SHORT NOTE

Derivation of Finite Medium Age-Diffusion Kernels in Source-Sink Theory

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HERESY-III(1) code is based on the source-sink theory originally developed by Feinberg (2) and Galanin(3). Their approach consisted of solving the age-diffusion equation in an infinite moderator containing an arbitrary arrangement of line elements. HERESY-III code was developed for D2O moderated systems, but the interactions between rods were specified by infinite medium age-diffusion kernels. As the heavy water moderated reactors are of finite dimension, it was necessary to develop age-diffusion kernels for finite medium. This note describes the derivation of age-diffusion kernels for finite medium, for line source of zero dimension and for a finite dimension line source.

1. Finite Medium Slowing Down Kernel

Consider a unit line source of neutrons placed eccentrically at a point $r_n$ in a medium of radius $R$. In this case, Fermi-age equation for a unit line source can be written as(4):

$$ F^2 q(r, T) - \frac{\partial q(r, T)}{\partial T} = \delta(r - r_n), \quad (T > 0), $$

where $q(r, T)$ is the number of neutrons slowing down past age $T$ per unit time per unit volume and $\delta(r - r_n)$ is Dirac delta function.

For an infinite medium, the solution of Eq. (1) for a unit line source becomes:

$$ q(r_n \rightarrow r, T) = \frac{e^{-\rho s/r}}{4\pi T}, $$

(This is the equation for the slowing down kernel used in HERESY-III code) where

$$ \rho s = r_n^2 + r^2 - 2rr_n \cos(\theta - \theta_n). $$

Figure 1 shows the co-ordinate representation.

Fig. 1 Co-ordinates of source rod and recipient rod

For finite medium, we assume that $q(r, T)$ is sum of solutions given by

$$ q(r, \theta, T) = u(r, \theta, T) + w(r, \theta, T), $$

where $u(r, \theta, T)$ is a regular solution and given by Eq. (2), and $w(r, \theta, T)$ an irregular solution and given by the solution of the differential equation (omitting vector notation for convenience):

$$ \frac{\partial^2 w(r, \theta, T)}{\partial r^2} + \frac{1}{r} \frac{\partial w(r, \theta, T)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w(r, \theta, T)}{\partial \theta^2} = \frac{\partial w(r, \theta, T)}{\partial T}. $$

The boundary conditions are

$$ w(r, \theta, T) = 0 \quad \text{when} \quad T = 0, \quad 0 \leq r \leq R, \quad (6) $$

$$ w(R, \theta, T) = -u(R, \theta, T), \quad T > 0. \quad (7) $$

Taking Laplace transform of Eq. (5), we obtain

$$ \frac{\partial^2 \tilde{w}(r, \theta, s)}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{w}(r, \theta, s)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \tilde{w}(r, \theta, s)}{\partial \theta^2} - \frac{s}{r} \tilde{w}(r, \theta, s) = 0, $$

where

$$ \tilde{w}(r, \theta, s) = \int_0^\infty w(r, \theta, T) e^{-sT} dT. $$

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Taking Laplace transform of Eq. (7), we obtain
\[ w(r, \theta, s) = \int_0^\infty \! u(r, \theta, T) e^{-sT} dT \]
or
\[ w(r, \theta, s) = -\frac{K_0(\sqrt{s} \rho)}{2\pi}, \quad \text{for } r=R, \quad (10) \]
where we have used Eq. (2) for \( u(r, \theta, T) \) and \( K_0 \) modified Bessel function of second kind of zero order.

The solution of Eq. (8) can be written as
\[ w(r, \theta, s) = \sum_{m=-\infty}^{+\infty} C_m I_m(\sqrt{s} r) \cos m(\theta - \theta_n), \quad (11) \]
where we have restricted the irregular solution to a symmetric function only, that is to one which depends only on the distance from the centre of the medium. \( C_m \) is a constant and \( I_m \) a modified Bessel function of first kind of order \( m \).

Using Eq. (10) which is second boundary condition, we obtain
\[ C_m = -\frac{1}{2\pi} \frac{I_m(\sqrt{s} R) K_m(\sqrt{s} R)}{I_m(\sqrt{s} R)}, \quad (12) \]
where we have used
\[ K_0(\sqrt{s} \rho) = \sum_{m=-\infty}^{+\infty} K_0(\sqrt{s} r) I_m(\sqrt{s} r) \cos m \theta, \quad \text{for } r > r_n, \quad (13) \]
\[ = \sum_{m=-\infty}^{+\infty} I_m(\sqrt{s} r) K_0(\sqrt{s} r) \cos m \theta, \quad \text{for } r < r_n, \quad (14) \]
\[ \int_0^{\frac{\pi}{2}} \cos m(\theta - \theta_n) \cos m'(\theta - \theta_n) d\theta = \pi, \quad \text{for } m=m', \]
\[ = 0, \quad \text{for } m \neq m'. \quad (15) \]

The solution of Eq. (1) becomes, for \( r < r_n \),
\[ q(r, \theta) = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \frac{K_m(\sqrt{s} R) I_m(\sqrt{s} R)}{I_m(\sqrt{s} R)} \]
\[ \cdot I_m(\sqrt{s} r) \cos m(\theta - \theta_n). \quad (16) \]
and for \( r > r_n \),
\[ q(r, \theta) = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} I_m(\sqrt{s} r) K_0(\sqrt{s} r) \cos m(\theta - \theta_n). \]

The inverse Laplace transform of Eq. (16) gives for \( r < r_n \),
\[ q(r_n \to r, \theta, T) = \frac{1}{\pi R^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e^{-\alpha_m r_n} J_m(c_n r) J_m(c_n r) \frac{I_m(\alpha_m r)}{I_m(\alpha_m R)} \]
\[ \cdot \cos m(\theta - \theta_n), \quad (18) \]
where \( \alpha_{mn} \) are zero's of \( J_m(\alpha_m R)=0 \).

For \( r > r_n \), Eq. (18) is symmetric in both \( r \) and \( r_n \) and hence is also solution for \( r > r_n \).

2. Finite Medium Diffusion Kernel

The diffusion equation for thermal neutron flux due to unit line source is given by
\[ -D \frac{\partial^2 \phi(r, \theta)}{\partial r^2} + \Sigma_a \phi(r, \theta) = \delta(r-r_n), \quad (19) \]
where \( D \) and \( \Sigma_a \) are diffusion coefficient and macroscopic absorption cross section for thermal neutrons, respectively.

For infinite medium, solution of Eq. (19) is given by
\[ \phi(r_n \to r, \theta) = A_0 K_0(\frac{r}{L}), \quad (20) \]
(This is the diffusion kernel used in HERESY-III code) where \( A_0 \) is constant and \( L \) the diffusion length for thermal neutrons.

For finite medium, the solution of Eq. (19) becomes
\[ \phi(r_n \to r, \theta) = A_0 K_0(\frac{r}{L}) \]
\[ + \sum_{m=-\infty}^{+\infty} C_m I_m(\frac{r}{L}) \cos m(\theta - \theta_n). \quad (21) \]

Boundary conditions are
\[ \phi(r_n \to R, \theta) = 0, \quad (22) \]
\[ -D \int_0^{\infty} \frac{\partial \phi}{\partial r} \rho d\theta = 1, \quad (23) \]
where \( \frac{\partial \phi}{\partial r} \) is evaluated for \( \rho = 0 \) for a line source of zero dimension and for \( \rho = a \) for a line source of finite dimension where \( a \) is the radius of the rod.

In order to apply boundary condition (i), Eq. (21) is written as (omitting vector notation for convenience)
\[ \phi(r_n \to r, \theta) = A_0 \sum_{m=0}^{\infty} K_m \left( \frac{r_n}{L} \right) I_m \left( \frac{r_n}{L} \right) \cos \nu \theta_n, \]  

where we have used the relation given by Eq. (13). Making use of Eq. (22), this gives

\[ \frac{C_m}{A_0} = \frac{I_m \left( \frac{r_n}{L} \right) K_m \left( \frac{R}{L} \right)}{I_m \left( \frac{R}{L} \right)}. \]  

The solution of Eq. (19) for a finite medium becomes

\[ \phi(r_n \to r, \theta) = A_n K_0 \left( \frac{r}{L} \right) \]  

\[ - \sum_{m=0}^{\infty} C_m \sum_{\nu=-\infty}^{\infty} (-1)^\nu \cdot I_{\nu+m} \left( \frac{r_n}{L} \right) I_\nu \left( \frac{r}{L} \right) \cos \nu \theta_n. \]  

The only constant to be determined is \( A_0 \) which for a line source of zero dimension has the value

\[ A_0 = \frac{1}{2\pi D}, \]  

and for a line source of radius \( a \),

\[ \frac{1}{A_0} = 2\pi a \sum_a \left[ K_1 (a/L) \right. \]  

\[ - \sum_{m=0}^{\infty} \frac{I_m (r_n/L) K_m (R/L)}{I_m (R/L)} \cdot I_1 (a/L) \right]. \]  

The solution of Eq. (19) for a finite medium becomes

\[ \phi(r_n \to r, \theta) = A_n K_0 \left( \frac{r}{L} \right) \]  

\[ - A_0 \sum_{m=0}^{\infty} \frac{I_m (r_n/L) K_m (R/L)}{I_m (R/L)} \cdot I_1 (a/L) \cos m(\theta - \theta_n). \]  

References