NOTES AND CORRESPONDENCE

Down-Slope Windstorms on an f-Plane

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Abstract

The effect of the earth's rotation on down-slope windstorms induced by a mountain range, which is uniform in the north-south direction, in an eastward uniform environmental flow with a constant buoyancy frequency is analytically investigated. The velocity field is assumed to be uniform in the north-south direction. After an appropriate non-dimensionalization, the non-dimensional Coriolis parameter \( f \) becomes small. To the \( O(f) \) approximation, the wind speed component in the east-west direction is not altered by the rotation, and the same down-slope windstorms occur as in the non-rotational case. On the other hand, the wind speed component in the north-south direction, which is everywhere uniform in the non-rotational case, is altered by the rotation. At the leeward foot of an east-west symmetric mountain profile, the southward wind component is increased by the rotation.

1. Introduction

Smith (1985) proposed a non-linear analytical model representing down-slope windstorms over a 2-dimensional mountain range with height \( M_m \). The basic assumption in Smith's model (1985) is the existence of an atmospheric height \( H \) at which the stream line splits into two branches over the mountain. One is horizontal and the other descending. Between the two stream lines, the potential temperature is uniform and the flow is stagnant. This region is called the dead region. Further, the buoyancy frequency \( N \) and horizontal velocity \( U \) of the environmental flow were assumed to be constant. On the assumptions, Smith (1985) showed that the downslope wind may have a magnitude of \( NH \), which is several times greater than \( U \). This occurs when \( H \) and \( M_m \) satisfy a certain required relation.

Gutman (1991) analytically examined the effect of density decrease with height on Smith (1985)'s model. His calculation showed that the neglect of density decrease leads to an overestimation of both the downslope wind speed and the mountain height \( M_m \).

Gutman and Apterman (1992) considered the case of an environmental flow with shear in Smith's model (1985). After reducing the problem to a tractable one by the method of Gutman (1957), they analytically calculated the shear effect. The result was that the atmospheric height \( H \) increases with the shear, while the downslope wind speed is not so altered by the shear.

In the above-mentioned studies, the earth's rotation was not taken into account. In this note, the rotational effect on down-slope windstorms is examined. After an appropriate non-dimensionalization, the non-dimensional Coriolis parameter \( f \) becomes small. Hence, only the first \( O(f) \) deviation from the non-rotational case is calculated.

The organization of this note is as follows. In Section 2, the formulation is presented. In Section 3, the non-rotational case is reviewed. In Section 4, the deviation due to the rotation is calculated. Concluding remarks are given in Section 5.

2. Basic equations

We consider down-slope windstorms caused by a mountain range, which is uniform in the north-south direction. The environmental flow is toward the east. The eastward, northward and upward coordinates are denoted by \( x, y \) and \( z \), respectively. The upstream and downstream foot site of the mountain are set at \( (z, x) = (0, 0) \) and \( (z, x) = (0, a) \), respectively. The mountain top is located at \( (z, x) = (M_m, m) \). The velocity field is assumed also to be 2-dimensional, i.e., uniform in the \( y \) direction (see Fig. 1).
The governing equations are the 2-dimensional steady Boussinesq hydrostatic isentropic equations on an f-plane,

\begin{align}
\frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial x} - f v &= 0, \\
\frac{\partial v}{\partial x} + w \frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial y} + f u &= 0, \\
\frac{\partial \phi}{\partial z} &= b, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{\partial b}{\partial x} + v \frac{\partial b}{\partial y} &= 0.
\end{align}

The vertical coordinate \( z \) is the pseudo-height of Hoskins and Bretherton (1972). \( u, v \) and \( w \) are the velocity components in the \( x, y \) and \( z \) directions, respectively. \( \phi \) is the geopotential and \( b \) is the buoyancy force which is proportional to the potential temperature. \( f \) is the Coriolis parameter.

By the conservation of mass (2-1d), there exists a stream function \( \psi \) such that

\[ w = -\frac{\partial \psi}{\partial x}, \quad u = \frac{\partial \psi}{\partial z}. \]

Differentiating (2-1a) with respect to \( y \), and using the 2-dimensional assumption, and further using (2-1c), one can see that \( \frac{\partial \phi}{\partial y} \) and \( \frac{\partial b}{\partial y} \) are independent of \( x \), i.e., equal to the far upstream inflow values,

\begin{align}
\frac{\partial \phi}{\partial y} &= \lim_{x \to -\infty} \frac{\partial \phi}{\partial y} = \frac{\partial \phi_{in}}{\partial y}, \\
\frac{\partial b}{\partial y} &= \lim_{x \to -\infty} \frac{\partial b}{\partial y} = \frac{\partial b_{in}}{\partial y}.
\end{align}

The far upstream state is assumed to be a uniform horizontal flow with a constant buoyancy frequency, i.e.,

\begin{align}
\lim_{x \to -\infty} \psi &= \psi_{in} = U z, \\
\lim_{x \to -\infty} v &= v_{in} = V, \\
\lim_{x \to -\infty} b &= b_{in} = N^2 z + b_{in}(0).
\end{align}

where \( U, V \) and \( N^2 \) are constant. It is easily seen that these assumptions are consistent with the governing equations (2-1a,b,c,d,e). Using the constants \( U \) and \( N \), the variables are non-dimensionalized as

\[ (x, y, z) \to (U/N)(x, y, z), \quad H \to (U/N)H, \]

\[ (u, v, w) \to U(u, v, w), \quad b \to (UN)b, \]

\[ \phi \to U^2 \phi, \quad \psi \to (U^2/N)\psi. \]

By the non-dimensionalization (2-5), the governing equations (2-1) become

\begin{align}
J(\psi, \frac{\partial \psi}{\partial z}) + \frac{\partial \phi}{\partial x} - f v &= 0, \\
J(\psi, v) + f(\frac{\partial \psi}{\partial z} - 1) &= 0, \\
\frac{\partial \phi}{\partial z} &= b, \\
J(\psi, b) &= 0,
\end{align}

where \( J(\cdot) \) is the Jacobian operator in \((z, x)\) plane. In deriving (2-6b-d), we used that \( \frac{\partial \phi}{\partial y} = \frac{\partial \phi_{in}}{\partial y} = -f \) and that \( \frac{\partial b}{\partial y} = \frac{\partial b_{in}}{\partial y} = 0 \). In (2-6), \( f \) is the non-dimensionalized Coriolis parameter, which is the original Coriolis parameter divided by \( N \). We can fairly assume that \( f \) is small,

\[ f \ll 1. \]

To the \( O(f^0) \) approximation, (2-6b) implies that \( v \) is conserved along the streamlines. Together with the assumption (2-4b), \( v \) becomes a constant,

\[ v = V \to the O(f^0) approximation. \]

From (2-6d), the buoyancy \( b \) is conserved along the streamlines,

\[ b(z, x) = b[\psi(z, x)]. \]

Using (2-9), from (2-6a,c), the following 2 equations are derived,

\begin{align}
J(\psi, (\frac{\partial \psi}{\partial z})^2/2 + \phi - zb) - f v J(\psi, x) &= 0, \\
J(\psi, \frac{\partial^2 \psi}{\partial z^2} - zdb/d\psi) - f \frac{\partial v}{\partial z} &= 0.
\end{align}

Substituting (2-8), to the \( O(f) \) approximation, (2-10a,b) become

\begin{align}
J(\psi, B) &= 0, \\
J(\psi, \Omega) &= 0,
\end{align}

where \( B = (\frac{\partial \psi}{\partial z})^2/2 + \Phi - zb, \Omega = \frac{\partial^2 \psi}{\partial z^2} - zdb/d\psi \) and \( \Phi = \phi - fVx \). (2-11a,b) imply that \( B \) and \( \Omega \) are conserved along the streamlines.

\begin{align}
B(z, x, y) &= B[\psi(z, x), \psi], \\
\Omega(z, x) &= \Omega[\psi(z, x)].
\end{align}
Three conserved quantities $b[\psi], B[\psi, y]$ and $\Omega[\psi]$ are fundamental tools to examine flows over 2-dimensional mountains. First, we review the case of $f = 0$.

3. Non-rotational case, $f = 0$

From the assumptions (2-4) and non-dimensionalization (2-5), far upstream, $\psi, b[\psi]$, and $\Omega[\psi]$ become

\[
\lim_{x \to -\infty} \psi = \psi_{in} = z, \quad \lim_{x \to -\infty} b[\psi] = b[\psi_{in}] = b_{in}(0) + z = b_{in}(0) + \psi_{in}, \quad \lim_{x \to -\infty} \Omega[\psi] = \Omega[\psi_{in}]
\]

\[= \partial^2 \phi_{in}/\partial z^2 - zdb[\psi_{in}]/d\psi = -\psi_{in}. \quad (3-1)\]

From (3-1), the functional forms of $b[\psi]$ and $\Omega[\psi]$ are determined as

\[
b[\psi] = b_{in}(0) + \psi, \quad (3-2a)
\]

\[
\partial^2 \psi/\partial z^2 - zdb/d\psi = \partial^2 \psi/\partial z^2 - z = \Omega[\psi] = -\psi. \quad (3-2b)
\]

From (3-2b), together with boundary conditions $\psi(M, x) = \psi_{in}(0) = 0$ and $\psi(M + F, x) = \psi_{in}(H) = H$, the stream function $\psi$ is given by

\[
\psi = z + \{(H - M - F) \sin(z - M)
-M \sin(M + F + z)} / \sin F, \quad (3-3)
\]

where $M = M(x)$ and $F = F(x)$ are, respectively, the mountain height and fluid depth. $H$ is the atmospheric height where the streamline splits into 2 branches, one is horizontal and the other descending. The flow above the horizontal branch is assumed to be undisturbed,

\[
u = u_{in} = U, \quad v = v_{in} = V \quad \text{and} \quad w = w_{in} = 0 \quad \text{for } z \geq H. \quad (3-4)
\]

The buoyancy between the 2 branches is assumed to be uniform,

\[
b(z, x) = b_{in}(H) \quad \text{for } M + F \leq z \leq H. \quad (3-5)
\]

From these assumptions, together with the hydrostatic equation (2-1c) and the conservation law $B = B[\psi] \quad (2-12a)$ along the streamline on $z = H$, the geopotential must satisfy

\[
\phi(H, x) - \phi(M + F, x) = (H - M - F)b_{in}(H), \quad \phi(H, x) = \phi_{in}(H). \quad (3-6)
\]

This, together with the conservation law $B = B[\psi] \quad (2-12a)$ along the streamline on $z = M + F$, implies

\[
\partial \psi/\partial z = \partial \phi_{in}/\partial z = 1 \quad \text{on } z = M + F. \quad (3-7)
\]

Substituting (3-7) into (3-3) yields

\[
G(M, F, H) = M + (H - M - F) \cos F = 0. \quad (3-8)
\]

Since $H, M, F$ and $H - M - F$ are positive, $F$ and $H$ must satisfy

\[
\pi/2 \leq F \leq H \leq 3\pi/2. \quad (3-9)
\]

Here we consider only $F$ and $H$ less than $2\pi$ for simplicity. In particular, at $(z, x) = (0, a)$, (3-8) is reduced to

\[
G(0, F(a), H) = \{H - F(a)\} \cos F(a) = 0. \quad (3-10)
\]

$F(a) = H$ is one solution to (3-10). In this case, the wind speed normal to the mountain at $(z, x) = (0, a)$ is given by

\[
u(0, a) = \partial \psi(0, a)/\partial z = 1 + \{H - F(a)\} / \sin F(a) = 1. \quad (3-11)
\]

This is equal to the inflow wind speed. On the other hand, the other solution $F(a) = \pi/2$ is possible, if the atmospheric height $H$ and maximum mountain height $M_m$ satisfy for some $F = F_m$ that

\[
G(M_m, F_m, H) = 0
\]

and

\[
\partial G(M_m, F_m, H)/\partial F = 0. \quad (3-12)
\]

In this case, the wind speed at $(z, x) = (0, a)$ is given by

\[
u(0, a) = \partial \psi(0, a)/\partial z = 1 + H - \pi/2
\]

or

\[
u(0, a) = NH - U(\pi/2 - 1)
\]

in the dimensional form. \quad (3-13)

$H$ is non-dimensional in the first equation of (3-13), and dimensional in the second. This may be several times greater than the inflow speed $U$. While, as mentioned in Section 2, the wind speed parallel to the mountain has an everywhere-constant inflow value $V$.

4. Rotational Case, $f \neq 0$

To the $O(f)$ approximation, $b = b[\psi]$ and $\Omega = \Omega[\psi]$ are exactly the same as in the non-rotational case. While, the geopotential $\phi$ in $B = B[\psi]$ for $f = 0$ is replaced by $\Phi = \phi - fVx$ in $B = B[\psi, y]$ for $f \neq 0$. However, using the same reasoning as deriving (3-6), we obtain the same equation as (3-6), but with $\phi$ replaced by $\Phi$,

\[
\Phi(H, x, y) - \Phi(M + F, x, y) = (H - M - F)b_{in}(H), \quad \Phi(H, x, y) = \Phi_{in}(H, y). \quad (4-1)
\]

Consequently, with respect to the wind speed normal to the mountain, to the $O(f)$ approximation, exactly the same results are obtained for $f \neq 0$ as
for \( f = 0 \). From (3-3) and (3-8), the wind speed \( u(z, x) \) along the mountain slope is given by

\[
\begin{align*}
\frac{\partial u}{\partial z} &= 1 + \{(H - M - F) + M \cos F\}/\sin F \\
&= 1 - M \tan F \\
&= 1 + (H - F) \sin F/(1 - \cos F). \quad (4-2)
\end{align*}
\]

On the other hand, the wind speed \( v \) parallel to the mountain is altered by the rotation. From (2-6b), \( v \) satisfies that

\[
\frac{\partial v}{\partial x} + f(1 - 1/u) = 0. \quad (4-3)
\]

Integrating (4-3) along a streamline \( \psi = \psi_{in}(0) = 0 \), the wind speed \( v(z, x) \) along the mountain slope is given by

\[
v(M, x) = V - f \int_{0}^{x} dx\{1 - 1/u(M, x)\}. \quad (4-4)
\]

Hereafter, only the solution representing down-slope windstorms is considered. First, we consider the case of \( \pi/2 < H \leq \pi \). In this case, under the condition (3-12), \( F(x) \) decreases monotonically from \( F(0) = H \leq \pi \) to \( F(a) = \pi/2 \). This together with (4-2), implies that \( u(M(x), x) \) is monotonically increasing along the slope,

\[
u(M(0), 0) = 1, \quad u(M(a), a) = 1 + H - \pi/2 > 1, \quad du(M(x), x)/dx > 0 \quad \text{for} \quad 0 \leq x \leq a. \quad (4-5)
\]

From (4-4), the \( v(M(x), x) \) is monotonically decreasing along the slope,

\[
v(M(0), 0) = V, \quad v(M(a), a) = V - f \int_{0}^{a} dx\{1 - 1/u(M, x)\} < V, \quad dv(M(x), x)/dx < 0 \quad \text{for} \quad 0 \leq x \leq a. \quad (4-6)
\]

Since \( u(M(x), x) \) depends on \( x \) only through \( M(x) \) from (3-8) and (4-2), the deviation from the inflow value \( V \) is dependent on the profile \( M(x) \). The deviation becomes smaller as the slope becomes steeper.

Second, we consider the case of \( \pi < H < 3\pi/2 \). In this case, under the condition (3-12), (3-8) and (4-2) imply the followings. \( F(x) \) is monotonically decreasing from \( F(0) = H > \pi \) to \( F(a) = \pi/2 \). \( u(M(x), x) \) is decreasing along part of the up-slope, and increasing along the rest. \( u(M(x), x) \) has a minimum at some value of \( x \) (say \( a \)), and recovers the inflow value at another some value of \( x \) (say \( m \)). \( 0 < a < \beta < m < a \),

\[
3\pi/2 > F(0) = H > F(\alpha) = (1 - \sin)^{-1}H > F(\beta) = \pi > F(m) = F_m > F(\alpha) = \pi/2, \quad dF(x)/dx < 0 \quad \text{for} \quad 0 \leq x \leq a, \quad M(0) = 0 < M(\alpha) = \{1 + \cos F(\alpha)\}/\tan F(\alpha) < \{M(\beta) = (H - \pi)/2
\]

\[
< M(m) = M_m > M(\alpha) = 0, \quad u(0, 0) = 1 > u(M(\alpha), \alpha) = -\cos F(\alpha) < u(M(\beta), \beta) = 1 < u(M(m), m) = 1 - M_m \tan F_m < u(0, a) = 1 + (H - \pi/2), \quad dv(M(x), x)/dx < 0 \quad \text{for} \quad 0 \leq x \leq a, \quad dv(M(x), x)/dx > 0 \quad \text{for} \quad a < x \leq m. \quad (4-7)
\]

In (4-7), \( (1 - \sin)^{-1} \) is the inverse function of \( 1 - \sin \), i.e., \( F(\alpha) - \sin F(\alpha) = H \). From (3-12), \( H, M_m \) and \( F_m \) must satisfy

\[
\sin F_m = -M_m/2 + \{M_m^2/4 + 1\}^{1/2} > 0
\]

and

\[
H = M_m + F_m - \cot F_m. \quad (4-8)
\]

While, from (4-4), \( v(M(x), x) \) is increasing along part of the up-slope and decreasing along the rest.

\[
v(M(0), 0) = V < v(M(\alpha), \alpha) < v(M(\beta), \beta) > v(M(m), m) > v(M(\alpha), a), \quad dv(M(x), x)/dx > 0 \quad \text{for} \quad 0 \leq x \leq \beta, \quad dv(M(x), x)/dx < 0 \quad \text{for} \quad \beta < x \leq a. \quad (4-9)
\]

Since \( v(M(x), x) \) is dependent on \( x \) only through \( M(x) \) from (3-8), (4-2) and (4-4), it depends on the profile \( M(x) \) whether the foot site value \( v(M(\alpha), a) \) is smaller than the inflow value \( V \) or not. In the following we show that \( v(M(\alpha), a) \) is smaller than \( V \) for any symmetric profile \( M(x) \). The symmetry implies that \( M(\alpha - \beta) = M(\beta) \). Then from (4-2), (4-4) and (4-7), the following inequality holds,

\[
v(M(\alpha), a) = V + f \int_{0}^{a} dx\{1/u(M, x) - 1\} - f \int_{a}^{\beta} dx\{1 - 1/u(M, x)\} < V + f \int_{0}^{\alpha} dx\{1/u(M, x) - 1\} - f \int_{a}^{\beta} dx\{1 - 1/u(M, x)\} < V + f\beta\{1/u(M(\alpha), \alpha) - 1\} + f\beta\{1/u(M(\beta), a - \beta) - 1\} = V + f\beta\{1/u(M(\alpha), \alpha) + 1/u(M(\beta), a - \beta) - 2\}. \quad (4-10)
\]

Using (3-8), (4-2) and (4-7), each term on the right hand side in (4-10) is estimated as

\[
1/u(M(\alpha), \alpha) = -1/\cos F(\alpha) < 1/\sin(1 - \sin)^{-1}H \mid \cos(1 - \sin)^{-1}H \mid \mid = 1.485 \ldots, \quad 1/u(M(\beta), a - \beta) \quad < 1/\sin(1 - \sin)^{-1}H \mid 1 - (H - \pi/2) \tan(F(\beta)) \mid = 0.389 \ldots. \quad (4-11)
\]
In (4-11), $F(a - \beta)$ is the smaller one of two solutions to (3-8) at $x = \beta$ and $x = a - \beta$, i.e., at $M = M(\beta) = M(a - \beta) = (H - \pi)/2$. Substituting (4-11) into (4-10) yields

$$V(M(a), a) < V + f\beta(1.485 + 0.389 - 2) = V - 0.126f\beta < V. \quad (4-12)$$

Equation (4-12) implies that the southward component of the velocity at the leeward foot is always increased by the rotation. In almost the same way, for the other solution not representing down-slope windstorm, the southward component is shown to be increased by the rotation.

In the linear approximation, from (2-6) one can easily derive the following perturbation equations:

$$u' = \partial u'/\partial z, \quad u' = -f\int_0^x dx'u',$$
$$\partial^4 u'/\partial x^4 + \partial^2 u'/\partial x^2 + f^2\partial^2 u'/\partial z^2 = 0,$$

where the primed variables represent the perturbations on the environmental flow. Under the linearized boundary conditions

$$\psi'(0, x) = -M(x), \quad \psi'(H, x) = H - F(x) - M(x),$$
$$\partial \psi'(H, x)/\partial z = 0,$$

the linear solution is given by

$$\psi'(z, x) = -M(x)\cos(H - z)/\cos H + O(f^2),$$
$$F(x) = H + M(x)(1 - \cos H)/\cos H + O(f^2).$$

(4-13)

(4-14)

(4-15)

From (4-15), the velocity components at the leeward foot become

$$u(0, a) = 1 + u'(0, a) = 1 + O(f^2),$$
$$v(0, a) = V + v'(0, a) = V - f(-\tan H)\int_0^a dxM(x) + O(f^2) < V. \quad (4-16)$$

To the $O(f)$ approximation, the eastward component is not affected by the rotation, while the southward component is increased. These linear considerations are compatible with the above obtained nonlinear results, though the linear solution cannot represent downslope windstorms.

5. Conclusion

To our knowledge, previously published analytical studies of downslope windstorms are restricted to a strictly 2-dimensional framework, and do not include the rotational effect of the earth. This effect on mesoscale phenomena such as downslope windstorms is indeed weak. However, it is desirable to include this effect in order to better understand the phenomena.

In this note, the rotational effect on downslope windstorms induced by a mountain range uniform in the north-south direction was examined. The environmental flow was assumed to be uniform, horizontal and of a constant buoyancy frequency. Since the Coriolis parameter, after an appropriate non-dimensionalization, is small, only the first order deviation from the non-rotational case was calculated.

The following results were obtained. To the first approximation, the velocity component normal to the mountain range is not affected by the rotation, and the same downslope windstorms occur as in the non-rotational case. This is compatible with the numerical result of Peng et al. (1995) that the Coriolis force has only a little influence on the downslope wind. On the other hand, the velocity component parallel to the mountain range, which is everywhere uniform in the non-rotational case, is altered by the rotation. This is not unexpected, since many published results (e.g., Bannon, 1988) say that the deviation due to the rotation is directly proportional to the Coriolis parameter. The deviation from the non-rotational case is dependent on the profile of the mountain. The wind speed parallel to the mountain range at the leeward foot may be greater or smaller than the inflow value. In particular, for an east-west symmetric profile, its southward component is increased by the rotation.

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References


一定の浮力振動数を持つ一様な一般風の中で、二次元的な山脈によって惹起される下ろし風に対する、地球回転の効果を解析的に調べた。速度場は山脈に平行な方向に一様と仮定した。適当な無次元化によってコリオリ因子は小さく成る。$O(f)$の近似で、山脈に垂直な速度成分は$f$の影響を受けず、非回転の場合と同じ下ろし風が起こる。山脈に平行な速度成分は、非回転の場合には至る所一様だが、$f$の影響を受ける。山脈の形が上流下流に対称である場合には、風下側の麓で、上流側から見て右方向の速度成分が増加する。