NOTES AND CORRESPONDENCE

Density Currents in Jet Shear Flows

By Hirotada Kanehisa

Meteorological Research Institute, Tsukuba, Ibaraki, 305 Japan
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Abstract

Density currents in jet shear flows are considered analytically in order to investigate the interaction between density currents and environmental shear. This study is an extension of Benjamin (1968) and Xu (1992), in which the environmental flows are uniform and of uniform shear, respectively. The environmental flows in this note have the same magnitude at both the upper and lower rigid boundaries, and an eastward maximum in the middle level. The result shows that, in spite of the eastwards negative shear in the upper layer, the depth and propagation speed of the density current increase as the shear increases.

1. Introduction

According to recent studies (e.g., Thorpe et al., 1982; Rotunno et al., 1988; Parsons, 1992; Weisman, 1992), the interaction between the cold pool of the outflow and the low-level shear of the environmental flow is of great importance for the longevity of convective systems. The density current mechanism of the interaction, however, has not yet been fully understood.

While Benjamin (1968) considered density currents in uniform environmental flows, Xu (1992) extended his study to a case of environmental flows of uniform shear in order to investigate the influence of the shear on density currents. According to his calculation, the depth and propagation speed of the density current increase (decrease) as the shear increases (decreases). On the other hand, Shapiro (1992) proposed a method to treat environmental flows of non-uniform shear.

In this note, in order to investigate the influence of low-level shear on density currents, jet shear flows are considered analytically, applying the method of Shapiro (1992) to the model of Xu (1992). The environmental flows have the same magnitude at both the upper and lower rigid boundaries, and an eastward maximum in the middle level. The shear is eastwards positive and negative in the lower and upper half layers, respectively. The result obtained is that the depth and propagation speed of the density currents increase as the shear increases, in spite of the eastwards negative shear in the upper layer. This result, together with the result of Xu (1992), shows that it is indeed the low-level shear which influences the behaviour of density currents.

The organization of this note is as follows. In Section 2, the formulation is presented. In Section 3, Bernoulli function is considered. In Section 4, the momentum-pressure balance equation is derived. In Section 5, the increase of the depth and propagation speed of the density currents is demonstrated. In Section 6, concluding remarks are given.

2. Formulation

The governing equation is the 2-dimensional Euler equation,

$$\frac{\partial u_i}{\partial t} + u_j \partial_j u_i = -(1/\rho) \partial_t P - \delta_{i3} g, \quad \partial_t u_i = 0, \quad (2-1)$$

where the subscripts $i,j=3$ and 1 denote the upward and eastward directions, respectively, with the convention of summation,

$$u_j \partial_j u_i = \Sigma_j u_j \partial_j u_i, \quad \partial_t u_i = \Sigma_i \partial_t u_i.$$

$\partial/\partial t$ and $\partial_i$ are temporal and spatial differential operators, respectively. $u_i$ is the velocity, $P$ the pressure, $\rho$ the density, $g$ the gravitational acceleration and $\delta_{ij}$ Kronecker's delta symbol. We consider the following situation (see Fig.1): A cold air mass of constant density $\rho + \Delta \rho$ propagates with a constant eastward velocity in a warm air mass of constant density $\rho$ between horizontal planes on $z=0$, and $z = H$, where $z$ is the upward coordinate.
In a coordinate system moving with the cold air mass, the system is stationary, so Eq. (2-1) becomes
\[ u_i \partial_j u_i = -(1/\rho) \partial_i P - \delta_{ij} g, \quad \partial_i u_i = 0. \] (2-2)

The origin \((x=0, z=0)\) is set at the nose of the cold air mass, where \(x\) is the eastward coordinate. At \(x=-\infty\), the height of the cold air mass is \(h \leq H\). At \(x=+\infty\) and \(-\infty\), \(u_i\) is horizontal, and denoted as \(u_{in}(z)\) and \(u_{out}(z)\), respectively. According to the non-divergence of velocity, there exists a stream function \(\psi\) such that
\[ u_i = -\varepsilon_{ij} \partial_j \psi, \] (2-3)
where \(\varepsilon_{33}=\varepsilon_{11}=0\) and \(\varepsilon_{31}=-\varepsilon_{13}=1\). Since, from Eq. (2-2), the vorticity \(\zeta=\varepsilon_{ij} \partial_j u_i\) is a function of the stream function \(\zeta=F[\psi]\), we consider such a state that \(\zeta\) is proportional to \(\psi\),
\[ \zeta = a^2 \psi, \quad a^2 \text{ is a positive constant.} \] (2-4)

At \(x=\pm \infty\), Eq. (2-4) becomes \(d^2 \psi_{in}(z)/dz^2 = a^2 \psi_{in}(z)\). In this note, we consider such \(u_{in}(z)\) that \(u_{in}(0)=u_{in}(H)=-U_0<0\). Then \(\psi_{in}(z)\) and \(u_{in}(z)\) are determined as
\[ \psi_{in}(z) = -(U/a) \{ \cosh az - \cosh a(z-H) \} \]
\[ / \sinh aH, \]
\[ u_{in}(z) = -U \{ \sinh az - \sinh a(z-H) \} / \sinh aH. \] (2-5)

The above Eq. (2-5) represents an eastward jet-shear inflow. If Eq. (2-4) is replaced by \(\zeta = -a^2 \psi\), where \(a^2\) is positive, then the inflow becomes westwardly jet-sheared. In this note, in order to investigate the effect of the low-level eastward positive shear, we restrict our attention to the case of Eq. (2-4). From Eq. (2-5), we can easily see that

\[ u_{in}(z) \to -U \{ 1 - a^2 z(H-z)/2 \} \text{ as } a \to 0. \]

In particular, \(u_{in}(z)\) becomes uniform \(-U\) for \(a=0\). At \(x=\infty\), Eq. (2-4) becomes \(d^2 \psi_{out}(z)/dz^2 = a^2 \psi_{out}(z)\), and \(\psi_{out}(z)\) can be written as \(\psi_{out}(z) = A \exp(-az) + B \exp(az)\). From the conditions \(\psi_{out}(H) = \psi_{in}(H)\) and \(\psi_{out}(H) = \psi_{in}(0)\), \(\psi_{out}(z)\) is determined as
\[ \psi_{out}(z) = -U \tanh(aH/2) \{ \exp(az) \]
\[ - \exp(aH + az - az) \} \]
\[ / \{ \exp(aH) - \exp(ah) \}. \] (2-6)

From Eq. (2-6), \(u_{out}(z)\) is given by
\[ u_{out}(z) = -U \tanh(aH/2) \{ \exp(az) \]
\[ + \exp(aH + az - az) \} \]
\[ / \{ \exp(aH) - \exp(aH) \}. \] (2-7)

3. Bernoulli function

From Eq. (2-2), the Bernoulli function is a function of the stream function,
\[ u_i u_i/2 + P/\rho + gz = B[\psi]. \] (3-1)

Applying Eq. (3-1) along the stream line from \((X=0, z=H)\) to \((x=-\infty, z=H)\) and assuming \(P_{in}(H)=0\), we obtain
\[ U^2/2 = U^2 \tanh(aH/2) \{ \exp(aH) + \exp(ah) \}^2 \]
\[ /2 \{ \exp(aH) - \exp(ah) \}^2 + P_{out}(H)/\rho. \] (3-2)

On the other hand, applying Eq. (3-1) along the stream line from \((x=0, z=0)\) to \((x=\infty, z=0)\), and using the fact that \(P(x=0, z=0) = P_{out}(x=-\infty, z=0)\), we obtain
\[ P_{\text{out}}(H)/\rho + (\Delta p/\rho)gh = U^2/2. \]  
(3-3)

From Eqs. (3-2) and (3-3), \( U \) is determined as
\[ U = \{2(\Delta p/\rho)gh\}^{1/2} \coth(aH/2)\{\exp(aH) - \exp(ah)\}/\{\exp(aH) + \exp(ah)\}. \]  
(3-4)

4. Momentum-pressure balance

Integrating Eq. (2-2) in the warm and cold air masses, respectively, and using Gauss' theorem yields
\[ \int_W \mathbf{d}S u_{i1} + \int_W \mathbf{d}S (P/\rho + gz) = 0 \quad \text{and} \quad \int_C \mathbf{d}S u_{i2} + \int_C \mathbf{d}S (P/\rho + gz) = 0, \]  
(4-1)
where the contours \( W \) and \( C \) are closed curves encircling the warm and cold air masses counter clockwise, respectively, and \( \mathbf{d}S \) is the line element vector on the contours directing normally outwards. Using the fact that \( \mathbf{d}S u_{i1} = 0 \) along stream lines, Eq. (4-1) are written more concretely as
\[ \int_0^H dz w_{in}(z)^2 + \int_0^h dz w_{out}(z)^2 + \int_0^H dz P_{\text{in}}(z)/\rho + \int_0^H dz P_{\text{out}}(z)/\rho = 0 \]  
and
\[ \int_0^h dz P_{\text{out}}(z)/\rho + \int_\Gamma dz P/\rho = 0, \]
where \( \Gamma \) is the interface between the warm and cold air masses. Eliminating the integral along \( \Gamma \) from the above equations, yields the following momentum-pressure balance equation:
\[ \int_0^H dz w_{in}(z)^2 - \int_h^H dz w_{out}(z)^2 + \int_0^H dz P_{\text{in}}(z)/\rho - \int_0^H dz P_{\text{out}}(z)/\rho = 0. \]  
(4-2)

By using Eqs. (2-5) and (2-7), Eq. (4-2) is calculated as
\[ U^2\{\sinh 2ah/2a - \sinh aH/a + H(\cosh aH - 1)/\sinh^2 aH \} + (\Delta p/\rho)gh^2/2 = 0. \]

Further substituting (3-2) and (3-3), we obtain
\[ D(a, h) = -H^2/2 + \{\sinh 2ah/2a - \sinh aH/a + H(\cosh aH - 1)/\sinh^2 aH \} + [\exp(aH) - \exp(ah)]^2 \times [(H/2 - h/4)\{\exp(aH) + \exp(ah)\}^2 - \exp(2aH) - \exp(2ah)/a - 2(H - h)\exp(aH + ah)] = 0. \]  
(4-3)

5. Increase of the depth and propagation speed

For small \( a \), Eqs. (3-4) and (4-3) are written as
\[ U = U(a^2, h) = \{2(\Delta p/\rho)gh\}^{1/2} \times \{(H - h)/H\}^{1/2} \{1 + a^2h(2H-h)/12\} + 0(a^4), \]  
(5-1)
\[ D(a^2, h) = Hh(2h - H)/4(H - h)^2 \]  
- \[ a^2h^2/24(2H - h)^2 + 0(a^4). \]  
(5-2)

From Eqs. (5-1) and (5-2), we can immediately see that for a uniform flow \( a = 0 \),
\[ h = H/2 \quad \text{and} \quad U = (1/2)\{(\Delta p/\rho)gH\}^{1/2}. \]  
(5-3)
Equation (5-3) recovers the result of Benjamin (1968). Further, we obtain from Eqs. (5-1) and (5-2) the following coefficients:
\[ \partial D(0, H/2)/\partial a^2 = -H^3/48, \]  
\[ \partial D(0, H/2)/\partial h = 1, \]
\[ \partial U(0, H/2)/\partial a^2 = (H^2/32)\{(\Delta p/\rho)gH\}^{1/2}, \]  
\[ \partial U(0, H/2)/\partial h = -(1/2H)\{(\Delta p/\rho)gH\}^{1/2}. \]

By using these coefficients, we finally obtain
\[ [dh/da^2]_{a=0} = H^3/48 > 0, \]  
\[ [dU/da^2]_{a=0} = \{(\Delta p/\rho)gH\}^{1/2} H^2/48 > 0. \]  
(5-4)

6. Conclusion

As was shown by Xu (1992), the depth and propagation speed of density currents increase (decrease) as the magnitude of uniform shear of the environmental flows increases (decreases). In this note, in order to investigate the interaction between density currents and low-level shear, density currents in jet shear flows were considered. The shear of the flows was eastwards positive and negative in the lower and upper half layers, respectively, and was zero on the average. In spite of the eastwards negative shear in the upper layer, the depth and propagation speed increased as the shear increased. This shows that not the total shear but the low-level shear has a primary influence on the behaviour of density currents. This result is consistent with recent studies of mesoscale convective systems (e.g., Rotunno et al., 1988; Weisman, 1992).
References


ジェット型シアを持つ一般流の中の重力流

金久博志
(気象研究所)

重力流と一般流との相互作用を調べる為に、ジェット型シアを持つ一般流の中の重力流を解析的に考察した。Benjamin (1986) と Xu (1992) は各々、一般流が一様流と一様シア流の場合を考察した。このノートはこれらの研究のジェット型シアへの拡張であり、一般流は大気の上端と下端で同じ大きさを持ち中層で東向き極大を持つ。上層での東向き負シアにも拘らず、重力流の厚さと伝播速度はシアの増大と共に増加する事が示される。