Horizontal Buoyancy Flux of Internal Gravity Waves in Vertical Shear

By Timothy J. Dunkerton

Northwest Research Associates, P.O. Box 3027, Bellevue, WA 98009, U.S.A.

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Abstract

The equations of motion describing two-dimensional internal gravity waves were analyzed to derive an expression for the horizontal buoyancy flux—i.e., the correlation between buoyancy (or temperature) and horizontal velocity—in various cases involving vertical shear. It is shown that, although this correlation vanishes at zeroth order for a single wave, its first order contribution is nonzero due to shear, even for steady, conservative, incompressible waves. Departures from steady, conservative motion or quasi-compressibility also cause a nonzero correlation. Several cases were analyzed and some numerical results obtained for waves approaching a critical layer of reduced intrinsic phase speed. With weak shear, the buoyancy flux is small relative to vertical momentum flux, as expected from the perturbation theory. Strong vertical shear enhances the buoyancy flux within the shear zone and causes partial reflection beneath, producing a nonzero correlation (at zeroth order) in this region. These effects may explain recent observations of zonal wind and temperature cospectra in the equatorial lower stratosphere.

1. Introduction

Gravity waves are important for the transport of momentum, heat and constituents in the middle atmosphere. Their vertical, rather than horizontal, transport is emphasized in the literature since the waves grow with height and are able, for example, to transport a relatively small amount of momentum from the lower to upper atmosphere, causing significant mean flow acceleration in the mesosphere (Lindzen, 1981; Fritts, 1984). Little attention has been paid to the horizontal transport of momentum by gravity waves, and even less to the horizontal transport of heat which is formally smaller (by an order of magnitude) in WKB theory.

Whether observations assign a significant role to the horizontal transport is uncertain, but the interpretation of observations may require an understanding of horizontal fluxes. Recently Sato *et al.* (1994) discovered that zonal wind (u') and temperature (T') fluctuations in the equatorial lower stratosphere, with period ~ 1 –3 days, display a remarkable correlation that changes sign according to the phase of the quasi-biennial oscillation (QBO). Equatorial Kelvin waves were previously identified from the quadrature spectrum $Q(u', T'; \omega)$ which has a significant peak at ~ 5 –20 days in the equatorial lower stratosphere (Wallace and Kousky, 1968; Hirota, 1978; Salby *et al.*, 1984; Maruyama, 1991). Re-

cent observations suggest an important contribution to the momentum flux from disturbances with periods below 5 days (Pfister *et al.*, 1993; Maruyama, 1994; Bergman and Salby, 1994). Noting the peculiar behavior of the cospectrum in their data, Sato *et al.* (1994) question whether the observed phase relation between u' and T' is consistent with Kelvin or internal gravity waves.

Sato *et al.* were mainly concerned with the interpretation of short-period disturbances, but it is obvious from their analysis that a similar problem arises with respect to ~ 5 –20 day disturbances traditionally identified as Kelvin waves. At Singapore, the integrated ~ 5 –20 day quadrature $Q(u', T')$ and cospectrum $C(u', T')$ power are greatly enhanced during the onset of the QBO westerly phase at each level.¹ Their ratio is typically 1.5:1. The spikes of ~ 5 –20 day quadrature power are consistent with those of eastward-propagating waves encountering a critical layer in descending QBO westerlies. Meridional velocity fluctuations (v') are uncorrelated (or negligible), supporting a Kelvin-wave interpretation. But the cospectrum of these waves, and that of short-period disturbances, requires explanation.

The purpose of this paper is to discuss the relation between horizontal wind and temperature (or, equivalently, buoyancy) for two-dimensional internal

1 As shown in Fig. 10 of Sato *et al.* (1994). Spectral analysis of Singapore rawinsonde data by the author supports their conclusions.

gravity waves in vertical shear. It is shown theoretically and numerically that the correlation $\overline{u'T'}$ is nonzero and proportional to the shear—even for steady, conservative, incompressible waves. For vertical shears typical of the QBO, the effect of compressibility is small, other than causing the waves to grow with height. Realistic damping likewise does not significantly alter the effect of shear. Numerical solutions demonstrate that superposition of upward and downward propagating waves creates a significant horizontal buoyancy flux beneath the shear zone, consistent with zeroth order wave solutions. Finally, we note that although the horizontal flux is formally nondivergent in the situation examined here, real wavepackets are of finite extent such that the horizontal transport of buoyancy is nonzero.

2. Theoretical results

The equations of motion for two-dimensional, nonrotating, hydrostatic, quasi-compressible, linear perturbations on a zonal-mean flow are

$$D_t u' + w' \bar{u}_z + \phi'_x = X' \quad (2.1a)$$

$$D_t \phi'_z + w' N^2 = Q' \quad (2.1b)$$

$$u'_x + \frac{1}{\rho_0} (\rho_0 w')_z = 0 \quad (2.1c)$$

in the notation of Holton (1975), where u , w are zonal and vertical velocity, ϕ is geopotential, $\rho_0 = \rho_s \exp(-z/H)$ is basic state density, and X , Q represent sources or sinks of momentum and heat. Primed quantities are deviations from the zonal mean, denoted by an overbar; subscripts x , z indicate differentiation, and

$$D_t \equiv \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x}. \quad (2.2)$$

a. Wave action law

Following Andrews and McIntyre (1976), parcel displacements and auxiliary variables may be defined as

$$D_t \xi' = u^\ell \quad (2.3a)$$

$$D_t \zeta' = w' \quad (2.3b)$$

$$D_t q' = Q' \quad (2.3c)$$

where

$$u^\ell = u' + \zeta' \bar{u}_z. \quad (2.4)$$

Assuming that zonally averaged quantities are temporally constant to leading order in wave amplitude, (2.1a–c) may be written

$$D_t u^\ell + \phi'_x = X' \quad (2.5a)$$

$$\phi'_z + \zeta' N^2 = q' \quad (2.5b)$$

$$\xi'_x + \frac{1}{\rho_0} (\rho_0 \zeta')_z = 0. \quad (2.5c)$$

Multiplication of (2.5a,b) by ξ'_x , ζ'_x gives

$$(\overline{\xi'_x u^\ell})_t + (\overline{\xi'_x \phi'_x} + \overline{\zeta'_x \phi'_z}) = \overline{\xi'_x X'} + \overline{\zeta'_x q'} \quad (2.6)$$

noting that

$$\begin{aligned} \overline{\xi'_x D_t u^\ell} &= (\overline{\xi'_x u^\ell})_t - \overline{D_t \xi'_x u^\ell} \\ &= (\overline{\xi'_x u^\ell})_t - \overline{u^\ell_x u^\ell} = (\overline{\xi'_x u^\ell})_t \end{aligned}$$

and $\overline{\zeta'_x \zeta'} = 0$. Substitution of (2.5c) in (2.6) implies that

$$(\overline{\xi'_x u^\ell})_t + \frac{1}{\rho_0} (\overline{\rho_0 \zeta'_x \phi'})_z = \overline{\xi'_x X'} + \overline{\zeta'_x q'}. \quad (2.7)$$

This equation describes conservation of wave action, a quadratic (second-order) quantity in wave amplitude. A three-dimensional version of this equation (in y, z, t) was derived by Andrews and McIntyre (1976):

$$\frac{\partial A}{\partial t} + \frac{1}{\rho_0} \nabla \cdot \mathbf{B} = \mathcal{F} \quad (2.8)$$

where A is wave action density, $\mathbf{B}$ is wave action flux, and $\mathcal{F}$ represents the departure from conservative motion. For steady, conservative waves, $\mathbf{B}$ is nondivergent. The famous theorem of Eliassen and Palm (1961), extended by Andrews and McIntyre, relates the mean flow tendency $\bar{u}_t$ to the divergence of $\mathbf{B}$ and other departures from steady, conservative motion.

The vertical flux in (2.7) is akin to a ‘radiation stress’ describing the impulse to an overlying layer of fluid by pressure forces ϕ' averaged over a wave-distorted fluid interface $z + \zeta'(x, z, t)$. (Refer to Andrews *et al.*, 1987, p. 325 for a diagram.) This stress is nonzero, since the quantities ζ'_x , ϕ' are in phase for internal gravity waves, but it does not vary in height if the waves are steady and conservative.

The radiation stress may be related to the vertical flux of horizontal momentum as

$$\begin{aligned} \overline{\zeta'_x \phi'} &= -\overline{\zeta' \phi'_x} \\ &= \overline{\zeta' (D_t u^\ell - X')} \\ &= (\overline{\zeta' u^\ell})_t - \overline{u^\ell w'} + \frac{1}{2} (\overline{\zeta'^2})_t \bar{u}_z - \overline{\zeta' X'} \end{aligned} \quad (2.9)$$

from which it is apparent that the momentum flux differs from the radiation stress only on account of departures from steady, conservative motion. In general, the components of Eliassen-Palm flux satisfy this type of relationship which—together with the wave action law (2.7) and mean tendency equations—leads to a ‘nonacceleration’ theorem (Andrews and McIntyre, 1976).

b. Correlation of zonal wind and temperature

For hydrostatic waves the temperature perturbation is proportional to ϕ'_z (Holton, 1975). The correlation of zonal wind and temperature is therefore proportional to

$$\rho_0 \overline{u' \phi'_z} = \rho_0 \overline{u' (q' - \zeta' N^2)}$$

$$\begin{aligned}
&= \rho_0 \overline{u'q'} - \frac{\rho_0 N^2}{k^2} \overline{u'_x \zeta'_x} \\
&= \rho_0 \overline{u'q'} - \frac{N^2}{\rho_0 k^2 \hat{c}} (\rho_0 w')_z \rho_0 w' \quad (2.10)
\end{aligned}$$

for each harmonic $\exp i(kx - \omega t)$, where k is zonal wavenumber, ω is frequency, $c = \omega/k$ is phase speed, and $\hat{c} = c - \bar{u}$ is intrinsic phase speed. (Use of the harmonic form is convenient, but unnecessary to the argument as shown at the end of this subsection.) The continuity equation (2.1c) was inserted on the last line of (2.10), together with the vertical displacement (2.3b), *i.e.*,

$$-\hat{c}\zeta'_x = w' \quad (2.11)$$

for steady waves.

A similar expression for $\rho_0 \overline{u'\phi'_z}$ may be derived using the streamfunction

$$-\rho_0 u' = (\rho_0 \psi')_z \quad (2.12a)$$

$$\rho_0 w' = (\rho_0 \psi')_x \quad (2.12b)$$

From (2.11), $-\hat{c}\zeta' = \psi'$ so that

$$\begin{aligned}
\rho_0 \overline{u'\phi'_z} &= \rho_0 \overline{u'(q' - \zeta' N^2)} \\
&= \rho_0 \overline{u'q'} - \frac{N^2}{\rho_0 \hat{c}} (\rho_0 \psi')_z \rho_0 \psi' \quad (2.13)
\end{aligned}$$

which agrees with (2.10) on account of the harmonic form assumed in the derivation.

Apart from a thermal damping term, the correlation of zonal wind and temperature evidently depends on the vertical variation of wave amplitude squared, as determined by $\rho_0 \psi'$ or $\rho_0 w'$:

$$\rho_0 \overline{u'\phi'_z} = -\frac{N^2}{2\rho_0 k^2 \hat{c}} (\rho_0^2 \overline{w'^2})_z + O(q') \quad (2.14)$$

Consider now the special case of a steady, conservative upward-propagating wave in a basic state $(\rho_0, \bar{u}, N)$ independent of height, for which

$$w' = -\frac{ku'}{m} = \frac{k\hat{c}}{N} u' \quad (2.15)$$

where m is vertical wavenumber. Substituting for a factor w' in (2.14),

$$\rho_0 \overline{u'\phi'_z} = -\frac{N^2}{2\rho_0 k^2 \hat{c}} (\rho_0^2 \overline{u'w'} k \hat{c} / N)_z \quad (2.16)$$

The quantity $\rho_0 \overline{u'w'}$ is independent of height by the wave action law (2.7) and (2.9), as are $\rho_0, k, \hat{c}$, and N , so the correlation $\rho_0 \overline{u'\phi'_z}$ vanishes identically. This is obvious from the perfect correlation (or anticorrelation) of u' and ϕ' to begin with, and the fact that ϕ'_z is exactly in quadrature with ϕ' .

Another special case involves the superposition of identical upward and downward waves with unequal

amplitude:

$$\begin{aligned}
\psi' &= \text{Re}[\psi_\uparrow \exp i(kx + mz - \omega t) \\
&\quad + \psi_\downarrow \exp i(kx - mz - \omega t)] \\
&= \text{Re}\psi_\uparrow [\exp i(kx + mz - \omega t) \\
&\quad + R \exp i(kx - mz - \omega t + \delta)] \quad (2.17a)
\end{aligned}$$

$$\overline{\psi'^2} = \frac{1}{2} |\psi_\uparrow|^2 [1 + R^2 + 2R \cos(2mz - \delta)] \quad (2.17b)$$

The correlation $\rho_0 \overline{u'\phi'_z}$ oscillates in the vertical about zero, and if $R = 1$, the momentum flux $\overline{u'w'}$ vanishes. Perfect reflection is analogous to a zonally propagating circulation cell in the $x - z$ plane confined between two rigid plates, such that vertical velocity vanishes on the boundaries. Suppose the propagation is eastward, and the motion adiabatic. Behind the upwelling there is cold air, due to adiabatic cooling, which partially overlaps the eastward (westward) motion at lower (upper) levels required by continuity. Warm air is ahead of the upwelling, overlapping the westward (eastward) motion at lower (upper) levels. The correlation $\overline{u'T'}$ is therefore negative (positive) in the lower (upper) layer, and zero at the midpoint and boundaries. By contrast, equal amounts of positive and negative momentum flux cancel identically in each layer when the zonal average is taken.

The WKB approximation provides a third example:

$$\psi'(x, z, t) = \text{Re} A(Z) \exp i\Phi(x, z, t) \quad (2.18)$$

where $k = \partial\Phi/\partial x$, $m = \partial\Phi/\partial z$, $\omega = -\partial\Phi/\partial t$, and $Z = \mu z$ is a 'slow' vertical coordinate such that A, m vary slowly in height, *i.e.*, as functions of Z . Then

$$\overline{\psi'_x \psi'_z} = \frac{k}{m} \left[\overline{\psi'_x \psi'_z} - \frac{1}{2} \mu k \text{Im}(A^* A') \right] \quad (2.19)$$

This is analogous to the first case, but using a local value of m , together with $O(\mu)$ correction that varies slowly in height. If departures from steady, conservative motion are $O(\mu^2)$ or less, the momentum flux is invariant to this order, and a useful result is

$$\rho_0 \overline{u'\phi'_z} = -\frac{N^2}{2\rho_0 k^2 \hat{c}} (\rho_0 \overline{u'w'}) \cdot (\rho_0 k \hat{c} / N)_z + O(\mu^2) \quad (2.20)$$

In (2.18–20) we have also assumed that the basic state density variation is $O(\mu^2)$, *i.e.*, that the mean flow varies rapidly over a density scale height, as observed in the QBO. Supposing for consistency that variations of N are likewise small, (2.20) reduces to

$$\rho_0 \overline{u'\phi'_z} = \frac{N}{2k\hat{c}} \rho_0 \overline{u'w'} \bar{u}_z + O(\mu^2). \quad (2.21)$$

The correlation between zonal wind and temperature is proportional to vertical shear, an order of magnitude smaller (in μ) than the momentum flux.

For upward waves, the ratio of momentum flux to intrinsic phase speed is positive, so the sign of $\rho_0 \overline{u' \phi'_z}$ in this case is determined by the vertical shear. Upward propagating waves with phase speeds of opposite sign will contribute in the same way to the correlation of zonal wind and temperature, but their momentum fluxes will tend to cancel. For downward waves, the sign of the correlation is unchanged since the sign of m in (2.15), and of $\overline{u' w' / \hat{c}}$, are both reversed.

Under the assumptions leading to (2.21) the same result may be obtained from the WKB approximation noting that $|\psi'| \propto |m|^{-1/2}$, substituting the value of m from (2.15) together with this proportionality into (2.14). If the shear is positive, $\overline{\psi'^2}$ decreases (increases) with height for waves with positive (negative) intrinsic phase speed; the correlation $\rho_0 \overline{u' T'}$ is therefore positive. If the shear is negative, $\overline{\psi'^2}$ increases (decreases) with height for waves with positive (negative) intrinsic phase speed; the correlation is then negative. It is assumed that the sign of $\hat{c}$ is invariant, i.e., that waves cannot propagate through critical levels.

If the unsteadiness, damping and density variation are $O(\mu)$, a more complicated expression for the single upward wave may be obtained from (2.14) by writing

$$\overline{w'^2} = \overline{u' w'} k \hat{c} / N + O(\mu) \quad (2.22)$$

so that

$$\begin{aligned} \rho_0 \overline{u' \phi'_z} &= -\frac{N^2}{2\rho_0 k^2 \hat{c}} (\rho_0^2 \overline{u' w'} k \hat{c} / N)_z \\ &\quad + \rho_0 \overline{u' q'} + O(\mu^2) \\ &= -\frac{N^2}{2k \hat{c}} \overline{u' w'} (\rho_0 \hat{c} / N)_z \\ &\quad - \frac{N}{2k} (\rho_0 \overline{u' w'})_z \\ &\quad + \rho_0 \overline{u' q'} + O(\mu^2) \end{aligned} \quad (2.23)$$

The first term represents variations of the basic state at $O(\mu)$; the second term contains all the departures from steady, conservative motion at $O(\mu)$ implied by (2.7) and (2.9).

Three simple examples demonstrate the effects of density variation, damping, and saturation at $O(\mu)$: 1) If the waves are steady and conservative, and $\bar{u}$, N are constant in height (but not ρ_0), then from (2.14,15)

$$\begin{aligned} \rho_0 \overline{u' \phi'_z} &= -\frac{N^2}{2\rho_0 k^2 \hat{c}} \rho_0 \overline{u' w'} \frac{k \hat{c}}{N} \rho_{0z} + O(\mu^2) \\ &= \frac{N}{2k H} \rho_0 \overline{u' w'} + O(\mu^2) \end{aligned} \quad (2.24)$$

The incompressible value of m was substituted in this equation. 2) If the waves are steady but damped by mechanical and thermal relaxation, with equal coefficients $\alpha_M = \alpha_T = \frac{1}{2}\alpha$, and the basic state is

again constant in height (including ρ_0), then it may be shown from the wave action law (2.7) and zeroth order wave solutions that

$$B(z) = B_0 \exp -P(z) \quad (2.25a)$$

$$P(z) = \int_0^z D(z') dz' \quad (2.25b)$$

$$D(z) = \alpha N / k \hat{c}^2 \quad (2.25c)$$

where $B = \rho_0 \overline{u' w'}$. In this case B decays with height, such that $B_z = -BP_z = -BD$, therefore

$$\begin{aligned} \rho_0 \overline{u' \phi'_z} &= -\frac{N^2}{2\rho_0 k^2 \hat{c}} \rho_0 B_z \frac{k \hat{c}}{N} + \rho_0 \overline{u' q'} + O(\mu^2) \\ &= \frac{N}{2k H_D} \rho_0 \overline{u' w'} + \rho_0 \overline{u' q'} + O(\mu^2) \end{aligned} \quad (2.26)$$

where $H_D = k \hat{c}^2 / N \alpha$ (Dunkerton, 1981). Note the similarity to the previous case, with H replaced by H_D . This formula could easily be extended to include $O(\mu)$ basic state variations. 3) Following Lindzen (1981), suppose that the waves are steady and conservative, growing with height until 'saturating' at some level, the criterion for saturation being that the waves are marginally unstable to convection. Above this level,

$$B = B_s = \rho_0 \frac{k \hat{c}^3}{N} \quad (2.27)$$

so that

$$\begin{aligned} \rho_0 \overline{u' \phi'_z} &= -\frac{N^2}{2\rho_0 k^2 \hat{c}} (\rho_0^2 k^2 \hat{c}^4 / N^2)_z + \rho_0 \overline{u' q'} \\ &\approx 2\rho_0 \bar{u}_z \hat{c}^2 + \rho_0 \overline{u' q'}. \end{aligned} \quad (2.28)$$

On the last line it was assumed for illustration that the mean flow varies rapidly over a density scale height. Note that the buoyancy flux at the breaking level, apart from the thermal damping term, is four times that implied by (2.21), but decreases rapidly above this level. More complete analysis of saturation may require consideration of the explicit forms of X' , Q' that cause the waves to saturate in the first place.

The contribution from $\rho_0 \overline{u' q'}$ may be evaluated for steady waves noting that $-\hat{c} q'_x = Q'$, or

$$k^2 \hat{c} q' = -\alpha_T \phi'_{zx} = \alpha_T N^2 \zeta'_x = -\alpha_T N^2 w' / \hat{c} \quad (2.29)$$

so that

$$\rho_0 \overline{u' q'} = -\frac{\alpha_T N^2}{k^2 \hat{c}^2} \rho_0 \overline{u' w'}. \quad (2.30)$$

The sign is opposite to the momentum flux. The effect of thermal damping is to shift temperature anomalies forward in the direction of intrinsic phase propagation, producing a negative correlation $\overline{u' T'} < 0$ if $\overline{u' w'} > 0$.

Finally, a nondimensional measure of the horizontal buoyancy flux may be obtained from the ratio

$$\begin{aligned}\frac{k\overline{u'\phi'_z}}{\overline{u'\phi'_{zx}}} &= \frac{k\hat{c}}{N^2} \frac{\overline{u'\phi'_z}}{\overline{u'w'}} + O(\mu^2) \\ &= \frac{\text{Re}(\psi^*\psi_z)}{\text{Im}(\psi^*\psi_z)} - \frac{\alpha_T}{k\hat{c}} + O(\mu^2)\end{aligned}\quad (2.31)$$

For a wave of harmonic form, this is equivalent to the ratio of cospectrum and quadrature power. In the three simple examples above, the respective nondimensional contributions are $(2mH)^{-1}$, $(2mH_D)^{-1}$, and $2\bar{u}_z/N$. In (2.21) the nondimensional value is $\bar{u}_z/2N$.

From the variety of cases analyzed here, it is obvious that a horizontal buoyancy flux can arise at $O(1)$ from wave reflection, or at $O(\mu)$ within a region of basic state variation, wave transience, saturation, or damping. The $O(\mu)$ contribution should not be downplayed, particularly if it is due to critical-level interaction within a zone of strong vertical shear. In this case, zonal-wind perturbations grow as $m^{1/2}$ up to the breaking level. The observed horizontal buoyancy flux might actually be larger within, rather than below, the shear zone on account of the factor of $\hat{c}$ appearing in the denominator of (2.21), or because of wave saturation (2.28). Saturation, however, would rapidly diminish its magnitude above the breaking level.

The harmonic form has been used for convenience, but is unnecessary. Starting from (2.4) the second term on the rhs of (2.10) may be expanded as follows:

$$\begin{aligned}-\rho_0\overline{\zeta'u'} &= \rho_0\overline{\zeta'^2}\bar{u}_z - \rho_0\overline{\zeta'u'\ell} \\ &= \rho_0\overline{\zeta'^2}\bar{u}_z - \rho_0\overline{\zeta'D_t\zeta'} \\ &= \rho_0\overline{\zeta'^2}\bar{u}_z - \rho_0(\overline{\xi'\zeta'})_t + \rho_0\overline{w'\xi'} \\ &= \rho_0\overline{\zeta'^2}\bar{u}_z - \rho_0(\overline{\xi'\zeta'})_t - \rho_0\overline{\psi'\xi'_x} \\ &= \rho_0\overline{\zeta'^2}\bar{u}_z - \rho_0(\overline{\xi'\zeta'})_t \\ &\quad + \frac{1}{\rho_0}(\overline{\rho_0\psi'}) (\overline{\rho_0\zeta'})_z\end{aligned}\quad (2.32)$$

Using (2.12a) it can be shown that

$$-2\rho_0\overline{\zeta'u'} = \rho_0\overline{\zeta'^2}\bar{u}_z - \rho_0(\overline{\xi'\zeta'})_t + \frac{1}{\rho_0} \frac{\partial}{\partial z} (\rho_0^2\overline{\psi'\zeta'})\quad (2.33)$$

Quadratic terms on the rhs are evaluated from the lowest order wave solutions. The last term contains the vertical momentum flux (proportional to $\rho_0 N \overline{\psi'\zeta'}$ for steady, conservative, slowly varying waves) and a basic state variation (ρ_0/N) inside the vertical derivative. Substitution of the harmonic form in this equation recovers (2.13).

c. Numerical solutions

The linear equations (2.1a–c) may be combined to give

$$\psi'_{zz} + m^2\psi' = 0\quad (2.34)$$

for a single harmonic, where

$$m^2 = \frac{N^2}{(c - \bar{u})^2} + \frac{\bar{u}_{zz}}{c - \bar{u}}\quad (2.35)$$

for hydrostatic, incompressible, steady, conservative waves. The nonhydrostatic and quasicompressible versions may be found in Booker and Bretherton (1967) and Holton (1975), respectively. If mechanical and thermal damping coefficients are equal and independent of height, then $c \rightarrow c + ic_0$ where $c_0 = \alpha/k$. To illustrate the effect of damping, a small imaginary part may be added to the phase speed. Eq. (2.32) was solved using the tridiagonal algorithm (with 801 equally spaced gridpoints) subject to boundary conditions

$$\psi(z_B) = 1\quad (2.36a)$$

$$\psi_z(z_T) = im(z_T)\psi(z_T)\quad (2.36b)$$

choosing the sign of m in (2.34b) to correspond to an upward propagating wave. Mean flow profiles were of the form

$$\bar{u} = \bar{u}_B + (\bar{u}_T - \bar{u}_B) \cdot \left[1 + \tanh\left(\frac{z - z_M}{D}\right) \right] / 2\quad (2.37)$$

and $N = .02 \text{ s}^{-1}$. In the following, $z_B = 0$, $z_M = 7.5 \text{ km}$, and $z_T = 15 \text{ km}$. For case 1 (weak shear):

$$\begin{aligned}u_B &= 0 \\ u_T &= 10 \text{ ms}^{-1} \\ D &= 2 \text{ ms}^{-1} \text{ km}^{-1} \\ c &= 15 \text{ ms}^{-1}.\end{aligned}\quad (2.38)$$

For case 2 (strong shear):

$$\begin{aligned}u_B &= -20 \text{ ms}^{-1} \\ u_T &= 20 \text{ ms}^{-1} \\ D &= 2 \text{ ms}^{-1} \text{ km}^{-1} \\ c &= 25 \text{ ms}^{-1}.\end{aligned}\quad (2.39)$$

These profiles are shown in Figs. 1a, 1c. Case 2 is representative of the QBO near 25 km, with maximum shear $10 \text{ ms}^{-1} \text{ km}^{-1}$, corresponding to a Richardson number $Ri = 4$.

The horizontal buoyancy flux is shown in Figs. 1b, 1d for cases 1, 2 respectively, along with (vertically constant) momentum flux. The WKB approximation (2.21) is also shown (dashed line). In case 1, there is a small buoyancy flux in the shear zone, and none outside. Nor is there any evidence of reflection. The WKB solution is, for practical purposes, exact. The momentum flux is .00133366, almost identical to the no-shear value ($m = .0013$). In case 2, there is a larger buoyancy flux in and beneath the shear zone, with wavy structure below due to interference

Fig. 1. Vertical profiles of mean zonal wind (units, ms^{-1}) for (a) case 1; (c) case 2; (e) variation of case 2. Horizontal buoyancy flux (units, m^{-1}) for (b) case 1; (d) case 2; (f) variation of case 2. Vertical flux of horizontal momentum (units, m^{-1}) is indicated by dotted line at right (constant in height). WKB solution for horizontal buoyancy flux is also shown (dashed line). Buoyancy (without $\hat{c}^{-1}$ factor) and momentum fluxes displayed in the figure correspond, respectively, to the real and imaginary parts of $-\psi^* \psi_z$.

of upward and downward waves in this region. The wavelength of the pattern is approximately half that of the wave itself (*cf.* 2.17). The WKB solution omits the small reflected component but is otherwise accurate. The momentum flux is .00039424, slightly less than the no-shear value ($m = .0004\bar{4}$).

Two variations of case 2 were examined. In the first, a small damping was added, $c_0 = 1 \text{ ms}^{-1}$. The momentum flux now decays with height, markedly so as $\hat{c}$ diminishes entering the shear zone (not shown). The buoyancy flux is slightly more positive than before; the only major difference is that it is relatively larger when measured against the momentum flux in the shear zone. In the second variation (shown in Figs. 1e, 1f), $D = 0.5 \text{ ms}^{-1}\text{km}^{-1}$; the shear is four times that of case 2 ($Ri \rightarrow 1/4$), with other parameters the same. The reflected wave is larger, and the momentum flux smaller (.000148514), so that the buoyancy flux is relatively much larger than in Fig. 1d. The ratio of horizontal buoyancy flux to vertical flux of horizontal momentum has no limit, since the momentum flux vanishes with perfect reflection. The amount of reflection observed in Fig. 1f is close to that predicted by Acheson's (1976) piecewise-continuous model ($R = 0.8$).

Other cases using the shear of Fig. 1e were analyzed for larger phase speeds ($|c| \gtrsim 50 \text{ ms}^{-1}$) that might be associated with fast Kelvin or gravity waves and tides. Buoyancy fluxes are sharply peaked in the shear zone, with less evidence of reflection. When mean flows are reversed, the buoyancy fluxes reverse sign, but are similar in magnitude. When phase speeds are reversed, the buoyancy fluxes do not change sign and are again similar in magnitude.

d. Comments

In order to apply these examples to the atmosphere, wave amplitudes must be weighted by a factor $\exp(z/2H)$ and $O(\mu)$ corrections due to compressibility should be retained. If the major effect is wave growth, the cospectrum and quadrature power corresponding to that shown in the figures will have an exponential envelope, growing with height. Moreover, observable fields like u' , T' will grow relative to ψ' , as $\hat{c} \rightarrow 0$, by a factor $\hat{c}^{-1}$. These two effects will tend to emphasize features in the shear zone over those below. Apart from reflection, $\overline{u'T'}$ will maximize between the base of the shear zone and some point just above the breaking level (if one exists). A more detailed analysis of observations of Kelvin and gravity waves in relation to the descending QBO shear zones is warranted before drawing specific conclusions. Concerning Kelvin waves, it should be noted that their latitudinal scale contracts approaching the critical layer, causing wave amplitude to pile up near the equator more dramatically than in the two-dimensional problem. This is

a WKB effect (with slowly varying approximation in time and height, but not latitude, as in Boyd, 1987) and does not require *vertical* reflection from sloping surfaces $\hat{\omega} = f$, although such reflection may be important if our two-dimensional examples are representative. Mean winds are sometimes asymmetric about the equator, so maximum wave amplitude may be shifted slightly off the equator as a result of reflection.

Time series of cospectrum power observed by Sato *et al.* (1994) reveal a qualitative difference between long and short-period disturbances at the equator. The cospectrum and quadrature power of ~ 5 –20 day disturbances are positively enhanced at the onset of QBO westerlies, but are much smaller at other times. By contrast, the cospectrum power of short-period disturbances oscillates with the QBO between small positive and negative values of nearly equal magnitude. Although the quadrature power at short periods is markedly less than that of long-period waves, the relatively large magnitude and oscillatory behavior of the cospectrum $C(u', T')$ suggests that it may represent a real phenomenon. If so, two possible interpretations are:

1) Fast Kelvin waves, although transparent to QBO westerlies (Hirota, 1978; Salby *et al.*, 1984; Maruyama, 1991), might contribute to a small horizontal buoyancy flux as they encounter QBO shear zones. In order for $C(u', T')$ to be enhanced apart from $Q(u', T')$, the vertical shear must be extremely strong (*e.g.*, Fig. 1f).

2) The gravity wave spectrum contains eastward and westward propagating waves (Pfister *et al.*, 1993).² If waves of equal but opposite phase speed are incoherent, they will contribute alike to the horizontal buoyancy flux (2.21), but momentum fluxes will tend to cancel. The vertical shear need not be strong.

A possible problem with either of these explanations is that, unless phase speeds are large compared to the QBO, waves will be selectively amplified (as measured by u') in the QBO shear zones as their intrinsic phase speed is reduced. For these waves, it is difficult to see how $C(u', T')$ could be amplified without a similar (though possibly smaller) amplification of $Q(u', T')$. (Downward propagating waves would be attenuated by damping or saturation at higher levels, unless phase speeds are large.) Consistent with the role of vertical shear is the observation in Fig. 11 of Sato *et al.* (1994) that short-period $C(u', T')$ is amplified at slightly different altitudes depending on the phase of the QBO.

Given the likelihood that many gravity waves are present at the same time, it may be worthwhile to construct spectral analogues of the monochro-

² The role of gravity waves in the QBO was emphasized by Dunkerton (1991) and Takahashi and Boville (1992) and is part of the original Lindzen and Holton (1968) theory.

matic formulae for $\overline{u'T'}$. One might assume, for example, that individual waves saturate just below critical levels, but that the resulting turbulence causes mechanical and thermal damping of other (non-breaking) waves. Presumably the waves most susceptible to damping in the westerly acceleration phase of the QBO would be those with phase speeds of positive sign and vice versa. Simultaneous observations of wind and temperature in the equatorial lower stratosphere, with better temporal resolution (e.g., radar, lidar), would provide valuable information with which to evaluate these effects in the context of a gravity wave spectrum.

3. Conclusion

The equations of motion describing two-dimensional internal gravity waves were analyzed to derive an expression for the horizontal buoyancy flux—that is, the correlation between buoyancy (or temperature) and horizontal velocity—in various cases involving vertical shear. It was shown that, although this correlation vanishes at zeroth order (in μ) for a single wave, its first order contribution is nonzero due to shear, even for steady, conservative, incompressible waves. Departures from steady, conservative motion or quasi-compressibility also cause a nonzero correlation. Several cases were analyzed and some numerical results obtained for waves approaching a critical layer of reduced intrinsic phase speed. With weak shear, the buoyancy flux is small relative to momentum flux, as expected from the perturbation theory. Strong vertical shear enhances the buoyancy flux within the shear zone and causes partial reflection beneath, producing a nonzero $O(1)$ correlation in this region.

These effects may explain recent observations of zonal wind and temperature cospectra in the equatorial lower stratosphere (Sato *et al.*, 1994). Such observations, at least, do not immediately preclude their interpretation as Kelvin or gravity waves. Taking the analysis of this paper into account, the behavior of cospectrum and quadrature power relative to the QBO seems qualitatively consistent with such an interpretation for long and short-period disturbances, respectively. For long-period disturbances, equatorial $C(u', T')$ and $Q(u, t')$ are selectively enhanced during the onset of QBO westerly phases, but otherwise small, as expected for Kelvin waves. For short-period disturbances, $C(u', T')$ alternates sign with the QBO, as expected of a gravity wave spectrum containing both eastward and westward propagating components (Pfister *et al.*, 1993).

Although the theoretical results suggest that the cospectrum will be measurable, particularly in a critical layer, further quantitative analysis of the ratio $C:Q$ is desirable as it may reveal useful information about the anisotropy of the spectrum or the nature of wave saturation and absorption. Simultane-

ous radar and lidar measurements in the equatorial lower stratosphere would be particularly valuable. A three-dimensional analysis of Kelvin waves in vertical shear and their associated horizontal buoyancy flux would significantly extend these results. Such topics are beyond the scope of this paper but fruitful avenues for exploration. A final point is that, although the horizontal buoyancy (or temperature) flux has been emphasized for its interpretive value—it is, after all, exactly nondivergent since independent of x —real wavepackets are of finite horizontal extent and may therefore contribute to a net horizontal transport under conditions more general than those considered here.

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References

- Acheson, D.J., 1976: On over-reflection. *J. Fluid Mech.*, **77**, 433–472.
- Andrews, D.G. and M.E. McIntyre, 1976: Planetary waves in horizontal and vertical shear: the generalized Eliassen-Palm relation and the mean zonal acceleration. *J. Atmos. Sci.*, **33**, 2031–2048.
- Andrews, D.G., J.R. Holton and C.B. Leovy, 1987: *Middle Atmosphere Dynamics*, Academic Press, 489 pp.
- Bergman, J.W. and M.L. Salby, 1994: Equatorial wave activity derived from fluctuations in observed convection. *J. Atmos. Sci.*, **51**, 3791–3806.
- Booker, J.R. and F.P. Bretherton, 1967: The critical layer for internal gravity waves in a shear flow. *J. Fluid Mech.*, **27**, 513–539.
- Boyd, J.P., 1987: The effects of latitudinal shear on equatorial waves, Part 1: theory and methods. *J. Atmos. Sci.*, **35**, 2236–2258.
- Dunkerton, T.J., 1981: Wave transience in a compressible atmosphere, part 1: transient internal wave, mean-flow interaction. *J. Atmos. Sci.*, **38**, 281–297.
- Dunkerton, T.J., 1991: Nonlinear propagation of zonal winds in an atmosphere with Newtonian cooling and equatorial wavelike driving. *J. Atmos. Sci.*, **48**, 236–263.
- Eliassen, A., and E. Palm, 1960: On the transfer of energy in stationary mountain waves. *Geophys. Publ.*, **22**, no. 3.
- Fritts, D.C., 1984: Gravity wave saturation in the middle atmosphere: a review of theory and observations. *Rev. Geophys. and Space Phys.*, **22**, 275–308.
- Hirota, I., 1978: Equatorial waves in the upper stratosphere and mesosphere in relation to the semi-annual oscillation of the zonal wind. *J. Atmos. Sci.*, **35**, 714–722.
- Holton, J.R., 1975: *The Dynamic Meteorology of the Stratosphere and Mesosphere*. Amer. Meteor. Soc., 319 pp.
- Lindzen, R.S., 1981: Turbulence and stress due to gravity wave and tidal breakdown. *J. Geophys. Res.*, **86C**, 9707–9714.

- Lindzen, R.S. and J.R. Holton, 1968: A theory of the quasi-biennial oscillation. *J. Atmos. Sci.*, **25**, 1095–1107.
- Maruyama, T., 1991: Annual variations and QBO-synchronized variations of the equatorial wave intensity in the lower stratosphere at Singapore during 1961–1989. *J. Meteor. Soc. Japan*, **69**, 219–232.
- Maruyama, T., 1994: Upward transport of westerly momentum due to disturbances of the equatorial lower stratosphere in the period range of about 2 days – a Singapore data analysis for 1983–1993. *J. Meteor. Soc. Japan*, **72**, 423–432.
- Pfister, L., K.R. Chan, T.P. Bui, S. Bowen, M. Legg, B. Gary, K. Kelly, M. Proffitt and W. Starr, 1993: Gravity waves generated by a tropical cyclone during the STEP tropical field program: a case study. *J. Geophys. Res.*, **98**, 8611–8638.
- Salby, M.L., D.L. Hartmann, P.L. Bailey and J.C. Gille, 1984: Evidence for equatorial Kelvin modes in Nimbus 7 LIMS. *J. Atmos. Sci.*, **41**, 220–235.
- Sato, K., F. Hasegawa and I. Hirota, 1994: Short-period disturbances in the equatorial lower stratosphere. *J. Meteor. Soc. Japan*, **72**, 895–872.
- Takahashi, M. and B.A. Boville, 1992: A three-dimensional simulation of the equatorial quasi-biennial oscillation. *J. Atmos. Sci.*, **49**, 1020–1035.
- Wallace, J.M. and V.E. Kousky, 1968: Observational evidence of Kelvin waves in the tropical stratosphere. *J. Atmos. Sci.*, **25**, 900–907.

鉛直シアー流中での内部重力波の水平温度フラックス

Timothy J. Dunkerton

(Northwest Research Associates, U. S. A.)

2次元内部重力波の式を用いて、鉛直シアーも含めたいろいろな場合の水平温度フラックスの式を導いた。水平温度フラックスはゼロ次のオーダーではゼロとなり、定常で保存する非圧縮的な波であっても、シアーの効果により1次のオーダーでゼロではなくなる。定常でなく保存的でない波のとき、また準圧縮性の波の場合もゼロでなくなる。いくつかの場合を解析し、また波が臨界層に近付いた場合の数値的な結果も示す。弱いシアーのとき、温度フラックスは鉛直運動量フラックスに比べて小さい。強い鉛直シアーのとき温度フラックスはシアーとともに大きくなり、部分反射を生み出しゼロでない値をもつ。これらの効果は熱帯下部成層圏での東西風と温度のコスベクトルに関する最近の観測結果を説明するであろう。