NOTES AND CORRESPONDENCE

A Relationship between the Bernoulli Function and Potential Vorticity

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Abstract

In 2-dimensional steady adiabatic unforced systems, both the Bernoulli function B and potential vorticity Q are functions of the stream function ψ , and Q is the derivative of B with respect to ψ , i.e., $Q = dB/d\psi$. In this note, this formula is extended to 3-dimensional steady adiabatic systems. Even if there exists dissipation, essentially the same formula holds as far as the inner product between the dissipative force and the velocity is everywhere negative. An application is presented to the problem of lee vorticity.

1. Introduction

In 2-dimensional steady adiabatic unforced systems, or almost 2-dimensional systems like the shallow-water system or the hydrostatic system written in isentropic coordinates, there exists a stream function ψ , and both the Bernoulli function B and potential vorticity Q are functions of ψ , i.e., $B = B[\psi]$ and $Q = Q[\psi]$. As is easily derived from the governing equations, Q is equal to the derivative of B with respect to ψ , i.e., $Q[\psi] = dB[\psi]/d\psi$.

In 3-dimensional steady adiabatic systems, the potential temperature θ is conserved along the stream lines. Conversely, the stream lines lie on the iso- θ surfaces. As a result, a stream function ψ can be defined on the iso- θ surfaces. In the absence of dissipation, both the Bernoulli function B and potential vorticity Q are functions of ψ and θ , i.e., $B = B[\psi, \theta]$ and $Q = Q[\psi, \theta]$ just as in 2-dimensional cases. In the presence of dissipation, they may be dependent on one more independent function, say ξ , i.e., $B = B[\psi, \theta, \xi]$ and $Q = Q[\psi, \theta, \xi]$.

In this note, the 2-dimensional formula is extended to 3-dimensional cases. In the particular case that the inner product between the dissipative force and the velocity is everywhere negative, it is shown that essentially the same formula holds, i.e., $Q[\psi, \theta, \xi] = \partial B[\psi, \theta, \xi]/\partial \psi$.

Recently, Schaer (1993) proposed a generalization of Bernoulli's theorem. Based on this generalized

theorem, Schaer and Smith (1993) interpreted the productive mechanism of lee vortices, which had formerly been attributed to the viscous boundary layer or tilting of horizontal vorticity. This interpretation is now called the pseudo-inviscid mechanism and further developed by Smith and Smith (1995) and Rotunno and Smolarkiewicz (1995). As an application of the formula derived in this note, we consider the problem of lee vorticity, along the line of pseudo-inviscid mechanism.

The organization of this note is as follows. In Section 2, the stream function on the iso- θ surface is introduced. In Section 3, the equations of the Bernoulli function and potential vorticity are reviewed. In Section 4, the formula between the Bernoulli function and potential vorticity is proved. In Section 5, an application to lee vorticity is presented. In Section 6, concluding remarks are given.

2. Stream function on iso- θ surface

We begin with the 3-dimensional steady adiabatic system of equations,

$$u_j \partial_j u_i + (\partial_i P)/\rho + \delta_{i3} g + \varepsilon_{ijk} f_j u_k = F_i, \quad (2-1a)$$

$$\partial_i(\rho u_i) = 0, \quad (2-1b)$$

$$u_i \partial_i \theta = 0, \quad (2-1c)$$

where the subscripts i, j, k denote the components of Cartesian coordinates, with the convention of summation, $u_j \partial_j u_i = \sum_{j=1}^3 u_j \partial_j u_i$, etc. The symmetric and antisymmetric tensors, δ_{ij} and ε_{ijk} , respectively, are defined as

$$\delta_{ij} = 1 \quad \text{if } i = j,$$

$$\delta_{ij} = 0 \quad \text{if } i \neq j,$$

$$\varepsilon_{ijk} = +1(-1) \quad \text{if } \{ijk\} \text{ is an even (odd) permutation of } \{123\},$$

$$\varepsilon_{ijk} = 0 \quad \text{otherwise.}$$

u_i is the velocity, ∂_i the partial differential operator, P the pressure, ρ the density, θ the potential temperature, g the gravitational acceleration and f_i the Coriolis parameter. F_i represents dissipation, and is assumed to satisfy $u_i F_i < 0$ everywhere. The stream lines are expressed as the intersections of two independent surfaces, $Y(x) = Y(x_1, x_2, x_3) = \text{constant}$ and $Z(x) = \text{constant}$, so that

$$\rho u_i = -A \varepsilon_{ijk} (\partial_j Y) (\partial_k Z)$$

with some function $A(x)$. (2-2)

Substituting (2-2) into the adiabatic equation (2-1c) yields

$$J(\theta, Y, Z) = 0,$$

where

$$J(\theta, Y, Z) = \varepsilon_{ijk} (\partial_i \theta) (\partial_j Y) (\partial_k Z). \quad (2-3)$$

Since (2-3) implies that $\theta(x)$, $Y(x)$ and $Z(x)$ are not independent, $Z(x)$ can be set equal to $\theta(x)$ without loss of generality. Then (2-2) becomes

$$\rho u_i = -A \varepsilon_{ijk} (\partial_j Y) (\partial_k \theta). \quad (2-4)$$

Substituting (2-4) into the mass conservation equation (2-1b) yields $J(A, Y, \theta) = 0$. This implies that $A(x)$ is a function of $Y(x)$ and $\theta(x)$, i.e., $A(x) = A[Y(x), \theta(x)]$. Let $\psi(x) = \psi[Y(x), \theta(x)]$ be the primitive of $A(x)$ with respect to $Y(x)$:

$$\psi(x) = \int^{Y(x)} dY A[Y, \theta(x)]. \quad (2-5)$$

In terms of $\psi(x)$, (2-4) is rewritten as

$$\rho u_i = -\varepsilon_{ijk} (\partial_j \psi) (\partial_k \theta). \quad (2-6)$$

$\psi(x)$ is the stream function on the iso- θ surface.

3. Bernoulli function and potential vorticity

Multiplying the momentum equation (2-1a) by u_i , and expressing the gradient of the pressure in terms of the gradient of enthalpy H and potential temperature, and further using (2-1c), we obtain the equation of the Bernoulli function $B(x)$;

$$u_i \partial_i B(x) = u_i F_i,$$

$$B(x) = u_i u_i / 2 + H + g x_3. \quad (3-1)$$

On the other hand, multiplying (2-1a) by $\varepsilon_{ijk} \partial_j$, and using (2-1b, c), we obtain the equation of the potential vorticity $Q(x)$,

$$u_i \partial_i Q(x) = \varepsilon_{ijk} (\partial_i \theta) (\partial_j F_k) / \rho,$$

$$Q(x) = (\omega_i + f_i) (\partial_i \theta) / \rho, \quad (3-2)$$

where $\omega_i = \varepsilon_{ijk} \partial_j u_k$ is the vorticity. In the free region, i.e., in the region of $F_i = 0$, both $B(x)$ and $Q(x)$ are conserved, and become functions of $\psi(x)$ and $\theta(x)$,

$$B = B[\psi, \theta]$$

and

$$Q = Q[\psi, \theta] \quad \text{in the region of } F_i = 0. \quad (3-3)$$

On the other hand, in the dissipative region, i.e., in the region of $F_i \neq 0$, $B(x)$ and $Q(x)$ are further dependent on some independent function $\xi(x)$,

$$B = B[\psi, \theta, \xi]$$

and

$$Q = Q[\psi, \theta, \xi] \quad \text{in the region of } F_i \neq 0, \quad (3-4)$$

Without loss of generality, the value of ξ can be assumed to increase along the stream lines, i.e.,

$$u_i \partial_i \xi > 0$$

and therefore

$$J(\psi, \theta, \xi) < 0. \quad (3-5)$$

Substituting (3-4) into (3-1) yields

$$\partial B / \partial \xi = -\rho u_i F_i / J(\psi, \theta, \xi). \quad (3-6)$$

This equation (3-6) together with (3-5) implies that the Bernoulli function B decreases along the stream lines in the dissipative region.

4. A relation between $B(x)$ and $Q(x)$

After some manipulation, (2-1a) is rewritten as

$$\partial_i B - (C_p T / \theta) \partial_i \theta + \varepsilon_{ijk} (\omega_j + f_j) u_k = F_i, \quad (4-1)$$

where C_p is the specific heat at constant pressure and T the temperature. Multiplying (4-1) by $\varepsilon_{ijk} (\partial_j \theta)$, and using (2-1c), we find

$$\varepsilon_{ijk} (\partial_j \theta) (\partial_k B) = Q \rho u_i + \varepsilon_{ijk} (\partial_j \theta) F_k. \quad (4-2)$$

Multiplying (4-2) by $\partial_i \xi$, and using (2-6), we obtain

$$\varepsilon_{ijk} (\partial_i B) (\partial_j \theta) (\partial_k \xi) = Q J(\psi, \theta, \xi) + \varepsilon_{ijk} F_i (\partial_j \theta) (\partial_k \xi). \quad (4-3)$$

From this equation, we obtain the following formula:

$$Q[\psi, \theta, \xi] = \partial B[\psi, \theta, \xi] / \partial \psi - \varepsilon_{ijk} F_i (\partial_j \theta) (\partial_k \xi) / J(\psi, \theta, \xi). \quad (4-4)$$

Since $u_i F_i < 0$ everywhere and $u_i \partial_i \xi > 0$, it is always possible to choose the function $\xi(x)$ so that $\partial_i \xi$ is anti-parallel to F_i . In this case, (4-4) is reduced to

$$Q[\psi, \theta, \xi] = \partial B[\psi, \theta, \xi] / \partial \psi, \quad (4-5)$$

In the free region, i.e., in the region of $F_i = 0$, as a matter of course, the argument ξ in (4-5) disappears.

Fig. 1. Production of a pair of potential vorticity

5. Application

As an application of the formula obtained above, we consider the problem of lee vorticity (see Fig. 1).

The flow is assumed toward the positive x -direction. This implies that the gradient of the stream function ψ is toward the negative y -direction. The Bernoulli function B and the potential vorticity Q are assumed to be uniform and zero far upstream, respectively. On the lee of the mountain, there is assumed to exist a dissipative region due to, *e.g.*, a jump. Downstream along the stream lines passing through the vicinity of the mountain top, B is less than the upstream value B_{in} . On the other hand, downstream along the stream lines far from the mountain top, B is nearly equal to B_{in} . Therefore, downstream $\partial B/\partial\psi$ is positive (negative) in the region of $y < 0$ ($y > 0$). As a result, from the formula obtained above, a pair of potential vorticity with opposite signs is produced in the leeward region.

6. Conclusion

In the above sections, the formula between the Bernoulli function B and potential vorticity Q in 2-dimensional steady systems is generalized into 3-dimensional steady adiabatic systems. In the free region, in which dissipation is absent, essentially the same formula holds that Q is the derivative of B with respect to the stream function ψ , *i.e.*, $Q = \partial B/\partial\psi$. Even in the dissipative region, the formula survives as far as the inner product between

the dissipative force and the velocity is everywhere negative. The negativity, a special form of which is the Newtonian dissipation, is not so far from the reality.

As an application, the production of lee vorticity was interpreted in terms of the formula. As a matter of course, this interpretation gives nothing new to the pseudo-inviscid mechanism of Schaer and Smith (1993), but only puts it in another way.

The formula may be useful to diagnose other meteorological phenomena. To investigate other applications belongs to our future study.

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ベルヌイ関数と渦位の間の一つの関係式

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2次元の定常の非強制の断熱の系では、ベルヌイ関数 B と渦位 Q は流線関数 ψ の関数であり、 Q は B の ψ に関する導関数で表わされる。即ち $Q = dB/d\psi$ 。このノートでは、この公式を3次元の定常の断熱系へ拡張する。散逸が存在する場合でも、散逸項と速度の内積が至る所で負である限り、本質的に同じ公式が成立する。この公式の山の風下渦の生成問題への適応例を示す。