NOTES AND CORRESPONDENCE

Downslope Windstorms in a Uniform Environmental Flow with Non-Uniform Potential Temperature Gradient

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Abstract

The effect of non-uniformity of vertical potential temperature gradient on downslope windstorms in a uniform environmental flow is analytically investigated by employing a steady 2-dimensional hydrostatic model. Both the potential temperature and a vorticity-like quantity become functions of the stream function. Their functional forms are chosen so that the environmental flow has a uniform horizontal velocity and non-uniform potential temperature gradient. Only the case of small parabolic deviation from uniform potential temperature gradient with the average value fixed is examined.

The calculation shows the following. For a prescribed atmospheric height, the concentration of the potential temperature gradient in the upper (lower) layer increases (decreases) both the downslope wind speed and required mountain height. This result, compared with the case of a uniform increase of potential temperature gradient, implies that the upper layer stability is more effective for downslope windstorms.

1. Introduction

Smith (1985) analytically solved a non-linear problem of downslope windstorms on the lee side of a mountain by employing a steady hydrostatic 2-dimensional Boussinesq model. In his model, the environmental flow was assumed to be uniform with a uniform buoyancy frequency N . There was further assumed to exist a critical atmospheric height H where the stream line splits into a horizontal and a descending branch over the mountain, between which the stratification is neutral. The neutral region is called the dead region. On the assumptions, Smith (1985) showed that the wind speed on the lee side foot can be greatly enhanced, and have a magnitude of NH .

Gutman (1991) extended Smith (1985)'s study to the case of an environmental flow having density decrease with height. Assuming the smallness of the density decrease, the problem could be analytically solved. The result showed that the neglect of the density decrease can yield an overestimation of both the downslope wind speed and the critical mountain height.

Gutman and Apterman (1992) extended Smith (1985)'s study to the case of a sheared environmen-

tal flow. The shear was assumed to be uniform. Although the equation to be solved was strongly non-linear, the problem was reduced to an analytically solvable one by the transformation of Gutman (1957). According to their result, the critical mountain height increases as the shear increases, while the speed of downslope winds is not so affected by the shear.

In this note, Smith (1985)'s study is extended to the case of a uniform environmental flow with non-uniform buoyancy frequency, *i.e.*, non-uniform potential temperature gradient. The model is 2-dimensional, hydrostatic, steady, Boussinesq and adiabatic. The potential temperature and a vorticity-like quantity become conserved along the stream lines. They are functions of the stream function. Their functional forms can be chosen so that the environmental flow has a uniform horizontal velocity and non-uniform potential temperature gradient. Since the problem is strongly non-linear, only the case of small deviation from uniform potential temperature gradient is examined.

The organization of this note is as follows. In Sections 2 and 3, the formulation is presented. In Section 4, the flow on the lower boundary of the dead region are considered. In Section 5, the downslope

Fig. 1. Schematic of downslope windstorms

windstorm equations are derived. In Section 6, the case of uniform potential temperature gradient is reviewed. In Section 7, the effect of non-uniformity of the potential temperature gradient is calculated. In Section 8, concluding remarks are given.

2. Basic equations

Essentially the same downslope windstorm configuration as Smith (1985) or Gutman and Apterman (1992) is considered (see Fig. 1).

The governing equations are the 2-dimensional steady Boussinesq hydrostatic isentropic equations,

$$u_i \partial_i u_1 + \partial_1 \phi = 0, \quad (2-1a)$$

$$\partial_3 \phi = g\theta/\Theta, \quad (2-1b)$$

$$\partial_i u_i = 0, \quad (2-1c)$$

$$u_i \partial_i \theta = 0, \quad (2-1d)$$

where subscripts $i = 3, 1$ denote the vertical and horizontal directions, respectively, with the convention of summation, *i.e.*,

$$u_i \partial_i u = \Sigma_i u_i \partial_i u, \quad \partial_i u_i = \Sigma_i \partial_i u_i, \quad u_i \partial_i \theta = \Sigma_i u_i \partial_i \theta.$$

u_i is the velocity, ∂_i the partial differential operator, ϕ the geopotential and θ the potential temperature. Θ is a reference value of θ and g is the gravitation acceleration. The vertical coordinate $x_3 = z$ is the pseudo-height of Hoskins *et al.* (1972). By the conservation of mass (2-1c), there exists a stream function ψ such that

$$u_i = -\varepsilon_{ij} \partial_j \psi, \quad (2-2)$$

where $\varepsilon_{33} = \varepsilon_{11} = 0$ and $\varepsilon_{31} = -\varepsilon_{13} = 1$. By (2-1d), the potential temperature becomes a function of the stream function.

$$\theta = \theta[\psi]. \quad (2-3)$$

There are two other quantities, a vorticity-like quantity and the Bernoulli function, conserved along the stream lines, which are derived from (2-1a,b) together with (2-3),

$$\begin{aligned} \partial u_1 / \partial z - (gz/\Theta) \theta'[\psi] \\ = \partial^2 \psi / \partial z^2 - (gz/\Theta) \theta'[\psi] = \Omega[\psi], \end{aligned} \quad (2-4)$$

$$u_1^2 / 2 + \phi - gz\theta/\Theta = B[\psi], \quad (2-5)$$

where primes denote the differentiations with respect to the argument.

3. Equation for ψ

θ and ψ are assumed to be prescribed far upstream,

$$\theta = \theta(z, x) \rightarrow \theta_{in}(z)$$

and

$$\psi = \psi(z, x) \rightarrow \psi_{in}(z) \quad \text{as } x \rightarrow -\infty, \quad (3-1)$$

where $x = x_1$. In particular, for a uniform environmental flow with a constant vertical potential temperature gradient, *i.e.*,

$$\psi_{in}(z) = Uz, \quad \theta_{in} = \Theta + \Gamma z, \quad (3-2)$$

the functional forms of $\theta[\psi]$ and $\Omega[\psi]$ become, respectively,

$$\theta[\psi] = \theta_{in}(\psi/U) = \Theta + (\Gamma/U)\psi$$

and

$$\Omega[\psi] = -(g\Gamma/\Theta U^2)\psi = -(N^2/U^2)\psi, \quad (3-3)$$

where $N^2 = g\Gamma/\Theta$. In this case, (2-4) is reduced to a linear differential equation with constant coefficients.

$$\partial^2 \psi / \partial z^2 + (N^2 U^2) \psi = (N^2 / U) z. \quad (3-4)$$

This is the equation considered by Smith (1985). Hereafter the variables are non-dimensionalized as

$$\begin{aligned} x_i &\rightarrow (U/N)x_i, \quad u_i \rightarrow Uu_i, \quad \psi \rightarrow (U^2/N)\psi, \\ \theta &\rightarrow (NU\theta/g)\theta, \quad \phi \rightarrow U^2\phi, \quad \Omega \rightarrow N\Omega. \end{aligned} \quad (3-5)$$

By the non-dimensionalization, (3-2), (3-3) and (3-4) become, respectively,

$$\psi_{in}(z) = z, \quad \theta_{in}(z) = g/NU + z, \quad (3-2')$$

$$\theta[\psi] = g/NU + \psi, \quad \Omega[\psi] = -\psi, \quad (3-3')$$

$$\partial^2\psi/\partial z^2 + \psi = z. \quad (3-4')$$

In this note, we consider a uniform environmental flow with a non-uniform potential temperature gradient. For this purpose, we choose the functional forms of $\theta[\psi]$ and $\Omega[\psi]$ as

$$\theta[\psi] = g/NU + \psi + \varepsilon G[\psi], \quad \Omega[\psi] = -\psi - \varepsilon \psi G'[\psi], \quad (3-6)$$

where ε is a parameter. In this case, (2-4) becomes

$$\partial^2\psi/\partial z^2 + \psi = z + \varepsilon(z - \psi)G'[\psi]. \quad (3-7)$$

At $x = -\infty$, (3-7) is reduced to the following ordinary differential equation,

$$\begin{aligned} d^2\psi_{in}(z)/dz^2 + \psi_{in}(z) \\ = z + \varepsilon\{z - \psi_{in}(z)\}G'[\psi_{in}(z)]. \end{aligned} \quad (3-8)$$

It is easily seen that $\psi_{in}(z) = z$ is the solution to (3-8) under boundary conditions $\psi_{in}(0) = 0$ and $d\psi_{in}(0)/dz = 1$. Thus, as desired, the horizontal velocity and potential temperature of the environmental flow characterized by (3-6) are determined as

$$u_{in}(z) = 1, \quad \theta_{in}(z) = g/NU + z + \varepsilon G[z]. \quad (3-9)$$

The parameter ε represents the non-uniformity of potential temperature gradient.

4. Velocity on the flow surface

In the dead region, the potential temperature is assumed well mixed so that

$$\theta(z, x) = \theta_{in}(H) = \text{constant}. \quad (4-1)$$

From (2-1b) and (4-1), the geopotential in the dead region is given by

$$\phi(z, x) = z\theta_{in}(H) + \phi_{in}(H) - H\theta_{in}(H). \quad (4-2)$$

On the other hand, from (2-5), on the lower boundary of the dead region, *i.e.*, on the flow surface $z = S(x)$

$$\begin{aligned} u_{in}(H)^2/2 + \phi_{in}(H) - H\theta_{in}(H) \\ = u_1^2/2 + \phi(S, x) - S\theta_{in}(H). \end{aligned} \quad (4-3)$$

Substituting (4-2) into (4-3) yields

$$\begin{aligned} u_1 = u_{in}(H) = 1 \\ \text{on the flow surface } z = S(x). \end{aligned} \quad (4-4)$$

5. Downslope windstorm equations

Let $M(x)$ and $F(x)$ be the mountain profile and the flow depth, respectively, and $(z, x) = (0, 0)$ be the lee side foot of the mountain. $M(x) + F(x) = S(x)$ and they satisfy

$$M(x) \rightarrow 0, \quad F(x) \rightarrow H \quad \text{as } x \rightarrow -\infty,$$

and

$$M(x) = 0 \quad \text{for } x \geq 0.$$

For $x < 0$, the solution to (3-7) under boundary conditions $\psi(M + F, x) = \psi_{in}(H) = H$ and $\partial\psi(M + F, x)/\partial z = u_{in}(H) = 1$ is formally given by

$$\begin{aligned} \psi(z, x) = z + (H - M - F) \cdot \cos(M + F - z) \\ - \varepsilon \int_z^{M+F} d\eta \cdot (\eta - \psi) \cdot G'[\psi] \cdot \sin(z - \eta). \end{aligned} \quad (5-1)$$

The stream function must further satisfy another boundary condition, $\psi(M, x) = \psi_{in}(0) = 0$. This gives a relation between $M(x)$, $F(x)$ and H ,

$$\begin{aligned} \Phi(M, F, H; \varepsilon) \\ = M + (H - M - F) \cdot \cos F \\ - \varepsilon \int_M^{M+F} dz \cdot (z - \psi) \cdot G'[\psi] \cdot \sin(M - z) = 0. \end{aligned} \quad (5-2)$$

From (5-1), the horizontal velocity at the lee foot is given by

$$\begin{aligned} u(0, 0) = 1 + \{H - F(0)\} \cdot \sin F(0) \\ - \varepsilon \int_0^{F(0)} dz \cdot (z - \psi) \cdot G'[\psi] \cdot \cos z. \end{aligned} \quad (5-3)$$

6. Case of uniform potential temperature gradient

In this section, we review the result of Smith (1985). In the case of $\varepsilon = 0$, *i.e.*, an environmental flow with a constant potential temperature gradient, (5-2) and (5-3) are reduced to

$$\begin{aligned} \Phi(M(x), F(x), H; 0) \\ = M(x) + \{H - M(x) - F(x)\} \cdot \cos F(x) = 0, \end{aligned} \quad (6-1a)$$

$$u(0, 0) = 1 + \{H - F(0)\} \cdot \sin F(0). \quad (6-1b)$$

Since $M(x) \geq 0$ and $H - M(x) - F(x) \geq 0$, (6-1a) implies that $\cos F(x)$ must be non-positive,

$$\cos F(x) \leq 0$$

or

$$\begin{aligned} \pi/2 + 2n\pi &\leq F(x) \leq 3\pi/2 + 2n\pi, \\ n &= 0, 1, 2, 3, \dots \end{aligned} \quad (6-2)$$

Because the dimensional atmospheric height for $H \geq 2\pi$ may become greater than that of the tropopause, hereafter we consider only such $F(x)$ and H that $0 \leq F(x) \leq H \leq 2\pi$. M takes the maximum M_m at some value $F = F_m$. As was shown in Smith (1985), F_m is not the extremum (minimum) of F for downslope windstorms to occur, i.e., $0 = [\partial M / \partial F]_{\Phi=0} = -[\partial \Phi / \partial F]_M / [\partial \Phi / \partial M]_F$ at $(M, F) = (M_m, F_m)$. This implies

$$\begin{aligned} \Phi(M_m, F_m, H; 0) \\ = M_m + (H - M_m - F_m) \cdot \cos F_m = 0, \end{aligned} \quad (6-3a)$$

$$\begin{aligned} \partial \Phi(M_m, F_m, H; 0) / \partial F \\ = -\cos F_m - (H - M_m - F_m) \cdot \sin F_m = 0. \end{aligned} \quad (6-3b)$$

From (6-3a,b), H , M_m and F_m must satisfy the following relations:

$$\begin{aligned} \sin F_m &= (M_m^2/4 + 1)^{1/2} - M_m/2, \\ \pi &> F_m > \pi/2, \\ H &= F_m + M_m(1 - \sec F_m) \\ &= M_m + F_m - \cot F_m. \end{aligned} \quad (6-4)$$

For a prescribed atmospheric height H , the required mountain height M_m is determined by (6-4). Since $M(0) = 0, F(0)$ in (6-1a) turns out to be $\pi/2$, and then $u(0, 0)$ in (6-1b) becomes

$$u(0, 0) = 1 + (H - \pi/2)$$

or

$$\begin{aligned} u(0, 0) &= U(1 - \pi/2) + NH \\ &\text{in dimensional form.} \end{aligned} \quad (6-5)$$

Since NH can become at least a few times greater than U in the real situation, (6-5) implies that the lee foot wind speed may be greatly enhanced.

7. Case of non-uniform potential temperature gradient

We consider the case of $|\varepsilon| \ll 1$. In this case, for a prescribed atmospheric height H , (5-2) and (5-3) become to the $0(\varepsilon)$ approximation

$$\begin{aligned} \Phi(M + \delta M, F + \delta F, H; \varepsilon) &= \delta M(1 - \cos F) \\ &- \delta F \{ \cos F + (H - M - F) \cdot \sin F \} \\ &- \varepsilon (H - M - F) \int_M^{M+F} dz \cdot \sin(z - M) \cdot \cos(z \\ &- M - F) \cdot G'[z + (H - M - F) \cdot \cos(z \\ &- M - F)] = 0, \\ u(0, 0) + \delta u(0, 0) \\ &= 1 + (H - \pi/2) - \delta F(0) + \varepsilon(H \end{aligned} \quad (7-1)$$

$$\begin{aligned} &- \pi/2) \int_0^{\pi/2} dz \cdot \cos z \cdot \sin z \cdot G'[z + (H \\ &- \pi/2) \cdot \sin z], \end{aligned} \quad (7-2)$$

where $F = [F]_{\varepsilon=0}$ and $\delta F = \varepsilon[dF/d\varepsilon]_{\varepsilon=0}$ etc. From (7-1) and (6-3b), $\delta F(0)$ and δM_m are calculated as

$$\begin{aligned} \delta F(0) \\ = -\varepsilon \int_0^{\pi/2} dz \cdot \sin^2 z \cdot G'[z + (H - \pi/2) \cdot \sin z], \end{aligned} \quad (7-3)$$

$$\begin{aligned} \delta M_m &= \varepsilon \{ (H - M_m - F_m) M_m \\ &/ (H - F_m) \} \int_0^{F_m} dz \cdot \sin z \cdot \cos z \\ &\cdot G'[z + M_m + (H - M_m - F_m) \cdot \cos(z - F_m)] \\ &+ \varepsilon \{ M_m / (H - F_m) \} \int_0^{F_m} dz \cdot \sin^2 z \\ &\cdot G'[z + M_m + (H - M_m - F_m) \cdot \cos(z - F_m)]. \end{aligned} \quad (7-4)$$

Substituting (7-3) into (7-2) yields

$$\begin{aligned} \delta u(0, 0) &= \\ \varepsilon \int_0^{\pi/2} dz \cdot \sin^2 z \cdot G'[z + (H - \pi/2) \cdot \sin z] \\ &+ \varepsilon (H - \pi/2) \int_0^{\pi/2} dz \cdot \cos z \cdot \sin z \cdot G'[z + (H \\ &- \pi/2) \cdot \sin z]. \end{aligned} \quad (7-5)$$

In particular, for $G' = \text{constant}$, (7-4) and (7-5) are reduced to

$$\delta M_m / \varepsilon G' = M_m F_m / 2(H - F_m) > 0$$

and

$$\delta u(0, 0) / \varepsilon G' = H/2 > 0 \quad (7-6)$$

For a parabolic deviation of potential temperature from uniform gradient,

$$\begin{aligned} \theta_{in}(z) &= g/NU + z + \varepsilon z(H - z), \text{ i.e.,} \\ G[z] &= z(H - z), \end{aligned} \quad (7-7)$$

(7-4) and (7-5) are concretely calculated as

$$\begin{aligned} (H - F_m) \delta M_m / \varepsilon M_m \\ = \{ 1/4 - (H - M_m - F_m) (5H - 2M_m \\ - 2F_m) / 12 \} \cos 2F_m \\ + \{ (8H - 5M_m - 5F_m) / 12 \} \sin 2F_m \\ + (2/3)(H - M_m - F_m)^2 \cos F_m \\ - (4/3)(H - M_m - F_m) \sin F_m \\ - F_m^2/2 + (H - M_m - 3F_m)(2M_m - H)/4 \\ - 1/4, \end{aligned} \quad (7-8)$$

$$\delta u(0,0) = -\varepsilon\{(1/6)(H - \pi/2)^2 + (4/3 - \pi/4)(H - \pi/2) + 1/2\}. \quad (7-9)$$

(7-9) implies that $\delta u(0,0)/\varepsilon$ is negative. By using (6-4), one can easily see (e.g., by a pocket computer) from (7-8) that $\delta M_m/\varepsilon$ is also negative.

8. Conclusion

In Smith (1985)'s analytic study of downslope windstorms, several unrealistic assumptions were made. The environmental flow had a constant density, a uniform horizontal velocity and a constant buoyancy frequency. The variations of density and horizontal velocity are considered by Gutman (1991) and Gutman and Apterman (1992), respectively. In this note, the effect of non-uniformity of buoyancy frequency with the average value fixed was examined.

The calculation showed that the concentration of the potential temperature gradient in the upper (lower) layer increases (decreases) both the downslope wind speed and required mountain height (see, (7-8) and (7-9) together with (7-7)). Compared with the case of a uniform increase of potential temperature gradient (see, (7-6)), we can conclude that the stability in the upper layer is more relevant to

downslope windstorms than that in the lower layer.

The present calculation was performed only for a small parabolic deviation from uniform potential temperature gradient. To examine whether qualitatively the same conclusion is obtained for other profiles of potential temperature belongs to future study.

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非一様な鉛直温位傾度を持つ一様な一般流の中でのおろし風

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一般流の鉛直温位傾度が一様でない場合のおろし風を、定常 2 次元静水圧模型を使い解析的に調べた。温位および渦度 (の様な物理量) が流線関数の関数と成る。これらの関数形を、一般流が一様な速度および非一様な温位傾度を持つ様に選んだ。一様な温位傾度から、平均値を一定にして、僅かに放物線的にずれた場合を調べた。結果は次の様に成った。指定された大気の高さに対して、温位傾度の上層 (下層) への集中は風下側での風速および要求される山の高さの両方を増加 (減少) させる。これは、風下側の強風に対しては下層の安定度よりも上層の安定度のほうがより効果的である事を意味する。