Separative Power of Ideal Cascade with Variable Separation Factors

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Concerning ideal cascades with separation factor varying from stage to stage, inter-stage flows are exactly determined by solution of the difference equation representing the conservation of the desired material flows. It is shown that, in such a cascade, the separative powers relevant to the desired and undesired materials are additive. This additive property is proved for the desired and undesired materials respectively by calculation establishing that the summation of separative powers of all stages is equal to the total separative power of the ideal cascade.

KEYWORDS: isotope separation, cascade, separation factor, value function, separative power, uranium, asymmetric process, flow rates

I. INTRODUCTION

The theory of ideal cascades for isotope separation has been developed mainly for designing the cascade with the separating elements having the same separation factor in all stages. The ideal cascade with symmetric separating elements was analyzed by Cohen(1) and by Benedict & Pigford(2), and that with asymmetric ones by Takashima(3), Kokubu(4) and others(5)(6). Recently, Olander(7) showed possibility of constructing "variable-α" ideal cascades with separation factor varying from stage to stage, and analyzed them under the assumption that the desired isotope fractions in all streams are much smaller than unity.

The additive property of separative power—i.e. the summation of separative powers over all stages represents the total separative power of the cascade—is characteristic of ideal cascades. The property was analytical demonstrated for ideal cascades with symmetric(1) and with asymmetric(6) elements, but not for "variable-α" ideal cascades. It is the purpose of the present report to verify this property in respect of a "variable-α" ideal cascade, based on exact calculations of interstage flows and separative power relevant to the desired and undesired materials(8).

II. CONCENTRATION DISTRIBUTION

We consider an isotope separation cascade consisting of \( n - f + 1 \) stages on the rectifier side and \( f - 1 \) stages on the stripper side. The stages are so numbered that the top and the bottom stages correspond to \( i = n \), and \( i = 1 \), respectively, as shown in Fig. 1. In an ideal cascade the inter-stage flows that merge at each confluent point possess identical compositions:

\[ N_i = N_{i-1}' = N_{i+1}' , \]  

where \( N_i \) is the mole fraction of the desired material (ex. \( ^{235}\text{UF}_6 \) in uranium enrichment) contained in the \( i \)-th stage feed, and primes ('), (") denote the enriched and depleted flows, while the symbol without prime pertains to the feed. The separation performed by the element is measured by the heads and tails separation factors \( \alpha \) and \( \beta \) defined by

\[ \alpha = R'/R , \quad \beta \]  

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where $R$ is the abundance ratio:

$$R = N/(1-N).$$  

Substituting Eqs. (2)~(4) into Eq. (1) gives

$$R_i = \alpha_{i-1} R_{i-1} = R_{i+1} / \beta_{i+1},$$  

from which

$$\alpha_i = \beta_{i+1}, \quad (i=1, 2, \ldots, n-1),$$  

$$R_i = \left( \prod_{j=i}^{n-1} \alpha_j \right) R_F, \quad (i=f, f+1, \ldots, n)$$  

$$R_i = R_F / \left( \prod_{j=i+1}^{n} \beta_j \right) = R_F / \left( \prod_{j=i}^{n} \alpha_j \right), \quad (i=1, 2, \ldots, f-1),$$

where $R_F$ is the abundance ratio of the feed flow to the cascade. Moreover, the product concentration

$$R_p = \left( \prod_{i=f}^{n} \alpha_i \right) R_F = \alpha_T R_F,$$

and the waste concentration

$$R_w = R_F / \left( \prod_{i=1}^{n} \beta_i \right) = R_F / \beta_T.$$  

### III. FLOW RELATIONS

The feed flow to the $i$-th stage

$$L_i = (G_b)_i + (G_a)_i,$$

where $G_b$ is the desired material flow (ex. $^{235}$UF$_6$ in uranium enrichment) and $G_a$ the undesired (ex. $^{238}$UF$_6$). The desired material flow is related to the feed flow to the cascade $F$ in the form

$$(G_b)_i = \gamma_{i-1} (G_b)_{i-1} + (1-\gamma_{i-1}) (G_b)_{i+1} + \delta_{iF} N_F F, \quad \delta_{iF} = \begin{cases} 1 & (i=f) \\ 0 & (i \neq f) \end{cases}$$

where the desired material cut

$$\gamma = \alpha (\beta - 1) / \alpha \beta - 1.$$  

Solution of Eq. (11) gives (see APPENDIX):

1. For the product

$$R_p = (G_b)_p \left( \prod_{i=1}^{n} \alpha_i \right) R_F / \beta_1 \left( \prod_{i=1}^{n} \alpha_i \right) - 1,$$

2. For the waste

$$R_w = (G_b)_w \left( \prod_{i=1}^{n} \alpha_i \right) \beta_1 \left( \prod_{i=1}^{n} \alpha_i \right) - 1.$$

3. For the enriching section

$$(G_b)_e = (G_b)_p \left( \prod_{i=1}^{n} \alpha_i \right) - 1 / (\alpha_i - 1)(\beta_i - 1).$$

4. For the stripping section

$$(G_b)_s = (G_b)_w \left( \prod_{i=1}^{n} \alpha_i \right) - 1 / (\alpha_i - 1)(\beta_i - 1).$$
In the above equations
\[
\begin{align*}
(G_a)_F & = N_p F, \\
(G_a)_p & = N_p P, \\
(G_a)_w & = N_w W,
\end{align*}
\] (17)

where \(P\) and \(W\) are the product and the waste flows of the cascade, respectively.

The undesired material flow \(G_a\) is governed by the equation
\[
(G_a)_i = \zeta_i (G_a)_{i-1} + (1 - \zeta_i) (G_a)_{i+1} + \delta_i (1 - N_p) F,
\] (18)

where the undesired material flow cut \(\zeta\)
\[
\zeta = \frac{\beta - 1}{\alpha \beta - 1}.
\] (19)

Solution of Eq. (18) gives \((G_a)_i\), but in an ideal cascade \((G_a)_i\) is more easily obtained such that
\[
(G_a)_i = (G_a)_i / R_i.
\] (20)

Substitution of Eqs. (6)~(9) and (13)~(16) into Eq. (20) yields:

(1) For the product
\[
(G_a)_p = (G_a)_F \frac{\beta_1 \prod_{i=1}^{n} \alpha_i - 1}{\beta_1 (\prod_{i=1}^{n} \alpha_i) - 1}.
\] (21)

(2) For the waste
\[
(G_a)_w = (G_a)_p \frac{\alpha_i \beta_i - 1}{(\alpha_i - 1)(\beta_i - 1)}.
\] (22)

(3) For the enriching section
\[
(G_a)_i = (G_a)_p \left[ \prod_{j=1}^{n} \alpha_j \right] \frac{\alpha_i \beta_i - 1}{(\alpha_i - 1)(\beta_i - 1)}.
\] (23)

(4) For the stripping section
\[
(G_a)_w = (G_a)_i \left[ \prod_{i=1}^{n} \alpha_i \right] \frac{\beta_i (\prod_{i=1}^{n} \alpha_i) - 1}{\alpha_i \beta_i - 1}.
\] (24)

In the above equations
\[
\begin{align*}
(G_a)_F & = (1 - N_p) F, \\
(G_a)_p & = (1 - N_p) P, \\
(G_a)_w & = (1 - N_w) W.
\end{align*}
\] (25)

IV. SEPARATIVE POWER

A separative power of a separating process is a change in the value of an isotope mixture which passes through the process and is measured by a value function.

In an “variable-\(\alpha\) ideal cascade”\(^{(1)}\), each stage is made up of asymmetric separating elements. Separative power for asymmetric process\(^{(10)}\) is defined by
\[
\delta U = L \varphi(a, \beta) \ln \beta - (\beta - 1) \ln \alpha,
\] (26)
or \[
\delta U = L \varphi(a, \beta) \ln \alpha - (\alpha - 1) \ln \beta.
\] (27)

The value function\(^{(10)}\) being consistent to Eqs. (26) and (27):
\[
V(R) = \varphi \left[ \frac{1}{\varphi(a, \beta)} \cdot \frac{R}{1 + R} \right] \ln R,
\] (28)

depends on separation factors. Consequently, it is not suitable for calculation of the total separative power of “variable-\(\alpha\) ideal cascade”, because there is no way of deciding which of the many separation factors to use in the value function.

The value function\(^{(1)}\) for symmetric processes, on the other hand, does not depend on separation factors:
\[
V_{sym}(R) = \frac{R - 1}{R + 1} \ln R,
\] (29)
but the function is not applicable to asymmetric processes, since it is associated with symmetric processes whose separative power

\[ \delta U = L \varphi_a(\alpha, \alpha) = L \varphi_a(\alpha, \alpha) \]

\[ = L \frac{\alpha - 1}{\alpha + 1} \ln \alpha . \]

When the function Eq. (29) is applied provisionally to asymmetric processes, it gives neither Eq. (26) nor Eq. (27) but the sum of a separative power \( dU_b \) relevant to the desired material and \( \delta U_a \) relevant to the undesired material in the form

\[ L'V_{sym}(\alpha R) + L'V_{sym}(R/\beta) - LV_{sym}(R) \]

\[ = \frac{R}{1+R} L \varphi_a(\alpha, \beta) + \frac{1}{1+R} L \varphi_a(\alpha, \beta) \]

\[ = G_b \varphi_a(\alpha, \beta) + G_a \varphi_a(\alpha, \beta) = \delta U_b + \delta U_a . \]

As a matter of convenience, \( \delta U_{sym} \) denotes the value change measured by Eq. (29), or

\[ \delta U_{sym} = \delta U_b + \delta U_a . \]

Similarly, the total value change by Eq. (29) over the cascade

\[ (\delta U_{sym})_T \]

\[ = PV_{sym}(R_p) + WV_{sym}(R_w) - FV_{sym}(R_F) \]

\[ = \frac{R_F}{1+R_F} F \varphi_b(\alpha_T, \beta_T) + \frac{1}{1+R_F} F \varphi_a(\alpha_T, \beta_T) \]

\[ = (G_b)_F \varphi_b(\alpha_T, \beta_T) + (G_a)_F \varphi_a(\alpha_T, \beta_T) \]

\[ = (\delta U_B)_T + (\delta U_A)_T . \]

Here the additive property of \( \delta U_b \):

\[ (\delta U_B)_T = \sum_{i=1}^{n} (\delta U_B)_i = \sum_{i=1}^{n} (G_b)_i \varphi_b(\alpha_i, \beta_i) , \]

and that of \( \delta U_a \):

\[ (\delta U_A)_T = \sum_{i=1}^{n} (\delta U_A)_i = \sum_{i=1}^{n} (G_a)_i \varphi_a(\alpha_i, \beta_i) \]

are analytically demonstrated as follows:

(1) Additive Property of \( \delta U_b \)

By using the value function \( V_b(R) \) for the desired material

\[ V_b(R) = \ln R , \]

the total separative power \( (\delta U_b)_T \) relevant to the desired material is calculated in the form

\[ (\delta U_b)_T = (G_b)_F \ln R_p + (G_b)_w \ln R_w \]

\[ - (G_b)_F \ln R_F \]

\[ = (G_b)_F \varphi_b(\alpha_T, \beta_T) , \]

or in terms of separation factors of each stage

\[ (\delta U_B)_T = (G_b)_F \sum_{i=f}^{n} \ln \alpha_i - (G_a)_w \sum_{i=1}^{f-1} \ln \beta_i , \]

where Eqs. (8) and (9) are used.

By using Eq. (15), \( \delta U_b \) of the \( i \)-th stage in the enriching section

\[ (\delta U_B)_i = (G_b)_i \varphi_b(\alpha_i, \beta_i) \]

\[ = (G_b)_F \left( \prod_{i=f}^{n} \alpha_i \right) - 1 \]

\[ \prod_{i=f}^{n} \alpha_i \]

\[ \cdot \left[ \frac{\alpha_i}{\alpha_{i-1}} \ln \alpha_i - \frac{1}{\beta_{i-1}} \ln \beta_i \right] , \]

\[ (i=f, f+1, ..., n) , \]

summed from the stage \( f \) to \( n \) to give

\[ \sum_{i=f}^{n} (\delta U_B)_i = (G_b)_F \sum_{i=f}^{n} \ln \alpha_i \]

\[ - (G_b)_F \left( \prod_{i=f}^{n} \alpha_i \right) \frac{1}{\alpha_{f-1}-1} . \]

In the stripping section,

\[ (\delta U_B)_i = (G_a)_w \left[ \beta_i \prod_{i=1}^{i-1} \alpha_i - 1 \right] \]

\[ \left[ \frac{\alpha_i}{\alpha_{i-1}} \ln \alpha_i - \frac{1}{\beta_{i-1}} \ln \beta_i \right] , \]

\[ (\delta U_A)_i = (G_a)_w \left[ \beta_i - 1 \right] \]

\[ \left[ \frac{\alpha_i}{\alpha_{i-1}} \ln \alpha_i - \frac{1}{\beta_{i-1}} \ln \beta_i \right] , \]

summed from the stage \( 1 \) to \( (f-1) \) to give

\[ \sum_{i=1}^{f-1} (\delta U_A)_i = -(G_a)_w \sum_{i=1}^{f-1} \ln \beta_i \]

\[ + (G_a)_w \left[ \prod_{i=1}^{i-1} \beta_i - 1 \right] \frac{\beta_f \ln \beta_f}{\beta_f - 1} . \]
Noting that
\[-(G_b)_p \left( \prod_{s=f}^{n} \alpha_s \right)^{-1} \ln \beta_f \]
\[+ (G_b)_w \left( \left( \prod_{s=f}^{n} \beta_s \right)^{-1} - 1 \right) \frac{\ln \beta_f}{\beta_f - 1} \]
\[= -(G_b)_w \ln \beta_f , \]
we obtain the relation
\[\sum_{i=1}^{n} (\delta U_b)_i = \left( \sum_{i=f}^{n} (\delta U_b)_i \right) + \sum_{i=f}^{n} (\delta U_b)_i \]
\[= (G_b)_p \sum_{i=f}^{n} \ln \alpha_s - (G_b)_w \sum_{i=f}^{n} \ln \beta_s , \]
compared by Eq. (35') to reduce to
\[\sum_{i=1}^{n} (\delta U_b)_i = (\delta U_b)_T . \]
(2) Additive Property of \( \delta U_a \)
By using the value function\(^{(8)} \) for the undesired material
\[V_a(R) = -\ln R , \]
the total separative power \((\delta U_a)_T \) relevant to the undesired material is calculated in the following manner:
\[(\delta U_a)_T = -(G_a)_p \ln R_p -(G_a)_w \ln R_w \]
\[+ (G_a)_p \ln R_p \]
\[= (G_a)_p \varphi_a(\alpha_T , \beta_T) , \]
(42)
or in terms of separation factors of each stage
\[(\delta U_a)_T = -(G_a)_p \sum_{i=f}^{n} \ln \alpha_s + (G_a)_w \sum_{i=f}^{n} \ln \beta_s , \]
(42')
In the enriching section, \( \delta U_a \) of the \( i \)-th stage
\[(\delta U_a)_i = (G_a)_p \varphi_a(\alpha_i , \beta_i) \]
\[= (G_a)_p \left( \prod_{s=i}^{n} \alpha_s \right)^{-1} \]
\[\times \left[ \frac{\beta_i}{\beta_i - 1} \ln \beta_i - \frac{1}{\alpha_i - 1} \ln \alpha_i \right] , \]
(43)
summed from the stage \( f \) to \( n \) to yield
\[\sum_{i=f}^{n} (\delta U_a)_i = -(G_a)_p \sum_{i=f}^{n} \ln \alpha_i , \]
\[+ (G_a)_p \left[ \left( \prod_{s=i}^{n} \alpha_s \right)^{-1} \right] \]
\[\times \frac{\alpha_i}{\alpha_i - 1} \ln \alpha_i . \]
(44)
Similar calculations in the stripping section give
\[\sum_{i=1}^{n} (\delta U_a)_i = -(G_a)_w \sum_{i=1}^{n} \ln \beta_s \]
\[+ (G_a)_w \left( \left( \prod_{s=i}^{n} \beta_s \right)^{-1} - 1 \right) \ln \beta_f \]
\[\times \frac{\beta_f}{\beta_f - 1} , \]
(45)
Thereupon the total sum of \((\delta U_a)_i \) over the cascade
\[\sum_{i=1}^{n} (\delta U_a)_i = -(G_a)_p \sum_{i=1}^{n} \ln \alpha_s , \]
\[+ (G_a)_w \sum_{i=1}^{n} \ln \beta_s , \]
(46)
which is compared by Eq. (42') to yield
\[\sum_{i=1}^{n} (\delta U_a)_i = (\delta U_a)_T . \]
(47)
From Eqs. (30')~(33), the value change measured by Eq. (29) also has the additive property:
\[(\delta U_{ym})_T = \sum_{i=1}^{n} (\delta U_{ym})_i \]
\[= \sum_{i=1}^{n} \left[ (G_a)_p \varphi_a(\alpha_i , \beta_i) + (G_a)_w \varphi_b(\alpha_i , \beta_i) \right] . \]
(48)
Thereupon, Olander's conjecture:
\[(\delta U_{ym})_T = \sum_{i=1}^{n} L_i \varphi_a(\alpha_i , \beta_i) \]
(49)
is not correct. The numerical values of both sides of Eq. (49), however, are almost equal in his example. It is clear from the fact that the deviation term:
\[\sum_{i=1}^{n} L_i \varphi_a(\alpha_i , \beta_i) - (\delta U_{ym})_T \]
\[= \sum_{i=1}^{n} (\delta U_{ym})_i \frac{R_i (e_i - 1)}{e_i + R_i} \]
(49)
is correct.
is very small compared with \((\delta U_{sym})_T\) in the example, because \(R_i < 1\) and because stages with \(e > 1\) and those with \(e < 1\) are arranged alternately so that adjacent stages' contributions counterbalance to some extent. On the other hand, the deviation

\[
\sum_{i=1}^{n} L_i \varphi_i(\alpha_i, \beta_i) - (\delta U_{sym})_T
\]

greater than that of Eq. (49) under the concentration condition \(R_i < 1\) because of the same reason for selecting the functional form in Ref. (10).

V. NUMERICAL EXAMPLE

Table 1 presents the main parameters of the model cascade containing 6 enriching stages and 2 stripping stages (i.e. \(n=8, f=3\)). The parameters have been chosen in common with those of Fig. 5 of Ref. (7).

| Table 1 Main parameters of model cascade |
|---|---|---|
| Stage number \(i\) | 1, 3, 5, 7 | 2, 4, 6, 8 |
| \(\alpha_i\) | 1.08 | 1.50 |
| \(\beta_i\) | 1.50 | 1.08 |
| \(\varphi_a(\alpha_i, \beta_i)\) | 1.6411762 \(\times 10^{-2}\) | 1.4712506 \(\times 10^{-2}\) |
| \(\varphi_b(\alpha_i, \beta_i)\) | 1.4712506 \(\times 10^{-2}\) | 1.6411762 \(\times 10^{-2}\) |
| \(e_i\) | 1.1154974 | 0.89846109 |

Tables 2 and 3 show the results obtained by the double precision calculations of FACOM 230-75 of Nagoya University Computation Center. Additive properties of separative powers are clearly found in Table 3, where

\[
\sum_{i=1}^{n} L_i \varphi_i(\alpha_i, \beta_i) = 1.0000079 (\delta U_{sym})_T
\]

VI. CONCLUSIONS

Conclusions obtained from the present study are summarized as follows:

1. Exact calculations were performed to determine the interstage flow rates of "variable-\(\alpha\)" ideal cascades in which separation factor varies from stage to stage.

2. Additive properties of separative power in "variable-\(\alpha\)" ideal cascades were analytically demonstrated for the desired and undesired materials, respectively.

REFERENCES

[APPENDIX]

Derivation of Eqs. (13)~(17)

The desired material flows are expressed in terms of $D(i, j)$ determinants. When neither leakage nor reflux exist between the $i$- and $j$-th stages:

$$
\eta_s = \tilde{\eta}_s = 1 \quad (s = i + 1, \ldots, j) \\
\eta_s = (G_s)/(G_s) \quad \tilde{\eta}_s = (G_s)/(G_s) ,
$$

(A1)

the determinant can be expressed explicitly by the separation factors in the following manner:

$$
D(i, j) = \left( \prod_{k=1}^{j-1} \eta_k \right) \left[ 1 + \sum_{k=i}^{j} \left( \prod_{s=k}^{j} v_s \right) \right] \\
= \left( \prod_{k=1}^{j} \tilde{\eta}_k \right) \left[ 1 + \sum_{k=i}^{j} \left( \prod_{s=k}^{j} \frac{1}{v_s} \right) \right] ,
$$

(A2)

Then the desired material flow rates are:

(1) For the product

$$
(G_p)_i = (G_p) \left( 1 + \frac{1}{v_i} \right) \\
1 + \sum_{k=1}^{i} \left( \prod_{s=k}^{i} v_s \right)
$$

(A3)

(2) For the enriching section

$$
(G_n)_i = (G_p) \left( 1 + \frac{1}{v_i} \right) \\
1 + \sum_{k=1}^{i} \left( \prod_{s=k}^{i} v_s \right) ,
$$

(i = n)

$$
(G_n)_i = (G_p) \left( 1 + \frac{1}{v_i} \right) \\
\left[ 1 + \sum_{k=1}^{i} \left( \prod_{s=k}^{i} v_s \right) \right] \left[ 1 + \sum_{k=i+1}^{n} \left( \prod_{s=k}^{i} v_s \right) \right] \\
1 + \sum_{k=1}^{i} \left( \prod_{s=k}^{i} v_s \right) ,
$$

(A4)

(3) For the stripping section

$$
(G_d)_i = (G_p) \left( 1 + \frac{1}{v_i} \right) \\
\left[ 1 + \sum_{k=i+1}^{n} \left( \prod_{s=k}^{n} \frac{1}{v_s} \right) \right] \left[ 1 + \sum_{k=i}^{n} \left( \prod_{s=k}^{i} \frac{1}{v_s} \right) \right] \\
1 + \sum_{k=1}^{i} \left( \prod_{s=k}^{i} \frac{1}{v_s} \right) ,
$$

(i = 2, 3, \ldots, f-1)

$$
(G_d)_i = (G_p) \left( 1 + \frac{1}{v_i} \right) \\
\left[ 1 + \sum_{k=i+1}^{n} \left( \prod_{s=k}^{n} \frac{1}{v_s} \right) \right] \left[ 1 + \sum_{k=i}^{n} \left( \prod_{s=k}^{i} \frac{1}{v_s} \right) \right] \\
1 + \sum_{k=1}^{i} \left( \prod_{s=k}^{i} \frac{1}{v_s} \right) ,
$$

(A5)

(4) For the waste

$$
(G_w)_i = (G_p) \left( 1 + \frac{1}{v_i} \right) \\
1 + \sum_{k=1}^{n} \left( \prod_{s=k}^{n} \frac{1}{v_s} \right) .
$$

(A6)

In the variable-$\alpha$ ideal cascade, the separation factors of adjacent stages are related each other such that

$$
\alpha_i = \beta_{i+1} \quad (\lambda = 1, 2, \ldots, n-1)
$$

(5)

we obtain

$$
1 + \sum_{k=1}^{n} \left( \prod_{s=k}^{n} v_s \right) = \frac{1}{(\beta_1-1) \prod_{s=1}^{n} \alpha_s} \left[ \beta_1 \left( \prod_{s=1}^{n} \alpha_s \right) - 1 \right] \\
1 + \sum_{k=1}^{n} \left( \prod_{s=k}^{n} \frac{1}{v_s} \right) = \frac{1}{\alpha_{n-1}} \left[ \beta_1 \left( \prod_{s=1}^{n} \alpha_s \right) - 1 \right] ,
$$

(A7)

and other similar relations, which are substituted into Eqs. (A3)~(A6) to yield Eqs. (13)~(17).