On the Mechanism of Overreflection of a Barotropic Rossby Wave

By Kiyoharu Takano

Climate Prediction Division, Japan Meteorological Agency, Tokyo 100-8122, Japan

(Manuscript received 5 January 1995, in revised form 13 January 1998)

Abstract

Using a simple model, we reexamine the mechanism of overreflection (OR) of a barotropic Rossby wave proposed by Lindzen and his colleagues. Though our model is somewhat artificial, it is advantageous to treat essential characters of a Rossby wave concerning the OR. It is clarified that the reversal of the relation between the direction of momentum flux and group velocity across an inflection point in the region where the WKB approximation is valid is essential for the occurrence of OR. As was demonstrated for the gravity wave case by Lindzen and Barker (1985), it is found that reflection layers behind the critical level is not important for the OR of a Rossby wave.

1. Introduction

Lindzen and his colleagues have investigated overreflection (hereafter referred to as OR) in relation to a basic mechanism of parallel flow instabilities (Lindzen, 1988). The OR is a phenomenon in which the amplitude of a reflected wave becomes larger than that of an incident wave. They argued that repetitions of OR in a cavity is nothing but the shear instability. According to their theory, the growth rate of an unstable wave can be calculated using the reflection coefficient and round-trip time of a wave in a cavity.

They investigated the necessary conditions for the occurrence of OR: In the case of a Rossby wave, for example, the presence of an inflection point and a critical level are necessary for OR. However, the physical role of inflection point and critical level for the OR was not shown enough as their proof is too mathematical. In this paper, we investigate the role of the inflection point for OR of barotropic Rossby wave and clarify the physical mechanism of OR.

2. Overreflection and barotropic instability.

In this section, we review Lindzen and his colleagues’ theories on the OR briefly, since they form a basis of our discussions in the following sections. The following arguments are based on Lindzen and Tung (1978).

When treat an OR of a barotropic Rossby wave, the linearized barotropic vorticity equation for Rossby waves which depend on \( x \) and \( t \) as \( e^{ik(x-ct)} \) is written as follows,

\[
\frac{d^2\psi}{dy^2} + \left( \beta - U_{yy} \right) \frac{U - C}{U - C} \psi = 0. \tag{1}
\]

In this paper the following notations are used:

- \( x \) : eastward coordinate,
- \( y \) : northward coordinate,
- \( t \) : time,
- \( U \) : basic zonal flow,
- \( u \) : perturbation zonal flow,
- \( v \) : perturbation meridional flow,
- \( p \) : perturbation pressure,
- \( \rho_0 \) : basic density,
- \( U \) : perturbation stream function,
- \( C \) : phase velocity in x-direction,
- \( f \) : Coriolis parameter,
- \( \beta = df/dy \), Rossby parameter,
- \( \beta' = \beta - U_{yy} \),
- \( k \) : wavenumber in x-direction,
- \( l \) : wavenumber in y-direction,
- \( C_{gy} \) : group velocity in y-direction.

The basic zonal flow \( U \) is assumed to satisfy a situation shown in Fig. 1. The same situation is, for example, realized by a tanh type profile. \( U \) is assumed to be a monotonically decreasing function of \( y \), and the gradient of zonal mean absolute vorticity
\[ Q = \frac{\beta - Uyy}{U - C} - k^2 > 0 \]

\[ Q = \frac{\beta - Uyy - k^2}{U - C} > 0 \]

\[ Q = \beta - Uyy - k^2 > 0 \]

\[ Q = \beta - Uyy - k^2 < 0 \]

\[ Q = 0 \] at \( y = y_t \); it is positive for \( y < y_t \), while negative for \( y > y_t \). We consider a Rossby wave incident from the left side in the figure. The critical level is located at \( y = y_c \) (\( y_c > y_t \)). If we put,

\[ U_{yy} = -2Q \]

\[ U - C = 0 \]

\[ Q \] can be regarded as the refractive index squared if \( Q \) is a sufficiently slowly varying function in the WKB sense. The wave is internal in the region where \( Q > 0 \), while it is external where \( Q < 0 \). Assuming that \( k \) is small enough, \( Q \) changes its sign approximately as follows;

\[ Q > 0 \quad \text{for} \quad y < y_t, \quad Q < 0 \quad \text{for} \quad y_t < y < y_c, \quad Q > 0 \quad \text{for} \quad y < y_c. \]

If \( U \) is assumed to be constant at \( y < y_t \), the wave field in this region can be expressed by a linear combination of an incident wave propagating toward the right and reflected wave propagating toward the left;

\[ \psi = \psi_i + R\psi_r, \]

where \( \psi_i \) and \( \psi_r \) represent the incident wave and the reflected wave, respectively. Note that \( \psi_i \) and \( \psi_r \) are normalized to have the same amplitude at \( y = -\infty \). \( R \) is the reflection coefficient. Multiplying Eq. (3) by \( \psi_r^* \) (the asterisk denotes the complex conjugate), we obtain

\[ 1 - |R|^2 = -|X|^2 \overline{\nu} = |X|^2 \overline{\nu} \left( \frac{\beta}{\rho_0 (U - C)} \right), \]

where \( |X|^2 \) is a positive constant. By checking the sign of \( \overline{\nu} \), we know whether \( |R| \) is larger or smaller than 1. If \( |R| > 1 \), it is called the OR. It is well known that the momentum flux \( \overline{\nu} \) is independent of \( y \) if \( C \neq U \), and it is discontinuous at \( C = U \) so far as \( C \) is real (Eliassen and Palm 1961, Lin 1956). The jump of \( \overline{\nu} \) at \( C = U \) is given by, (provided that \( U_y < 0 \))

\[ \overline{\nu}^+ - \overline{\nu}^- = -\pi \left[ \frac{k\beta_c}{2U_y} \right] |Y|^2, \]

where \( \overline{\nu}^+ (\overline{\nu}^-) \) means \( \overline{\nu} \) for \( y_c > y_c \) (\( y_c < y_c \)), and \( \beta_c \) stands for the value of \( \beta - U_{yy} \) at \( y = y_c \). \( |Y|^2 \) is a positive constant. If there is a rigid boundary somewhere in the region \( y > y_c \), \( \psi \) must vanish at the boundary. Therefore, we have that

\[ \overline{\nu}^- = 0 \quad \text{at} \quad y > y_c. \]

In this case, as \( U_y (y_c) < 0 \) and \( \beta_c < 0 \) evidently, we obtain that

\[ \overline{\nu}^- = \pi \left[ \frac{k\beta_c}{2U_y} \right] |Y|^2 > 0. \]

It then follows from Eq. (4) that an OR occurs in this case. On the other hand, if \( y_c < y_t \), with the other conditions fixed, then the sign of momentum flux jump is reversed, i.e.,

\[ \overline{\nu}^- = \pi \left[ \frac{k\beta_c}{2U_y} \right] |Y|^2 < 0, \]

because of \( \beta_c > 0 \). Then, a partial reflection (PR) occurs. The reflection coefficient can be calculated numerically in the way similar to solve a boundary value problem (Lindzen et al., 1980).

Next, if there is a rigid wall (or partial reflector) at \( y = y_b \prec y_t \), a kind of cavity is formed between two reflectors at \( y = y_b \) and \( y = y_t \). If the reflection coefficient at \( y = y_b \) is denoted by \( r \), the wave amplitude changes by a factor \( |rR| \) after a round trip in the cavity. If \( |rR| \) is larger then 1, the wave amplifies exponentially through successive reflections. Lindzen and his colleagues considered that this is nothing but an instability. The growth rate of the unstable wave can be given by

\[ kC_t = \frac{1}{\tau} \ln |Rr|, \]
where $r$ is a round trip time of the wave in the cavity. This formula is named the Laser Formula (Lindzen et al., 1980). They applied their interpretation to various kinds of instability (Lindzen and Rothental (1981), Lindzen and Rothental (1983a, b, c), Lindzen et al. (1983)).

Using a numerical model, Lindzen and Barker (1985) investigated time dependent behavior of the OR in the case of an internal gravity wave and found the following conditions for the occurrence of the OR. These are,

1) There exists a wave propagation region in front of a critical level,

2) There exists a critical level (or a steering level) and some condition that the wave is not absorbed there is satisfied. (In the case of an internal gravity wave case for example, it is necessary that Richardson number is less than 1/4 near the critical level).

3) There exists an internal region or a wave-through region behind the critical level.

These are rigorous conditions for the occurrence of OR but their physical meanings are not easily understood except for the first one. For the OR of a Rossby wave for example, the relationship between existence of an inflection point and the condition 2) or 3) mentioned above is not evident. Furthermore, the boundary condition $w_T=0$ (Eq. (6)) which seems to be necessary for OR of a Rossby wave is not included in conditions 1)-3). Lindzen and Barker (1985) showed that, in the case of a gravity wave, the reflection layer behind the critical level is not necessary for the OR and it even obstructs the OR in a certain situation.

In this paper, we will investigate the physical mechanism of OR of a Rossby wave and consider the role of inflection point and boundary conditions in the following sections.

3. Model description

3.1 The model and the basic equation.

In the following sections, we investigate the OR of a barotropic Rossby wave using a model equation which is simplified, but does not lose the essential characteristics of the barotropic Rossby wave.

As shown in the previous section, the important factors for the OR and a barotropic instability are a critical level (or a steering level) and an inflection point. Therefore, we investigate their role in the OR. Ideally speaking, we would like to consider a model that has one critical level and one inflection point. However, this turns out to be a difficult task. In particular, it is impossible to make such a model in case of $\beta \neq 0$, and $U = \text{const}$ at $y = \pm \infty$. For example, the tanh type basic flow has two inflection points and one critical level (see Section 2) and is difficult to deal with such a profile analytically. Therefore, we will deal with a boldly simplified model: The simplest way to make a model with one inflection point and one critical level is to assume that $U$ and $\beta - U_{yy} = \beta_c$ are both to be proportional to $y$, i.e.,

$$\beta - U_{yy} = \beta_c = -ay$$

$$U = -by.$$  \hspace{1cm} (a, b > 0)  \hspace{1cm} (10)

Then the equation is

$$\frac{d^2\psi}{dy^2} + \left(\frac{-ay}{-by - C} - k^2\right) \psi = 0.$$  \hspace{1cm} (11)

The condition (10) are not satisfied by any profile of $U$. However, it is shown that the Eq. (11) possesses a wave solution similar to a normal Rossby wave.

The wave solution (hereafter it is referred to as pseud-Rossby wave) shows behaviors analogous to those of Rossby wave near the inflection point and critical level. In this model, $U - C$ is proportional to the distance from the critical level, and $\beta_c$ is also proportional to the distance from the inflection point. These features are similar to those of normal barotropic Rossby wave near the critical level and the inflection point. Therefore the wave solution near the inflection point and the critical level in this model is the same as those of normal Rossby wave.

We define $Q$ as

$$Q = \frac{-ay}{-by - C} - k^2.$$  \hspace{1cm} (12)

Depending on the sign of $Q$, the region is classified into an internal wave region and an external wave region. The sign of $Q$ at $y = \pm \infty$ is determined by the sign of $a/b - k^2$. In the case of $a/b - k^2 < 0$, the distribution of $Q$ is given by Fig. 2 with (a) for $C > 0$ and (b) for $C < 0$. The wave is internal only between the critical level and the inflection point. This situation is irrelevant to the OR. On the other hand, in the case of $a/b - k^2 > 0$, the distribution of $Q$ is given by Fig. 3. At $y = \pm \infty$, the wave is internal.

Next, we examine the validity of WKB approximation. The WKB approximation is valid only if

$$\left|\frac{dQ}{dy}/4Q^{3/2}\right| = \left|\frac{aC}{(-by - C)^2}/4Q^{3/2}\right| \ll 1.$$  \hspace{1cm} (13)

This condition is satisfied except near the inflection point and the critical level. Therefore, the solution of Eq. (11) is either a plane wave type or an exponential type except the critical level and the inflection point.
The behavior of momentum flux derived from Eq. (11) is,
\[ \text{uv} = \text{const} \quad \text{for} \quad U \neq C, \]
\[ \text{uv}_+ - \text{uv}_- = -\pi \left( \frac{k\beta_e}{2U_y} \right) |Y|^2 \quad \text{at} \quad U = C, \]
which coincides with that for the normal barotropic Rossby wave.

The local group velocity in the y direction in our artificial system is
\[ C_{gy} = \frac{2\beta ek l}{(k^2 + l^2)^2}, \]
where \( l^2 = \frac{-ay}{-by - C} - k^2 \quad \text{and} \quad \beta_e = -ay. \)
which is again similar to that of normal barotropic Rossby wave.

We note the following points, which will be used in the later discussion: If \( \beta_e > 0 \), the Rossby wave propagates only westward relative to the fluid. This is a common situation in the real atmosphere. If \( \beta_e < 0 \), on the other hand, the Rossby wave propagates only eastward. The marginal point for these two regions is the inflection point where \( \beta_e = 0 \). These facts show that the relations between \( C_{gy} \) and momentum flux \( \text{uv} \) are as follows,

\[ \text{if} \quad \beta_e > 0, \quad C_{gy} > 0 \quad \text{when} \quad \text{uv} < 0, \quad (16) \]
\[ \text{if} \quad \beta_e < 0, \quad C_{gy} < 0 \quad \text{when} \quad \text{uv} < 0, \quad (17) \]
Thus, the propagating direction of information changes as the sign of \( \beta_e \) changes for constant momentum flux. The OR of a barotropic Rossby wave is closely related to characteristics which will be shown in the following sections.

Fig. 2. Distribution of Q for simple artificial model when \( a/b - k^2 < 0 \). (a) \( C > 0 \). (b) \( C < 0 \).
As mentioned above, this pseudo-Rossby wave has basic characteristics similar to those of a normal Rossby wave, yet the simplicity of its governing equation allows an analytical solution, which is useful to clarify the mechanism of the OR of the Rossby wave. We will analyze the solution of this simple model and solve the scattering problems in the following sections.

3.2 Transformation of the equation and asymptotic solutions.

If we put \( a/b = A \), \( a/b - k^2 = B \), \( C/b = C' \), and \( C' + y = z \), Eq. (11) can be transformed into the standard form of confluent hyper-geometric equation (e.g., Bateman 1953) as follows,

\[
\frac{d^2 V}{dz^2} + \left( B + \frac{A C'}{-z} \right) \psi = 0. \tag{18}
\]

1) In case of \( B > 0 \), Eq. (18) can be rewritten, by the transformation \( \psi = e^{iBz} V(z) \), with \( \xi = -2i\sqrt{B} z \), as follows,

\[
\frac{d^2 V}{d\xi^2} - \frac{dV}{d\xi} - \frac{\alpha V}{\xi} = 0,
\]

where \( \alpha = -\frac{A}{2i\sqrt{B} C'} \).

Two independent solutions \( V_1 \) and \( V_2 \) of this equation are written as follows,

\[
V_1 = \Phi(\alpha, 1, 2; \xi), \tag{20}
\]

where

\[
\Phi(a, c; z) = 1 + \frac{a z}{c 1!} + \frac{a(a+1) z^2}{c(c+1) 2!} + \cdots \tag{21}
\]

\[
V_2 = \Psi(\alpha, 0; \xi), \tag{22}
\]
where
\[
\psi(a, c; z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Gamma(a + s) \Gamma(-s) \Gamma(1 - c - s) z^s}{\Gamma(a) \Gamma(a + c + 1)} ds,
\]
\[
-\frac{3}{2} \pi < \arg z < \frac{3}{2} \pi,
\]
\[-\text{Re}(a) < r < \min(0, 1 - \text{Re}(C)).
\] (23)

These solutions have the following asymptotic forms at \(z \to \infty\).

\[
V_1 \sim \frac{\Gamma(2)}{\Gamma(1 - \alpha)} e^{i\pi(\alpha + 1)} \xi^{-\alpha} + \frac{\Gamma(2)}{\Gamma(1 + \alpha)} e^{i\pi\alpha},
\] (24)

\[
V_2 \sim \xi^{-\alpha}, \quad \text{where} \quad \epsilon = 1 \quad (\epsilon = -1)
\]

\[\text{for} \quad \text{Im}(\xi) > 0 \quad (\text{Im}(\xi) < 0).
\] (25)

In terms of \(z\), Eqs. (24) and (25) are expressed as

\[
\psi_1 \sim \frac{\Gamma(2)}{\Gamma(1 - \alpha)} e^{i\pi(\alpha + 1) + \frac{\pi}{2} \text{arg}(2i\sqrt{B}z)} + \frac{\Gamma(2)}{\Gamma(1 + \alpha)} e^{i\pi\alpha},
\] (26)

and

\[
\psi_2 \sim e^{i\pi\alpha} e^{-2i\pi(\alpha + 1) + \pi \text{arg}(2i\sqrt{B}z)},
\] (27)

as \(z \to \infty\),

\[\text{where} \quad \epsilon = -1 (\epsilon = 1) \quad \text{for} \quad z = +\infty (z = -\infty)
\]

\[\text{and} \quad \hat{C} = \frac{A}{\sqrt{B}} \quad \alpha' = -\frac{\hat{C}}{2}.
\] (28)

The solutions are of exponential type at \(z \to \infty\).

4. The scattering problems

We solve the scattering problems to investigate how the OR occurs. If \(B < 0\), the solutions are almost of exponential type except a narrow region around \(y = 0\) (see Fig. 2). Thus we only deal with the case of \(B > 0\).

4.1 The case with no boundaries

Now, we consider a wave incident onto \(y = 0\) from negative \(y\) direction, and scattered near \(y = 0\). The wave is partly reflected and partly transmitted there (see Fig. 4). We assume that there is no boundary anywhere in the region considered. We impose the condition that only a transmitted wave exists at \(y = +\infty\). The solution of Eq. (11) can be written as

\[
\psi = A_1\psi_1 + A_2\psi_2,
\] (30)

where the asymptotic expansions of \(\psi_1\) and \(\psi_2\) are given by Eqs. (26) and (27), respectively. In \(y < 0\) region, where \(\beta_0 > 0\), the wave propagating toward the positive \(y\) direction is given by

\[
\psi \sim e^{i\pi y},
\] (l > 0).

On the other hand, in \(y > 0\) region where \(\beta_0 < 0\), the transmitted wave propagating toward the positive \(y\) direction is expressed by

\[
\psi \sim e^{-i\pi y}.
\] (32)
Thus, we see that the group velocity with a fixed sign of \( l \) changes its sign at \( y = 0 \) (This is another expression of the change of the relation between group velocity and momentum flux at \( y = 0 \) as shown in the previous section). If the asymptotic expansion (26) and (27) are considered, the relation between \( A_1 \) and \( A_2 \) becomes,

\[
A_2 = A_1 \frac{\Gamma(2)}{\Gamma(1-\alpha)} e^{\frac{\pi}{\alpha}}.
\]  

(33)

The reflection coefficient \( R \), the ratio of the amplitudes of the incident wave \( \sim e^{il\psi} \) and the reflected wave \( \sim e^{-il\psi} \) at \( y = -\infty \) is given by,

\[
|R| = \left| \frac{A_1 \frac{\Gamma(2)}{\Gamma(1+\alpha)} e^{\frac{\pi}{\alpha}}}{A_2 e^{\frac{\pi}{\alpha}} - A_1 \frac{\Gamma(2)}{\Gamma(1-\alpha)}} \right| = \left| \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} e^{\frac{\pi}{\alpha}} \right|.
\]  

(34)

It is noted that \( |\Gamma(1-\alpha)| = |\Gamma(1+\alpha)| \) when \( \alpha \) is real. Figure 5 shows \( |R| \) as a function of \( \alpha \). When the critical level coincides with the inflection point \( (\alpha = 0) \), \( |R| \) becomes infinite. A region with large \( |R| \) exists near \( \alpha = 0 \). When the critical level is behind the inflection point \( \alpha < 0 \), \( |R| \) decreases as \( |\alpha| \) increases, but it remains larger than 1. When the critical level is in front of the inflection point \( (\alpha > 0) \), \( |R| \) decreases as \( |\alpha| \) increases. It is interesting to note that, for \( \alpha < \ln(2/\pi) (\approx 0.22) \), \( |R| \) remains larger than 1.

We consider the results obtained above. Lindzen and Tung (1978) used the following equations to discuss the OR as shown before,

\[
1 - |R|^2 = -|X|^2 \bar{\nu} \bar{\nu},
\]  

(35)

provide that \( \beta_e \) is positive at the observation point, and

\[
\bar{\nu} \bar{\nu} = \pi \left| \frac{k_\alpha}{2U_y} \right|^2 \left| Y\right|^2 + \bar{\nu} \bar{\nu} +.
\]  

(36)

To use these relations, it is necessary to remember that the relation between the direction of group velocity and the sign of momentum flux changes according to the sign of \( \beta_e \) as shown in the previous section. We consider the role of the second term in the r.h.s. of Eq. (36). If a radiation condition at \( y = \pm \infty \) is applied, the momentum flux behind the critical level is

\[
\bar{\nu} \bar{\nu} > 0,
\]  

(37)

because \( \beta_e < 0 \) for \( y > 0 \). This is advantageous for OR in our scattering problem because \( \beta_e > 0 \) at the observation point \( y = -\infty \) (see Eqs. (16) and (17)).

Next, we consider the role of the momentum flux jump at the critical level (the first term in the r.h.s. of Eq. (36)). As shown before, the momentum flux jumps at the critical level, so that the wave propagates to the critical level is absorbed at the critical level corresponding to the sign of \( \beta_e \). However, for the wave that exists in the region where the sign of \( \beta_e \) is opposite to that at the critical level, the momentum flux jump at the critical level act to enhance the wave which travels away from the critical level. This is nothing but what Lindzen and Tung (1978) pointed out. On the basis of the above consideration, we analyze our results of the scattering problem.

First, we consider the case of \( \alpha < 0 \), where the critical level is behind the inflection point. The radiation condition at \( y = \infty \), \( \bar{\nu} \bar{\nu} > 0 \) is advantageous for the OR as shown in the previous paragraph. The momentum flux jump at the critical point, \( \pi \left| \frac{k_\alpha}{2U_y} \right|^2 \left| B\right|^2 > 0 \) is also advantageous for the OR because the sign of \( \beta_e \) for \( y < 0 \) is different from that at the critical level (i.e., \( \beta_e < 0 \) at \( y = C \)).

Accordingly, we have

\[
\bar{\nu} \bar{\nu} > 0.
\]  

(38)

The OR always occurs in this case.

Secondly, we consider the case of \( \alpha > 0 \), where the critical level is in front of the inflection point. In this case, the radiation condition at \( y = +\infty \) is disadvantageous for the OR. However, the momentum jump at the critical level suppresses the OR, because the sign of \( \beta_e \) at the critical level is the same as that of an observation point in \( y > 0 \). Then the first term in the r.h.s. of Eq. (36) has the opposite sign of the second term. Whether OR or PR occurs
depends on the magnitude of these terms. If the critical level is near the inflection point, the absolute value of the momentum flux jump at the critical level is relatively small. Therefore the OR occurs.

The marginal value of \( C \) which separates the OR or FR is \( \ln(2/r) = 0.22 \), as shown before.

Lastly, we treat the case of \( C = 0 \), where the critical level coincides with the inflection point. The equation has no singular point and no momentum flux jump. Therefore \( \bar{w} = \text{const} \) in the whole region. The OR, however, occurs because the sign of group velocity changes as the sign of \( \beta_e \) changes at \( y = 0 \) as discussed above. In this case, the equation is very simple and is given by

\[
\frac{d^2 \psi}{dy^2} + \left( \frac{a}{b} - k^2 \right) \psi = 0.
\] (39)

The solution which satisfies the radiation condition at \( y = +\infty \) is

\[
\psi = e^{iy+\delta'}, \quad \left( l = \sqrt{\frac{a}{b} - k^2} \right),
\] (40)

where \( \delta' \) is a constant. This solution represents the wave which propagates to the positive \( y \) direction where \( \beta_e < 0 \), and the wave which propagates to the negative \( y \) direction where \( \beta_e > 0 \). In \( y < 0 \) region, there is only a reflected wave in our scattering problem. Therefore, \( |R| = \infty \). This solution is a normal mode with the radiation conditions at \( y = \pm\infty \).

4.2 The case with rigid wall in the region \( y > 0 \)

It is difficult to calculate the reflection coefficient \( R \) analytically if a rigid wall is located somewhere in \( y > 0 \). However if the rigid boundary is far from the inflection point, we can calculate \( R \) approximately as follows. As in the previous subsection, we write the solution as,

\[
\psi = A_1 \psi_1 + A_2 \psi_2.
\] (41)

The refractive index squared is written as,

\[
Q = \frac{-a\gamma}{-b\gamma - C - k^2}.
\] (42)

If \( y \) is large enough, \( Q \) is almost independent of \( y \) and \( C \). Therefore the approximate solution of Eq. (11) is,

\[
\psi = A' e^{ilz} + A'e^{-ilz},
\] (43)

where \( l = \sqrt{\frac{a}{b} - k^2} \), for large \( z = y + \bar{C}' \). Then, the solution will be written as follows at large \( y > 0 \),

\[
\psi \sim D \left[ e^{ilz} + e^{-ilz+is} \right].
\] (44)

The solution is a superposition of the incident wave and the reflected wave with the same amplitude but different phase. The difference of phase is determined only by the position of the rigid boundary. Then we use the following condition as the boundary condition at \( y = +\infty \).

\[
A_2' = A_1'e^{-is}.
\] (45)

This condition is approximately equivalent with the condition that there is a rigid boundary at very large but finite positive \( y \). By using Eq. (45), we get

\[
A_2e^{\bar{\gamma}z} - A_1e^{-\bar{\gamma}z} = \frac{\Gamma(2)}{\Gamma(1-\alpha)} \frac{\Gamma(2)}{\Gamma(1+\alpha)} e^{is}.
\] (46)

Then, the reflection coefficient at \( y = -\infty \) becomes

\[
|R| = \frac{1}{\sqrt{2e^{\bar{\gamma}x}(e^{\bar{\gamma}x} - 1) [1 + \cos(2\theta - \delta)] + 1}},
\] (47)

where \( \theta = \arg(\Gamma(1+\alpha)) \).

In Fig. 6, \(|R|\) is shown as a function of \( \bar{C} \) for four different values of \( \delta \). It is seen that \(|R| < 1 \) for \( \bar{C} > 0 \) and \(|R| > 1 \) for \( \bar{C} < 0 \). As \( \bar{C} \) is increased from \( -\infty \), \(|R| \) reaches a maximum at a certain negative value of \( \bar{C} \), and then decreases monotonically. \(|R| \) for negative \( \bar{C} \) shows a strong dependence on \( \delta \). It is noted that the configuration of the basic flow in our case with an equivalent rigid wall completely coincides with Lindzen and Tung's (1978) wave geometry which was shown in the previous section. Yamada and Okamura (1984) calculated reflection coefficient for a Rossby wave incident on the tanh type basic flow. The dependence of \(|R| \) on \( \bar{C} \) in our result is similar to that of their calculation.

The difference between the present case and the case with no boundaries (Subsection 4.1) is

\[
\bar{w} = 0,
\] (48)

in Eq. (36) for the former. This means that, due to the rigid wall condition, there are two waves that have same amplitude and propagate in the opposite directions, in the region beyond the critical level. The condition whether the OR occurs or not is determined solely by the sign of momentum flux jump at the critical level.

First, we consider the case of \( \bar{C} < 0 \). In this case, the critical level is located in \( y > 0 \) region where \( \beta_e < 0 \). Thus, for a wave that is incident from \( y = -\infty \) where \( \beta_e > 0 \), the momentum flux jump at the critical level is advantageous for the OR. Since \( \bar{w} = 0 \), the OR always occurs by Eq. (36).

When \( \bar{C} > 0 \), on the other hand, the critical level located in \( y < 0 \) region where \( \beta_e > 0 \). Thus the momentum flux jump at the critical level is disadvantageous for the OR and no OR occurs by Eq. (36).
In the case of $\dot{C} = 0$, there is no momentum flux jump. Then $\overline{uv} = 0$ and $|R| = 1$. In this case, the equation is the same as Eq. (39). The solution which satisfies the boundary condition is
\begin{equation}
\psi = e^{-\mu y} + e^{i\mu y} + i\eta.
\end{equation}
(49)
The first term in the r.h.s. is the same solution in the case of no boundaries (Subsection 4.1) and it expresses a radiating solution from the inflection point. The second term expresses the wave that is an absorbed solution at the inflection point. The ratio of these two wave amplitudes is 1, and $R = 1$.

5. Time dependent behaviors of the OR

In the previous section, we investigated the OR as scattering problems. In this section we investigate the time dependent behavior of the OR with a numerical model. We consider the following configuration of $\beta_c$ and $U$:
\begin{align}
\beta_c &= -ay, \\
U &= -by, \text{ for } |y| \leq y_0,
\end{align}
(50)
\begin{align}
\beta_c &= -ay_0, \\
U &= -by_0, \text{ for } |y| > y_0.
\end{align}
(51)
The basic field parameters are $a = 4$, $b = 1$, and $y_0 = 4$ in numerical calculation. This configuration is the same as that used in the previous sections except for $|y| > y_0$. The reason we assume Eq. (51) is to avoid an increase of group velocity away from the inflection point (see Eq. (13)). The increase of group velocity increases economical cost of the numerical calculation. The numerical calculations were done with a usual finite deference scheme in space and time. The interval of finite difference in space changes from 0.001 near the inflection point to 0.1 outside $y_0$. Time integration is done by the leap-frog scheme. The boundaries are located far from the inflection point (at $y = \pm y_1$, $y_1 > 0$). The boundary condition at $y = -y_1$ is $\psi = 0$ and that at $y = y_1$ is $\psi = \psi_1(t)$ for wave train and $\psi = 0$ for wave packet. $\psi_1(t)$ is forcing of incident wave train whose amplitude is a tanh type function of time with $0$ at $t = 0$ and constant for large $t$. In Fig. 7, shown

![Graphs showing time dependent behaviors of the OR.](image)
are results when a wave train with $k^2 = 3$ and $\hat{C} = 0$ is incident to the inflection point from negative $y$ region. In this case, the critical level coincides with the inflection point ($y = 0$) the reflection coefficient $R$ is infinite as shown before, i.e.,

$$|R| = \frac{1}{\left| e^{\hat{C}\pi} - 1 \right|} = \frac{1}{\hat{C}\pi + \frac{1}{2} \hat{C}^2 \pi^2 + \cdots} \to \infty.$$ (52)
At $t=0$ the wave train is incident from the left side. After the incidence of the wave train, the amplitudes of both reflected and transmitted waves decay linearly with the distance from $y=0$. As the group velocity is constant for $|y|>y_0$, this shape of amplitude of wave means a linear growth of the amplitude. This is a kind of resonance as pointed out by McIntyre and Weissman (1978) for the case of internal gravity wave. As is well known, the amplitude of the wave grows linearly with time when resonance occurs. This can also be shown in our case as follows. The solution at $y=-\infty$ (far from $y=0$) can be given by a linear combination of a free solution and a forced solution:

$$
\psi = \psi_0 + E R e^{-iy} + F e^{-iy},
$$

(53)

where the first term in the r.h.s. is the incident wave, the second term, the reflected wave, and the third term, a normal mode solution term (see Subsection 4.1). In other words, the first two terms express forced solutions and the last term a free solution. $E$ is a complex constant with $|E|=1$ and $F$ is also a constant. Using the usual procedure, and considering the limit of $C \to 0$ in Eq. (54) we obtain that

$$
\psi = \psi_0 + E' \frac{klb}{\pi(k^2 + L^2)} |\psi_0| e^{-iy} + F' e^{-iy},
$$

(54)

where a use has been made of

$$
l^2 = \frac{a}{b} - k^2, \quad \text{and} \quad \hat{C} = \frac{AC}{b\sqrt{B}} = \frac{A}{b\sqrt{B k}}.
$$

(55)

The definitions of $A$ and $B$ are the same as in Section 3 and $E'$ is some constant with $|E'|=1$. This theoretical prediction of the linear growth rate coincides with that derived from our numerical initial value problem. If we put a wave packet with the same phase velocity into the zonal flow configuration above, after the incidence of wave packet into the inflection point, a reflected and a transmitted wave grow linearly with time. Eventually, waves with constant amplitudes are radiated toward both sides forever (not shown). This demonstrates an excitation of a normal mode.

In the case of the OR with $\hat{C}$ slightly difference form $\hat{C} = 0$, the behaviors of reflected and transmitted waves are almost the same as those in the normal near-resonant wave. As such a wave has some wave components with $\hat{C} = 0$ unless the incident wave has a constant amplitude everywhere, the reflected wave is also expressed as a superposition of the over-reflected wave and the resonant reflected wave with $\hat{C} = 0$ (Fig. 8). If the phase velocity of the incident wave is largely different from 0, the influence of the resonant wave is small (not shown). Lastly, we point out that we found any unstable mode neither in the eigenvalue problem with radiation condition at $y = \pm \infty$ nor in the initial value problem. This is because of lack of cavity as pointed out by Lindzen and his colleagues.

6. Conclusion

Based on a simplified model equation which is somewhat artificial but supports pseudo-Rossby waves similar to Rossby wave in many respects, the mechanism of the overreflection of Rossby waves are explored. The model equation has advantage of having a single inflection point in contrast to the existence of more than two inflection points in an ordinary barotropic vorticity equation. We have shown that the reversal of the sign of $\beta$, at the inflection point, which reverses the relation between momentum flux and the direction of group velocity, is the most essential for the occurrence of OR. Because of this reversal, the normal wave propagation, attenuation or absorption subjected by the causality in the region of a sign of $\beta$, acts inversely for the wave in the region of opposite sign of $\beta$. This is implicitly shown in Lindzen and Tung (1978) but not explicitly. In our simplified model, two special waves whose critical levels coincide with the inflection point exist. They are a radiating wave and an absorbed wave at the inflection point. The former is a normal mode with radiation conditions at $y = \pm \infty$. If a wave train whose critical level coincides with the inflection point is incident on the critical level, a kind of a resonance occurs: the reflected and transmitted waves grow linearly with time.

Since the distribution of $\beta - U_{yy}$ in a real basic flow is more complex than the one studied in this paper, our conclusion must be confirmed for real Rossby waves. In a numerical initial value problem, however, the incident wave is affected by one of inflection points and one of critical levels before it is influenced by other inflection points, critical levels or boundaries. The results will be presented elsewhere.

Our analysis has been restricted for the case of a barotropic Rossby wave. However, our conclusion is likely to hold for other waves, such as internal gravity waves and should be extended to these waves if the overreflection concept has universality in waves in the parallel shear flows.

Acknowledgments

The author would like to thank deeply the late Professor M. Uryu for his guidance and kind encouragement. The author also wishes to thanks to Dr. M. Takahashi, Tokyo University and Dr. Y. Wakata, Kyushu University for their critical discussions and suggestions. His thanks are extended to Dr. Morita, Professor Miyahara, Kyushu University, and Professor Takefu, Saga, Medical College
Fig. 8. As in Fig. 7 except for $C = -0.075$. 
and all other members of atmospheric physics group of Kyushu University for their suggestions and encouragements. In addition, very useful comments and criticisms by two anonymous reviewers certainly have helped to improve the paper.

References

パロトロビックロスピー波の過剰反射のメカニズムについて

高野清治
（気象庁気候情報課）

Lindzenらによって提案されたパロトロビックロスピー波の過剰反射のメカニズムについて簡単化したモデルを使い再検討した。我々の簡單化したモデルは人工的だが、過剰反射に果たす変曲点、クリティカル・レベル、境界条件の果たす役割を調べるのに適している。我々の結果によれば、過剰反射には変曲点を境として、運動量フラックスと群速度の向きの関係が反転することが本質的であることがわかった。また、クリティカル・レベルの向こう側の反射層は Lindzen と Barker (1985) が重力波の場合について示したようにロスピー波の場合も過剰反射にとって重要でないことがわかった。