NOTES AND CORRESPONDENCE

An Effective SVD Calculation Method for Climate Analysis

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Abstract

We propose an effective method for treating singular value decomposition (SVD) analysis. This method is economical in the sense that the problem can be reduced to a matrix problem of the temporal size, not spatial size. Thus, this method is efficient when the spatial size is very large and the number of the ensemble is much smaller, as in a typical climatological problem. With this method, it is very simple to change the area of analysis by the use of weights, which are introduced to deal with the inhomogeneity of the data.

1. Introduction

Singular value decomposition (SVD) analysis is a useful method for examining relationships between meteorological data. This analysis method is a natural extension of the empirical orthogonal function (EOF) analysis method (*e.g.*, North *et al.*, 1982) employed for two different fields. The essence of this method is to extract the specified patterns, which are related to each other, from huge meteorological variations. Recently, many works have been performed using this analysis method (*e.g.*, Wallace *et al.*, 1992).

However, the calculations using this method require more computational resources when the spatial sizes become larger. When we deal specifically with climate data, the spatial sizes are usually enormous. Compared to the scale of the spatial array, the size of the ensemble is usually not so large for climate analysis. In the case of EOF, Hirose and Kutzbach (1969) proposed a simple method to overcome this difficulty. In their method, the speed of the calculation depends only on the temporal, not spatial, size of the data. Their method then becomes very fast when we deal with climate analysis. Therefore, a generalization of the Hirose and Kutzbach (1969) method to calculate SVD analysis appears hopeful.

The purpose of this note is to show an economical and convenient method of SVD analysis by generalizing the method by Hirose and Kutzbach (1969).

In Section 2, we propose an alternative method of SVD analysis, by means of the SVD variational principle with respect to the temporal coefficients. We compare the performance of the new method to the direct SVD one for concrete meteorological problems in Section 3. Section 4 is devoted to a summary and discussions.

2. An alternative method of SVD analysis

Let us consider two anomalous sets of data, such as an N_z -component vector representing the 500 hPa height anomaly z_t and an N_s -component vector representing the sea surface temperature (SST) anomaly s_t . Here suffix t represents the t -th observation with $t = 1$ to N_t . Let $\mathbf{Z}$ be an $N_z \times N_t$ matrix and $\mathbf{S}$ be an $N_s \times N_t$ matrix, whose t -th columns are the observational vectors z_t and s_t . The t -th observation of the i -th variable of $\mathbf{Z}$ is denoted by z_{it} and the k -th variable of $\mathbf{S}$ by s_{kt} . We wish to determine the two vectors $\mathbf{e}$ and $\mathbf{f}$ such that the time series of the projections of the data have the highest resemblance. We define the resemblance by the covariance of these two time series as

$$I = N_t^{-1} \mathbf{e}' \mathbf{Z} \mathbf{S}' \mathbf{f} \equiv \mathbf{e}' \mathbf{V} \mathbf{f}, \quad (1)$$

where the prime denotes the matrix transpose, and $\mathbf{V}$ denotes the $N_z \times N_s$ variance matrix whose ik component is given by

$$V_{ik} = N_t^{-1} \sum_{t=1}^{N_t} z_{it} s_{kt}. \quad (2)$$

If we consider the maximizing problem under the constraint $J = \mathbf{e}' \mathbf{e} = 1$ and $K = \mathbf{f}' \mathbf{f} = 1$, we can

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derive the Euler-Lagrange equations for vectors $\mathbf{e}$ and $\mathbf{f}$, written as

$$\begin{aligned} \mathbf{V}\mathbf{f} &= \lambda\mathbf{e} \\ \mathbf{V}'\mathbf{e} &= \lambda\mathbf{f}. \end{aligned} \quad (3)$$

These equations show that the vectors $\mathbf{e}$ and $\mathbf{f}$ are just the left and right singular vectors of the singular value decomposition (SVD) of the matrix $\mathbf{V}$, which is given by

$$\mathbf{V} = \sum_{\alpha=1}^R \lambda_{\alpha} \mathbf{e}_{\alpha} \mathbf{f}_{\alpha}' \equiv \mathbf{E} \mathbf{L} \mathbf{F}', \quad (4)$$

where $\mathbf{e}_{\alpha}$ and $\mathbf{f}_{\alpha}$ are orthonormal sets of R ($= \min(N_s, N_z, N_t - 1)$) vectors, $\mathbf{E}$ ($\mathbf{F}$) is an $N_z \times R$ ($N_s \times R$) matrix whose α -th column is $\mathbf{e}_{\alpha}$ ($\mathbf{f}_{\alpha}$), and $\mathbf{L}$ is a diagonal matrix whose α -th diagonal element is a singular value λ_{α} . Note that the value of λ in Eq. (3) equals that of I itself, and therefore the solution that maximizes I is $\mathbf{e}_1$, $\mathbf{f}_1$, which are associated only with λ_1 , the largest singular value of Eq. (4). However, we usually treat the set of modes in Eq. (4) for SVD analysis. We shall order the singular values such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_R$.

Next, let us expand the anomalous fields $\mathbf{Z}$ and $\mathbf{S}$ by the singular vectors. The two fields are given by

$$\begin{aligned} \mathbf{Z} &= \sum_{\alpha=1}^R \mathbf{e}_{\alpha} \mathbf{a}_{\alpha}' \equiv \mathbf{E} \mathbf{A}' \\ \mathbf{S} &= \sum_{\alpha=1}^R \mathbf{f}_{\alpha} \mathbf{b}_{\alpha}' \equiv \mathbf{F} \mathbf{B}', \end{aligned} \quad (5)$$

where $\mathbf{A}$ and $\mathbf{B}$ are $N_t \times R$ time-coefficient matrices whose α -th columns are $\mathbf{a}_{\alpha}$ and $\mathbf{b}_{\alpha}$, respectively. It is easy to show that these time coefficients satisfy the orthogonal relationship

$$N_t^{-1} \mathbf{A}' \mathbf{B} = \mathbf{L} \quad (6)$$

by putting Eq. (5) into Eq. (3). By using the orthonormality of the singular vectors, Eq. (5) can be solved for the coefficients to give

$$\begin{aligned} \mathbf{A} &= \mathbf{Z}' \mathbf{E} \\ \mathbf{B} &= \mathbf{S}' \mathbf{F}. \end{aligned} \quad (7)$$

By using Eq. (6), we also have the equations

$$\begin{aligned} \mathbf{E} &= \mathbf{Z} \mathbf{B} (N_t \mathbf{L})^{-1} \\ \mathbf{F} &= \mathbf{S} \mathbf{A} (N_t \mathbf{L})^{-1}. \end{aligned} \quad (8)$$

Equations (7) and (8) show the dual relationship between the time coefficients and the singular vectors. Bearing in mind this duality, we can construct an alternative form of the variational problem as follows: if we define the time coefficients of the vectors $\mathbf{e}$ and $\mathbf{f}$ by

$$\begin{aligned} \mathbf{a} &= \mathbf{Z}' \mathbf{e} \\ \mathbf{b} &= \mathbf{S}' \mathbf{f}, \end{aligned} \quad (9)$$

the covariance in Eq. (1) can be rewritten as

$$I = N_t^{-1} \mathbf{a} \mathbf{b}'. \quad (10)$$

Similarly, we can relate the vectors $\mathbf{e}$ and $\mathbf{f}$ to time coefficients as

$$\begin{aligned} \mathbf{e} &= \Lambda^{-1} \mathbf{Z} \mathbf{b} \\ \mathbf{f} &= \Lambda^{-1} \mathbf{S} \mathbf{a}, \end{aligned} \quad (11)$$

if we consider Eq. (8) and use $\Lambda = N_t \lambda$. Then we can consider the maximizing problem of I under the constraint $J = \mathbf{e}^2 = 1$ and $K = \mathbf{f}^2 = 1$ for time coefficients $\mathbf{a}$ and $\mathbf{b}$. The Euler-Lagrange equations now become

$$\begin{aligned} \mathbf{Y} \mathbf{a} &= \Lambda \mathbf{b} \\ \mathbf{X} \mathbf{b} &= \Lambda \mathbf{a}, \end{aligned} \quad (12)$$

with

$$\begin{aligned} \mathbf{X} &\equiv \mathbf{Z}' \mathbf{Z} \\ \mathbf{Y} &\equiv \mathbf{S}' \mathbf{S}. \end{aligned} \quad (13)$$

It can be seen that Eq. (12) is equivalent to Eq. (3) if Eq. (11) is true and so the solutions to Eq. (12) can be distinguished by the mode α as before. Equation (12) can be written as the eigenvalue problem

$$\begin{aligned} \mathbf{U} \mathbf{a} &= \Lambda^2 \mathbf{a} \\ \mathbf{U}' \mathbf{b} &= \Lambda^2 \mathbf{b}, \end{aligned} \quad (14)$$

with

$$\mathbf{U} = \mathbf{X} \mathbf{Y}. \quad (15)$$

Equation (14) shows that the time coefficients $\mathbf{a}$ and $\mathbf{b}$ are right and left eigenvectors of the $N_t \times N_t$ matrix $\mathbf{U}$, respectively. This also means that the matrix $\mathbf{U}$ can be decomposed as

$$\mathbf{U} = N_t \mathbf{A} \mathbf{L} \mathbf{B}'. \quad (16)$$

Note the difference between Eq. (16) and the SVD in Eq. (4). It is also clear from Eqs. (6) and (16) that the total squared variance can be easily calculated from the relation

$$\text{Tr} \mathbf{U} = N_t^2 \sum_{\alpha=1}^R \lambda_{\alpha}^2 = N_t^2 \sum_{i=1}^{N_z} \sum_{k=1}^{N_s} V_{ik}^2. \quad (17)$$

Although solving Eqs. (3) and (12) are equivalent, the SVD algorithm in Eq. (4) for solving Eq. (3) requires a large amount of computational time when the spatial sizes N_z or N_s are very large. In fact, with the method we perform about $O(N_z N_s^2)$ floating point calculations for the case $N_z \geq N_s$ (Golub and Van Loan, 1996). This is due to the huge size of the matrix $\mathbf{V}$ in Eq. (3). Compared to those of Eq. (3), the matrix sizes in Eq. (12) are determined by the size of N_t alone. Therefore, if the condition $N_z, N_s \gg N_t$ is satisfied, solving Eq. (12) will take much less computational time, because solving

the eigenvalue problem in (14) requires only $O(N_t^3)$ floating point calculations, and it is also clear that, in principle, the memory size can become very small. Although actual speed depends on the algorithm for solving the eigenvalue problem (14), the power method with a deflation technique, which subtracts matrix components with smaller modes from U by using Eq. (16), is sufficient to solve Eq. (14) as we will show in an example.

As shown in the appendix, data inhomogeneity can be considered merely by introducing weights when spatial vectors or matrices are multiplied. This characteristic of the weights can be used to consider the importance of each area of analysis, by setting larger weights in more important areas and smaller weights in less important areas. For example, we can mask an area not used within the SVD analysis simply by setting the weight for that area to zero. Treating zero-weight is somewhat troublesome for the direct method. On the other hand, the new method is based on the matrix of time array, and the operation is affected little by the spatial weight.

To see this, suppose that basic datasets are ones arranged with a regular order of longitude-latitude, and ones that partly contain areas of SVD analysis. If we follow the direct method with zero-weight by using these datasets, the matrix size becomes very large due to the area not used for the SVD analysis. The operation in Eq. (A.8) is also troublesome when the weight is zero. So, in practice, we assign the spatial array only to the area of SVD analysis. In this case, however, the order of the array is generally not that of longitude and latitude. Thus, the spatial components of the pattern vector must be rearranged to make figures. Changing the area of SVD analysis affects not only the size of the matrix but also the order of the array. On the other hand, when we follow the new method, we need not do any calculation concerning the spatial weight except for making the matrix of the time array by considering the weight. The pattern vector obtained is now of the same regular order as that of the basic datasets without any additional operation. Changing the area of SVD analysis can be easily done by altering the weight without affecting the matrix size. Even when the basic dataset is rather large compared to the area of SVD analysis, multiplying zero-weight uses little computational resources compared to the matrix calculation. In this sense, this new method is also superior to the direct one in respect to treating the weight.

3. Examples

In this section, we shall compare the performance of our new method with the direct SVD one by examining two examples within the meteorological context. Although the performance will depend on the actual program, these examples will highlight

some differences between the two methods. A power method with a deflation technique, of which the program is written by the author, is used to solve the eigenvalue problem (14), whereas the program for the SVD in Eq. (4) is adapted from Numerical Recipes in Fortran (Press *et al.*, 1992). Calculations are performed on the SUN Ultra1 workstation, model 200E.

The first example is that of a PNA pattern in the Northern Hemisphere winter (see *e.g.*, Wallace *et al.*, 1992). SVD analysis is carried out for the sea surface temperature (SST) of the northern Pacific and 500 hPa geopotential height of the same area. The SST data used in this study comes from GISST data compiled by the UK Meteorological Office (Parker *et al.*, 1995). The original $1^\circ \times 1^\circ$ data are converted to a set of 192 points on a $5^\circ \times 5^\circ$ grid from 20°N to 65°N . The 500 hPa geopotential height data used in this study are based on NMC analysis data, and are also converted to $5^\circ \times 5^\circ$ grid data. The 192 grid points used for the SVD analysis are the same as those of the SST. The time series of both data sets are normalized by their variances, and the 48 winter periods (December–February) from 1946 to 1994 are used. As these data exist inhomogeneously on a sphere, we have used the modification technique as shown in the appendix. Calculations using the new method are based on the whole northern hemisphere dataset for 500 hPa geopotential height and whole global dataset for SST, with non-zero weight in the northern Pacific area. On the other hand, calculations using the direct method are carried out by assigning the spatial array to the northern Pacific area only. SVD patterns are then calculated for the 4 largest singular values. Figure 1 shows the heterogeneous correlation pattern for the first mode obtained by the new method. The same result is obtained by the direct SVD method (not shown). It can be seen that the patterns obtained are very similar to that in Fig. 6 of Wallace *et al.* (1992), though the methods and periods of analysis are different. The computational times are shown in Table 1, which reveals that the new method is 2.5 times faster than the direct method. If we compare only the central routine (SVD routine or eigenvalue problem routine), the new method is 25 times faster, reflecting on the difference between the size of the time array (48) and that of the spatial array (192). Of course, the memory size is larger for the direct method because of the large covariance matrix.

The second example is that of the mode of interannual variability in the Southern Hemisphere winter shown by Kuroda and Kodera (1998). They showed that the zonal-mean zonal wind from May to October in the Southern Hemisphere winter has a preferable mode of variability, and this variability is closely related to the wave activities at that time. Although they have extracted the mode by

S1(SST)

$r=0.81$

48%

S1(Z500)

Fig. 1. Heterogeneous correlation ($\times 10$) pattern for the first mode of the SVD analysis between the SST (upper panel) and 500-hPa geopotential height (lower panel) of the northern Pacific in winter time. Squared covariance fraction between two fields is expressed as a percentage and is printed on the right. Temporal correlation coefficients between the expansion coefficients r are printed on the left. The percentages of the variance of respective fields explained by this mode are printed on the upper right-hand corner of each panel.

the multiple empirical orthogonal function (M-EOF) method, it can be also extracted by using the SVD analysis method between the zonal-mean zonal wind and the vertical or horizontal component of the E-P flux (Kuroda and Kodera, 1998). Hence, we shall perform here the SVD analysis between the zonal-

Table 1. CPU time (in units of seconds) for the execution of the programs for the PNA pattern in the Northern Pacific winter (PNA) and the interannual mode of variability in the Southern Hemisphere winter (SH), by means of the direct SVD method (SVD) and the new method (new) on the SUN Ultra1 workstation. The numbers inside the brackets show the execution time for the central routine (SVD or eigenvalue problem) only.

	PNA	SH
SVD	16.0 (12.6)	921.4 (905.8)
new	6.4 (0.5)	0.5 (0.1)

mean zonal wind and the vertical component of the E-P flux as an example of the application to a very large spatial array. The data used in this analysis are the monthly mean zonal-mean zonal wind and the vertical component of the E-P flux, averaged from the 5-day mean fields of NMC troposphere-stratosphere data. Both time series are normalized by their variances, and 16-year data from 1979 to 1994 are used. Thirteen-grid points from 80°S to 20°S are selected in the meridional direction, and 10-grid points from 850 hPa to 1 hPa are selected in the vertical. As each period of analysis consists of 6 consecutive months, the total spatial size for this multiple SVD is 780 ($= 13 \times 10 \times 6$). All weights are set to be unity, and SVD patterns are calculated for the 4 largest singular values. The resultant heterogeneous correlation patterns for the first mode obtained by the new method is shown in Fig. 2. The same result is also obtained by the direct method (not shown). However, the computational times differ greatly, as seen in Table 1. This shows that the new method is 2,000 times faster than the direct method, and 10,000 times faster in the central routine. Evidently, this results from the enormous difference between the size of the time array (16) and that of the spatial array (780).

4. Summary and discussions

We have proposed a simple, economical and convenient method of performing singular value decomposition (SVD) analysis. In this method, the computational speed does not depend on the spatial size but on the ensemble size only. Another advantage of this method is the ease with which changing the area of analysis, by utilizing weights, can be accomplished. This naturally keeps the computational time small. In this sense, the method is efficient for climate analysis.

We have demonstrated the performance of this method by looking at two typical examples. These examples apparently show that the new method becomes more and more efficient when the spatial size

Fig. 2. The same as Fig. 1, except for the multiple SVD analysis between the zonal-mean zonal wind (upper panel) and the vertical component of the E-P flux (lower panel) of the Southern Hemisphere from May to October.

becomes very large compared to that of the ensembles. Of course, if the ensemble size is very large compared to that of the spatial one, then the direct SVD method is more efficient. However, this is not usually the case because we often deal with two-dimensional patterns in meteorological fields. Time is of a one-dimensional nature and, compared to the two-dimensional patterns, can be generally considered to be small.

The use of the power method with a deflation technique as in the examples is not crucial for our method. Any technique suffices if the eigenvalue problem in (14) can be solved quickly. However, the use of the power method has advantages here. First, the matrix U in Eq. (14) has only positive eigenvalues. This guarantees convergence for the power method. Second, we usually wish to know modes associated with the first few largest singular values only. The power method with deflation automatically yields a solution with an order of the singular values. In this respect, the method is more efficient computationally. Of course, the power method is not always perfect. If it happens that a singular value is very close to the largest one below it, then convergence becomes very slow.

If the two fields are identical, the usual EOF and SVD analysis methods become identical to one another. In this case, matrix V in Eq. (2) becomes symmetric, and singular vectors e_α and f_α become identical. Then, Eq. (3) reduces to only one eigenvalue equation and its solution is an EOF. Similarly, a and b in Eq. (12) become identical and this equation results in one eigenvalue equation. Solving this equation to obtain the EOF is achieved simply by following the method by Hirose and Kutzbach (1969). Therefore, the method presented here can be regarded as a generalization of the method of Hirose and Kutzbach (1969) applied to the SVD analysis problem.

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Appendix

SVD on a curved surface with inhomogeneous sampling data

To consider the SVD problem on a curved surface with inhomogeneous sampling data, it is convenient to deal with the continuous field problem as a first step. Let us consider two anomalous scalar fields with the t -th observation, $\Phi(\mathbf{x})^t$ and $\Psi(\mathbf{y})^t$, on curved surfaces $\mathcal{D}_x$ and $\mathcal{D}_y$. Here $\mathbf{x}$ and $\mathbf{y}$ represent spatial points on each surfaces. If we define two

scalar fields $\hat{e}(\mathbf{x})$ and $\hat{f}(\mathbf{y})$, the covariance in (1) is generalized, in this case, as

$$I = N_t^{-1} \iint_{\mathcal{D}_x \times \mathcal{D}_y} dS(\mathbf{x}) d\tilde{S}(\mathbf{y}) \hat{e}(\mathbf{x}) \hat{f}(\mathbf{y}) \times \sum_{t=1}^{N_t} \Phi(\mathbf{x})^t \Psi(\mathbf{y})^t, \quad (\text{A.1})$$

where we have denoted the area elements of two spaces as $dS(\mathbf{x})$ and $d\tilde{S}(\mathbf{y})$. Note that I is a scalar and, thus, invariant under a coordinate transformation.

The unit vector condition is generalized to the constraint that the squared integral of $\hat{e}(\mathbf{x})$ and $\hat{f}(\mathbf{y})$ be unity. Then the Euler-Lagrange equations become

$$\begin{aligned} \int_{\mathcal{D}_y} d\tilde{S}(\mathbf{y}) V(\mathbf{x}, \mathbf{y}) \hat{f}(\mathbf{y}) &= \sigma \hat{e}(\mathbf{x}) \\ \int_{\mathcal{D}_x} dS(\mathbf{x}) V(\mathbf{x}, \mathbf{y}) \hat{e}(\mathbf{x}) &= \sigma \hat{f}(\mathbf{y}), \end{aligned} \quad (\text{A.2})$$

with

$$V(\mathbf{x}, \mathbf{y}) = N_t^{-1} \sum_{t=1}^{N_t} \Phi(\mathbf{x})^t \Psi(\mathbf{y})^t. \quad (\text{A.3})$$

Suppose we take the points $\mathbf{x}_i$ ($i = 1 \cdots N_x$) and $\mathbf{y}_k$ ($k = 1 \cdots N_y$) from the continuous surfaces as the mean values within each of their surrounding areas, and rewrite the symbols $z_{it} = \Phi(\mathbf{x}_i)^t$ and $s_{kt} = \Psi(\mathbf{y}_k)^t$. In addition, if we define ΔS_i ($\Delta \tilde{S}_k$) to be the area patch including $\mathbf{x}_i$ ($\mathbf{y}_k$), and introduce constant scaling factors A_1 and A_2 such that

$$\begin{aligned} \Delta S_i &= w_i A_1^2 \\ \Delta \tilde{S}_k &= \tilde{w}_k A_2^2, \\ e_i &= A_1 \hat{e}(\mathbf{x}_i), \\ f_k &= A_2 \hat{f}(\mathbf{y}_k), \\ \lambda &= \frac{\sigma}{A_1 A_2}, \end{aligned} \quad (\text{A.4})$$

then Eq. (A.2) can be rewritten as

$$\begin{aligned} \sum_{k=1}^{N_y} \tilde{w}_k V_{ik} f_k &= \lambda e_i \\ \sum_{i=1}^{N_x} w_i e_i V_{ik} &= \lambda f_k, \end{aligned} \quad (\text{A.5})$$

with unit vector conditions

$$\begin{aligned} \sum_{i=1}^{N_x} w_i e_i^2 &= 1 \\ \sum_{k=1}^{N_y} \tilde{w}_k f_k^2 &= 1. \end{aligned} \quad (\text{A.6})$$

Here, the definition of V_{ik} is the same as that in Eq. (2). Note how the arbitrary scaling of the factors

A_1 and A_2 corresponds to how e_i and f_k are normalized. This arbitrary scaling can be used to set the weights to $O(1)$. For example, if the data exist on the same longitude and latitude interval with longitudinal width $\Delta\lambda$ and latitudinal width $\Delta\phi$ on the Earth of radius $\hat{a}$, we can set w_i to be the cosine of the latitude by putting $A_1^2 = \hat{a}^2 \Delta\phi \Delta\lambda$.

If Eq. (3) and its corresponding unit vector conditions are compared with Eqs. (A.5) and (A.6), it is found that the effect of the inhomogeneity of the data is included only by considering weights when multiplying the vector or matrix with the spatial array as in

$$(\mathbf{V}' \cdot \mathbf{e})_k \equiv \sum_{i=1}^{N_z} w_i V_{ik} e_i, \quad (\text{A.7})$$

where the dot symbol is introduced to clarify the inclusion of the spatial weight in the matrix product. Note that the vectors $\mathbf{e}$ and $\mathbf{f}$, whose components are given by e_i and f_k respectively, results in the proper invariant pattern under data sampling due to the coordinate invariance in Eq. (A.1).

If we define matrix $\tilde{\mathbf{V}}$ and vectors $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{f}}$ whose components are given by

$$\begin{aligned} \tilde{V}_{ik} &= \sqrt{w_i \tilde{w}_k} V_{ik}, \\ \tilde{e}_i &= \sqrt{w_i} e_i, \\ \tilde{f}_k &= \sqrt{\tilde{w}_k} f_k, \end{aligned} \quad (\text{A.8})$$

then the matrix and vectors obey the same equation as (3) and the unit vector conditions $\tilde{\mathbf{e}}^2 = 1$ and $\tilde{\mathbf{f}}^2 = 1$. Hence, the solution of $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{f}}$ can be obtained by the direct SVD of matrix $\tilde{\mathbf{V}}$. The proper invariant vectors $\mathbf{e}$ and $\mathbf{f}$ are then obtained from Eq. (A.8).

In the case of the alternative method, the only difference when considering inhomogeneity is the in-

clusion of the weight when the matrix product with respect to the spatial array is achieved. So Eqs. (11) and (12) remain unchanged, but the definition of the matrices $\mathbf{X}$ and $\mathbf{Y}$ are modified as

$$\begin{aligned} \mathbf{X} &\equiv \mathbf{Z}' \cdot \mathbf{Z} \\ \mathbf{Y} &\equiv \mathbf{S}' \cdot \mathbf{S}, \end{aligned} \quad (\text{13'})$$

in order to include the weights. This comes from the inclusion of the weight for the unit vector conditions for solving the variational problem.

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気候解析における効果的な特異値分解の計算法について

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気候データの特異値分解 (SVD) 解析法を行う場合に有用な経済的で便利な計算法を見出した。この方法は、行列の大きさが空間配列の大きさではなくデータ数で決まるので、気候データの解析のような空間は大きいデータ数はさほどではない場合に極めて有効になる。また、データ不均一の効果を取り入れる重みの利用を一般化することによってこの方法はデータの取り扱いにおいても便利である。