level. The standard grid number with respect to time (τ_{max}) is 80 in AUTO. Therefore, the number of the unknowns can be estimated as $80N+1$, while the number of the elements of the Jacobian matrix which must be retained can be counted as about $400N^2$, using this package. Then, this system can be numerically solved under an initial guess which is made from the old solution $(x_{k-1,\tau}, T_{k-1})$.

A different method is necessary at Hopf bifurcation points, because there is no periodic solution which has already been obtained. AUTO makes an initial guess for a periodic solution from the eigenvalues and eigenfunctions of the Jacobian matrix at the bifurcation point. Then, the first periodic solution can be obtained on this periodic solution branch.

The stability of periodic solutions is automatically examined using this package. This is done by calculating *Floquet multipliers* (see, e.g., Guckenheimer and Holmes, 1983). Period doubling bifurcations and bifurcations to invariant tori can also be detected.

c. Results

As in the primitive equation model, we will examine how fine a resolution is necessary. Figure 6 shows the dependence of the stationary solutions for the $\psi_{0,5}$ mode at level 1 on γ . The curves of T15 and T17 are almost the same, including the locations of the bifurcation points. Truncation of about T15 will also give good results in this model.

Figure 7 illustrates the horizontal structure at 400 hPa for the T15 model. The value of γ is 1.2, which corresponds to the winter situation. This figure shows results similar to observations, as understood from the fact that there are troughs on the lee sides of the Tibetan Plateau and the Rocky Mountains, although this solution is unstable as seen from

Fig. 6. Amplitude of the $\psi_{0,5}$ mode at $p=800$ hPa as a function of γ from T9 to T17 in the quasi-geostrophic model. T9 (—), T11 (---), T13 (-·-·-), T15 (- - - -), T17 (—). Letter B represents bifurcation points. Here also, bifurcation is always Hopf bifurcation.

Fig. 6.

Figure 8 shows stationary and periodic solutions for the T11 model as a function of γ , although only one periodic solution branch bifurcating from the first bifurcation point is drawn. Inspecting the linear stability of the periodic solutions, we can see that

Fig. 7. Horizontal structure of the stream function at 400 hPa when $\gamma=1.2$. The contour interval is $2 \times 10^7 \text{ m}^2/\text{sec}$.

they are stable just after the bifurcation, but they become unstable at γ less than 1.0. A torus appears at this bifurcation point.

These bifurcation diagrams are interesting and will give much information about the property of the model, but the object of this paper is just to show several examples resulting from the present method. Further investigation will follow in the near future.

5. Applications to other problems

By using stationary solutions or Jacobian matrices of the primitive equation model, there are various applications to other problems, in addition to the problems mentioned in the Introduction.

First, large-scale phenomena which may be considered as the realization of 'multiple equilibria' will be examined as to whether they are true or not. One example is the result of Yoden *et al.* (1987). Investigating the zonal mean flow in winter Southern Hemisphere, they concluded that two preferred patterns, *i.e.*, single jet and double jet patterns exist. They thought that this phenomenon may be a result of 'multiple equilibria'. Utilizing our method, we can obtain stationary solutions of the model by which the two preferred patterns can be simulated. Thus, it will be concluded whether their hypothesis is true or not.

The second problem is the role of non-stationarity. For instance, it is known that, comparing the mid-latitude response to tropical heating in general circulation models with that in linear stationary models, the amplitude of the former is larger than that

Fig. 8. Periodic solution branch ($\circ$) as well as stationary solution branch (solid and dashed line) of $\psi_{0,1}$ at $p=800 \text{ hPa}$ for T11. Only one periodic solution branch bifurcating from the first Hopf bifurcation point is illustrated. Mark $\circ$ shows the largest value of each periodic solution. The upper panel illustrates the period of the periodic solution. Units are non-dimensionalized.

of the latter (*e.g.*, Nigam *et al.*, 1986). This seems curious, because transient components have a tendency to destroy stationary components in observations (*e.g.*, Lau, 1979) and many models, where stationary components are defined as time mean fields and transient components as deviations from the stationary components. Kasahara (1991) tackled this problem from the view of the interaction of stationary components with baroclinic waves, and pointed out that baroclinic waves result in the amplification of Rossby wavetrains. However, since his model is linear, only the interaction at the initial stage is treated. A method of studying this problem systematically is to construct a non-linear stationary model and to investigate the difference between it and a corresponding time integration model. When the vertical shear in the mid-latitude is weak, the model atmosphere is stable, and solutions are stationary. Then, as the vertical shear increases, stationary solutions become unstable, and behavior will be different between the stationary and time integration models. Investigating what happens at this point, we will be able to answer this problem. The

present method can be directly applied to such problems.

Next, using the Jacobian matrix constructed in the present paper, we can obtain linear stationary solutions of the primitive equation model for arbitrary basic states. This is essentially the same as the model described in Appendix b in Ting and Held (1990).

The linear stability of basic states as eigenvalue problems has been studied for the quasi-geostrophic model (*e.g.*, Simmons *et al.*, 1983). On the other hand, for the primitive equation model, it has been investigated as initial value problems (*e.g.*, Simmons and Hoskins, 1976), although the eigenvalue problem was adopted for a simplified primitive equation model (Tanaka and Kung, 1989). By using the Jacobian matrix in this paper, linear stability analyses for arbitrary basic states are possible as eigenvalue problems, even for the general primitive equation model.

6. Conclusions and future outlook

Using the spectral model in which non-linear terms are calculated by the interaction coefficient method, stationary solutions of the primitive equation model and stationary and periodic solutions of the quasi-geostrophic model can be obtained with a resolution which simulates the real atmosphere. Also, a linear stability analysis for these solutions can be made. It can be said that we now possess a tool to investigate 'order in chaos'.

Although the above examples have shown only stationary solutions and periodic solutions, it is easy to obtain non-stationary local minimum points and pseudo-periodic solutions. This can be done just by relaxing the convergence criteria in Newton's method, after obtaining one solution on each solution branch by using methods for non-linear least squares, such as the Levenberg-Marquardt method (Marquardt, 1963).

One of the future developments is to incorporate various physical processes. The first one will be the incorporation of surface fluxes. In the bulk method, the product of surface wind speeds, V , times other variables must be treated. Its spectral expression is not possible. The present paper therefore simplifies or neglects these terms. However, if the grid values of V and the other variables associated with the surface flux are introduced as dependent variables, the surface flux can be rigorously expressed in the sense of the bulk method. In this system, therefore, dependent variables are composed of the spectral coefficients described above and these grid values at the lowest level. Also the equations expressing the relation between the spectral coefficients and the grid values are added. In short, the transform method is introduced in the boundary layer. Further, if cu-

mulus convection is related to σ at the top of the boundary layer, it is possible to treat the effect of cumulus convection in a similar manner. In these formulations, since the grid variables are needed at boundary layer levels only, the total number of the unknowns does not increase so much. Therefore, these developments will be possible in the near future.

Kikuchi (1969) made an early attempt to incorporate the effect of radiation. His method was based on the linearization of the radiative transfer equation. Various modifications will be possible. This is also a significant future problem.

Another important task will be the development of algorithms for saving CPU time and memory. Various algorithms for time integration have been proposed. We must study similar problems for obtaining stationary and periodic solutions.

Acknowledgements

I would like to express my appreciation to E. Doedel for providing his package AUTO and kindly answering my many questions. I am also indebted to H. Mukougawa for discussing his recent research. Thanks are extended to M. Kimoto and J. Mizushima for discussing this subject. The original manuscript was greatly improved by many appropriate comments from anonymous referees.

The computations were performed on the FACOM VP-2000 at the Data Processing Center of Kyoto University and Data General MV40000 at the Information Processing Center of Wakayama University. This work was supported by a Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture.

Appendix

A. Real and imaginary parts of nonlinear terms

Since it is desirable that calculations are as effective as possible in large systems, complex numbers should not be treated. Real and imaginary parts are calculated separately. It is no problem to divide linear terms into real and imaginary parts. For non-linear terms, further explanations are necessary. Take the case of non-linear terms $A \cdot B$ which yield mode α ($m_\alpha \geq 0$). This can be written as $\sum_{\beta, \gamma} C_{\alpha\beta\gamma} A_\beta B_\gamma$, where $m_\alpha = m_\beta + m_\gamma$. In this calculation, the summation by m_β or m_γ must be taken from $-M$ to M , which means that excessive variables, *i.e.*, variables with negative m , are used. Since calculations in this form are inefficient, it should be transformed into the form in which m_β and m_γ are positive or zero, utilizing the relation $A_{\beta^*} = A_\beta^*$, $C_{\alpha\beta^*\gamma} = C_{\gamma\beta\alpha}$, *etc.*;

$$\sum_{m_\beta=-M}^M \sum_{m_\gamma=-M}^M C_{\alpha\beta\gamma} A_\beta B_\gamma$$

$$\begin{aligned}
&= \sum_{m_\beta=0}^M \sum_{m_\gamma=0}^M C_{\alpha\beta\gamma} A_\beta B_\gamma \\
&\quad + \sum_{m_\beta=-1}^{-M} \sum_{m_\gamma=0}^M C_{\alpha\beta\gamma} A_\beta B_\gamma \\
&\quad + \sum_{m_\beta=0}^M \sum_{m_\gamma=-1}^{-M} C_{\alpha\beta\gamma} A_\beta B_\gamma \\
&= \sum_{m_\beta=0}^M \sum_{m_\gamma=0}^M C_{\alpha\beta\gamma} A_\beta B_\gamma \\
&\quad + \sum_{m_\beta=1}^M \sum_{m_\gamma=0}^M C_{\alpha\beta\gamma} A_\beta^* B_\gamma \\
&\quad + \sum_{m_\beta=0}^M \sum_{m_\gamma=1}^M C_{\alpha\beta\gamma} A_\beta B_\gamma^* \\
&= \sum_{m_\beta=0}^M \sum_{m_\gamma=0}^M C_{\alpha\beta\gamma} A_\beta B_\gamma \\
&\quad + \sum_{m_\beta=1}^M \sum_{m_\gamma=0}^M C_{\gamma\beta\alpha} A_\beta^* B_\gamma \\
&\quad + \sum_{m_\beta=0}^M \sum_{m_\gamma=1}^M C_{\beta\alpha\gamma} A_\beta B_\gamma^*, \quad (A1)
\end{aligned}$$

where the summation by n_β and n_γ is omitted for simplicity. Then, the real and imaginary parts can be derived as follows;

$$\begin{aligned}
&(C_{\alpha\beta\gamma} + C_{\gamma\beta\alpha} + C_{\beta\alpha\gamma}) a_r b_r \\
&\quad + (-C_{\alpha\beta\gamma} + C_{\gamma\beta\alpha} + C_{\beta\alpha\gamma}) a_i b_i \\
&\equiv C1_{\alpha\beta\gamma} a_r b_r + C2_{\alpha\beta\gamma} a_i b_i, \quad (A2)
\end{aligned}$$

$$\begin{aligned}
&(C_{\alpha\beta\gamma} - C_{\gamma\beta\alpha} + C_{\beta\alpha\gamma}) a_i b_r \\
&\quad + (C_{\alpha\beta\gamma} + C_{\gamma\beta\alpha} - C_{\beta\alpha\gamma}) a_r b_i \\
&\equiv C3_{\alpha\beta\gamma} a_i b_r + C4_{\alpha\beta\gamma} a_r b_i, \quad (A3)
\end{aligned}$$

where a_r (b_r) and a_i (b_i) are the real and imaginary parts of A_β (B_γ), respectively. When m_β (m_γ) is zero, the second (third) term in each parenthesis should be eliminated. For the other non-linear terms, $\nabla A \cdot \nabla B$ and $J(A, B)$, the procedure is the same.

B. Basic equations of the primitive equation model

The vertical coordinate is σ ($=p/p_s$), where p represents pressure and subscript s indicates surface values. The primitive equation system contains four prognostic equations; the vorticity equation, the divergence equation, the thermodynamic equation, and the tendency equation for the surface pressure. They are

$$\frac{\partial}{\partial t} \nabla^2 \psi = -J(\psi, \nabla^2 \psi) - \nabla \chi \cdot \nabla (\nabla^2 \psi)$$

$$\begin{aligned}
&-\nabla^2 \psi \nabla^2 \chi - J(\psi, f) - \nabla \cdot (f \nabla \chi) \\
&-\dot{\sigma} \frac{\partial \nabla^2 \psi}{\partial \sigma} - J\left(\dot{\sigma}, \frac{\partial \chi}{\partial \sigma}\right) \\
&-\nabla \dot{\sigma} \cdot \nabla \frac{\partial \psi}{\partial \sigma} - J(RT, \pi) \\
&-\nu_H \left(\nabla^4 - \frac{4}{a^4}\right) \nabla^2 \psi + F_\psi, \quad (B1)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t} \nabla^2 \chi &= -\nabla^2 \left(\frac{\nabla \psi \cdot \nabla \psi}{2} \right) - \nabla^2 \left(\frac{\nabla \chi \cdot \nabla \chi}{2} \right) \\
&-\nabla^2 J(\psi, \chi) + \nabla \psi \cdot \nabla (\nabla^2 \psi) \\
&-J(\chi, \nabla^2 \psi) + (\nabla^2 \psi)^2 - J(\chi, f) \\
&+\nabla \cdot (f \nabla \psi) - \dot{\sigma} \frac{\partial \nabla^2 \chi}{\partial \sigma} + J\left(\dot{\sigma}, \frac{\partial \psi}{\partial \sigma}\right) \\
&-\nabla \dot{\sigma} \cdot \nabla \frac{\partial \chi}{\partial \sigma} - RT \nabla^2 \pi - \nabla (RT) \cdot \nabla \pi \\
&-\nabla^2 \Phi - \nu_H \left(\nabla^4 - \frac{4}{a^4}\right) \nabla^2 \chi + F_\chi, \quad (B2)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial T}{\partial t} &= -J(\psi, T) - \nabla \chi \cdot \nabla T - \dot{\sigma} \frac{\partial T}{\partial \sigma} \\
&+\frac{\kappa T \dot{\sigma}}{\sigma} - \kappa T \frac{\partial \dot{\sigma}}{\partial \sigma} - \kappa T \nabla^2 \chi \\
&+\frac{\dot{Q}}{C_p} - \nu_H \nabla^4 T - \mu(T - T_e), \quad (B3)
\end{aligned}$$

$$\frac{\partial \pi}{\partial t} = \int_0^1 [-J(\psi, \pi) - \nabla \chi \cdot \nabla \pi - \nabla^2 \chi] d\sigma, \quad (B4)$$

respectively, where ψ , χ , T , and Φ denote stream function, velocity potential, temperature and geopotential, respectively, and $\pi = \ln p_s$. The other constants are as follows; f the Coriolis parameter, $\dot{Q}$ the heating rate, C_p the specific heat at constant pressure, R the gas constant for air, T_e the radiative equilibrium temperature, ν_H the horizontal diffusion coefficient, μ the Newtonian heating/cooling rate, a the radius of the earth, and $\kappa = R/C_p$. Further, F represents the vertical diffusion term. This term for physical quantity A can be expressed as

$$F_A = \frac{g}{p_s} \frac{\partial \tau_A}{\partial \sigma}, \quad (B5)$$

where $\tau_A = -\rho_s C_D V A$ at the surface, and $\tau_A = (\rho^2 g / p_s) \nu_V (\partial A / \partial \sigma)$ at other levels. Here ρ indicates air density, C_D drag coefficient, V wind speed at the surface, g gravity acceleration, and ν_V vertical diffusion coefficient.

The hydrostatic equation is necessary for obtaining the geopotential Φ . This can be written as

$$\frac{\partial \Phi}{\partial \sigma} = -\frac{RT}{\sigma}, \quad (B6)$$

$$\frac{\partial \Phi}{\partial \sigma^\kappa} = -C_p T \sigma^{-\kappa}. \quad (B7)$$

Equation (B6) is applied only to the lowest layer, while (B7) is applied to other layers.

The upper and lower boundary conditions are $\dot{\sigma} = 0$ at $\sigma = 0$ and 1. These boundary conditions, together with the continuity equation, give $\dot{\sigma}$ at arbitrary levels as follows;

$$\dot{\sigma} = - \int_0^\sigma \left[\frac{\partial \pi}{\partial t} + J(\psi, \pi) + \nabla \chi \cdot \nabla \pi + \nabla^2 \chi \right] d\sigma. \quad (\text{B8})$$

This alternative is

$$\frac{\partial \dot{\sigma}}{\partial \sigma} = - \frac{\partial \pi}{\partial t} - J(\psi, \pi) - \nabla \chi \cdot \nabla \pi - \nabla^2 \chi. \quad (\text{B9})$$

As stated in Section 2, the model will be designed simply. Four levels are taken in the vertical at the same interval. The levels where $\dot{\sigma}$ is defined are $\sigma = 1.00, 0.75, 0.50, 0.25$, and 0, which are named level $\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}$, and $4\frac{1}{2}$, respectively. On the other hand, the other variables are defined at $\sigma = 0.875, 0.625, 0.375$, and 0.125, which are called level 1, 2, 3, and 4, respectively. It is assumed that the surface is flat and there is no heat flux between the air and the earth's surface. The radiative equilibrium temperature has no gradient in the zonal direction. It is possible without any problem that the surface topography and the zonal variation of the radiative equilibrium temperature are taken into account in this model. The formulation of the heat flux between the air and the earth's surface is mentioned in Section 6.

Concerning the radiation, as was seen in (B3), the Newtonian heating/cooling scheme is adopted. The radiative equilibrium temperature can be written as

$$T_e(\phi, \sigma) = [T_e](\sigma) + \gamma \overline{T_e}(\phi, \sigma), \quad (\text{B10})$$

where $[T_e]$ expresses averages on the σ surface, $\overline{T_e}$ deviations from $[T_e]$, and γ a parameter. By changing γ , various meridional gradients of the temperature, *i.e.*, various strengths of the zonal flow, can be produced. The radiative equilibrium temperature used in this study is presented in Table B.

Tropical heating is given as an external parameter in the following form;

$$\dot{Q}/C_p = q(\sigma)G(\phi, \lambda), \quad (\text{B11})$$

where

$$q(\sigma) = \begin{cases} q_0 \sin(\pi_0 \sigma_D) \exp(-\sigma_D) & \text{when } \sigma_U < \sigma < \sigma_L \\ 0 & \text{when } \sigma \leq \sigma_U \text{ or } \sigma \geq \sigma_L, \end{cases}$$

$$G(\phi, \lambda) = \begin{cases} \cos \frac{\pi_0 \phi}{30^\circ} \cos \frac{\pi_0 (\lambda - 90^\circ)}{60^\circ} & \text{when } |\phi| < 15^\circ \text{ and } 60^\circ < \lambda < 120^\circ \\ 0 & \text{otherwise,} \end{cases}$$

Table B. Radiative equilibrium temperature (K) used in the primitive equation model. Other components are zero.

level (σ)	0.125	0.375	0.625	0.875
$T_{e0,0}$	220	238	258	277
$T_{e0,2}$	0	-6	-14	-22

$q_0 = 6.4$ K/day, $\sigma_D = (\sigma - \sigma_U)/(\sigma_L - \sigma_U)$, and σ_U and σ_L are 0.15 and 0.95, respectively.

Further, the Rayleigh friction of time constant 15 days is applied to the uppermost level for (B1) and (B2), although the terms are not presented in these equations. This process is incorporated to weaken the reflection from the upper boundary, and may be physically considered as a substitute for the wave propagation into the stratosphere.

References

- Abraham, R.H. and C.D. Shaw, 1992: *Dynamics—The Geometry of behavior*. Addison-Wesley, 642 pp.
- Cehelsky, P. and K.K. Tung, 1987: Theories of multiple equilibria and weather regimes—A critical reexamination. Part II: Baroclinic two-layer models. *J. Atmos. Sci.*, **44**, 3282–3303.
- Charney, J.G. and J.G. DeVore, 1979: Multiple flow equilibria in the atmosphere and blocking. *J. Atmos. Sci.*, **36**, 1205–1216.
- Charney, J.G. and D.M. Straus, 1980: Form-drag instability, multiple equilibria and propagating planetary waves in baroclinic, orographically forced, planetary wave systems. *J. Atmos. Sci.*, **37**, 1157–1176.
- Deuffhard, P., 1984: Computation of periodic solutions of nonlinear ODEs. *BIT*, **24**, 456–466.
- Doedel, E.J., 1986: *AUTO: Software for continuation and bifurcation problems in ordinary differential equations*. Department of Applied Mathematics, California Institute of Technology, 226 pp.
- Ellsaesser, H.W., 1966: Expansion of hemispheric meteorological data in antisymmetric spherical harmonic (Laplace) series. *J. Appl. Meteor.*, **5**, 263–276.
- Ghil, M. and K. Mo, 1991: Intraseasonal oscillations in the global atmosphere. Part I: Northern Hemisphere and tropics. *J. Atmos. Sci.*, **48**, 752–779.
- Guckenheimer, J. and P. Holmes, 1983: *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Applied Mathematical Sciences, Vol. 42, Springer-Verlag, 453 pp.
- Hassard, B.D., N.D. Kazarinoff and Y.-H. Wan, 1981: *Theory and Applications of Hopf bifurcation*, Cambridge University Press, 311 pp.
- Jin, F.-F. and M. Ghil, 1990: Intraseasonal oscillations in the extratropics: Hopf bifurcation and topographic instabilities. *J. Atmos. Sci.*, **47**, 3007–3022.
- Kasahara, A., 1991: Transient response of planetary waves to tropical heating: Role of baroclinic instability. *J. Meteor. Soc. Japan*, **69**, 293–309.
- Keller, H.B., 1977: Numerical solution of bifurcation and nonlinear eigenvalue problems. *Applications of Bifurcation Theory*, P.H. Rabinowitz, Ed., Academic Press, 359–384.

- Keller, H.B., 1978: Global homotopies and Newton methods. *Recent Advances in Numerical Analysis*, C. de Boor and G.H. Golub, Eds., Academic Press, 73–94.
- Kikuchi, Y., 1969: Numerical simulation of the blocking process. *J. Meteor. Soc. Japan*, **47**, 29–54.
- Kikuchi, Y., 1972: Spectral methods. *Meteor. Res. Note*, **110**, 83–105 (in Japanese).
- Kimoto, M., 1989: Multiple flow regimes in the Northern Hemisphere winter. Ph. D. Thesis, University of California, Los Angeles, 210pp.
- Kubíček, M. and M. Marek, 1983: *Computational methods in bifurcation theory and dissipative structures*. Springer Verlag, 243 pp.
- Lau, N.-C., 1979: The observed structure of tropospheric stationary waves and the local balance of vorticity and heat. *J. Atmos. Sci.*, **36**, 996–1016.
- Legras, B. and M. Ghil, 1985: Persistent anomalies, blocking and variations in atmospheric predictability. *J. Atmos. Sci.*, **42**, 433–471.
- Lorenz, E.N., 1960: Energy and numerical weather prediction. *Tellus*, **12**, 364–373.
- Marquardt, D.W., 1963: An algorithm for least squares estimation of nonlinear parameters. *SIAM J. Appl. Math.*, **11**, 431–441.
- Mukougawa, H., 1988: A dynamical model of “quasi-stationary” states in large-scale atmospheric motions. *J. Atmos. Sci.*, **45**, 2868–2888.
- Nigam, S., I.M. Held and S.W. Lyons, 1986: Linear simulation of the stationary eddies in a general circulation model. Part I: The no-mountain model. *J. Atmos. Sci.*, **43**, 2944–2961.
- Orszag, S.A., 1970: Transform method for the calculation of vector-coupled sums: Application to the spectral form of the vorticity equation. *J. Atmos. Sci.*, **27**, 890–895.
- Reinhold, B.B. and R.T. Pierrehumbert, 1982: Dynamics of weather regimes: quasi-stationary waves and blocking. *Mon. Wea. Rev.*, **110**, 1105–1145.
- Reinhold, B.B. and R.T. Pierrehumbert, 1985: Corrigendum to “Dynamics of weather regimes: quasi-stationary waves and blocking”. *Mon. Wea. Rev.*, **113**, 2055–2056.
- Schneider, E.K., 1989: A method for direct solution of a steady linearized spectral general circulation model. *Mon. Wea. Rev.*, **117**, 2137–2141.
- Silberman, I., 1954: Planetary waves in the atmosphere. *J. Meteor.*, **11**, 27–34.
- Simmons, A.J. and B.J. Hoskins, 1976: Baroclinic instability on the sphere: Normal modes of the primitive and quasi-geostrophic equations. *J. Atmos. Sci.*, **33**, 1454–1477.
- Simmons, A.J., J.M. Wallace and G.W. Branstator, 1983: Barotropic wave propagation and instability, and atmospheric teleconnection patterns. *J. Atmos. Sci.*, **40**, 1363–1392.
- Tanaka, H.L. and E.C. Kung, 1989: A study of low-frequency unstable planetary waves in realistic zonal and zonally varying basic states. *Tellus*, **41A**, 179–199.
- Thompson, J.M.T. and H.B. Stewart, 1986: *Nonlinear Dynamics and Chaos*. John Wiley and Sons, 376 pp.
- Ting, M. and I.M. Held, 1990: The stationary wave response to a tropical SST anomaly in an idealized GCM. *J. Atmos. Sci.*, **47**, 2546–2566.
- Yoden, S., M. Shiotani and I. Hirota, 1987: Multiple planetary flow regimes in the Southern Hemisphere. *J. Meteor. Soc. Japan*, **65**, 571–586.

プリミティブ・モデルの定常解と準地衡風モデルの定常解、周期解を求める方法

伊藤久徳

(和歌山大学教育学部)

プリミティブ・モデルの定常解と準地衡風モデルの定常解、周期解を求める方法を提案する。両モデルともスペクトル係数法に基づくスペクトル・モデルとして構築する。非線形項も相互作用係数を用いてスペクトル形で計算する。このことは非線形方程式系をニュートン法によって解くときに必要とされるヤコビアン行列を陽に得るために要請されることである。このことによって現実大気を念頭においた解像度で解を求めることができた。またこれらの解に対する線形安定性も求めることができる。これら一連の計算においては、モデルに診断方程式が含まれることや非線形項が複雑なことから、いくつかの計算上の工夫が必要とされる。種々の問題に対する応用や今後の発展についても議論する。