Thanks are due to Dr. T. Tamai for the reactor irradiations of U$_3$Si$_2$ samples at KUR in the Visiting Research Program, and to Mr. K. Katori for the experimental help.

---REFERENCES---

---


**Short Note**

**Unbiasedness Evaluation of Statistical Error in Staged Monte Carlo Calculation**

Hiromasa Iida and Yasushi Seki

Japan Atomic Energy Research Institute*

Received August 25, 1981
Revised September 30, 1981

**KEYWORDS:** Tokamak devices, Monte Carlo method, transport theory, errors, comparative evaluations, radiation streaming, neutron streaming, neutral beam injection, computer calculations, neutron flux, computer codes

In a deep penetration problem of radiation transport, staged calculation is often employed using a discrete ordinate transport code. It is a useful technique for saving memory storage and machine time. In a radiation streaming problem with a complex geometry, a staged calculation based on the Monte Carlo method is also conceivable and there are some applied examples\(^{(1)-(5)}\).

One disadvantage in the staged Monte Carlo method is said to be the difficulty in defining the absolute statistical error of the final result\(^{(5)}\).

In this note we prove the unbiasedness of the statistical error value obtained in the final calculation. The method employed in our paper\(^{(1)}\) calculating neutron streaming through a neutral beam injection (NBI) port of a Tokamak type fusion reactor is as follows:

1. The whole reactor is divided into two blocks by the plane containing the port mouth. In the first Monte Carlo calculation, in the first block, neutron histories are followed and required data of neutrons which reach the mouth are stored on a magnetic tape for the subsequent use.

2. In the second Monte Carlo calculation neutron fluxes in the NBI are calculated using the particle data, i.e. energy, weight, direction and location, stored on the tape as neutron source.

A Monte Carlo transport code MORSE\(^{(5)}\)
is used in this analysis. Although point detector estimator is used as well as track-length estimator in the neutron streaming analysis, in this note only the latter estimator is assumed for simplicity.

1. Unbiasedness Evaluation of Fractional Standard Deviation Obtained by Second Run of Staged Calculation

In an ordinary once-through calculation, the MORSE code calculates variance of the mean as,

$$
\sigma^2 = \frac{1}{N-1} \left[ \frac{1}{n_1} \sum_{i=1}^{n_1} m x_i^2 - \frac{1}{n_1^2} \left( \sum_{i=1}^{n_1} m x_i \right)^2 \right],
$$

where $N$: Number of batches
$n_1$: Total number of independent histories
$m$: Number of independent histories in a batch
$x_i$: Accumulated estimate in $i$-th batch.

Note that

$$
n_1 = Nm,
$$

where $x_{ij}$ is the estimate from the $j$-th history in the $i$-th batch,

$$
\bar{x} = \frac{1}{n_1} \sum_{i=1}^{n_1} m x_i,
$$

where $\bar{x}$ is the mean averaged over $n_1$ histories.

The fractional standard deviation $F_0$ is

$$
F_0 = \sigma / \bar{x}.
$$

When we use the same routine in the second run of the staged Monte Carlo method, the following variance is obtained:

$$
\sigma_1^2 = \frac{1}{N_1-1} \left[ \frac{1}{n_{11}} \sum_{i=1}^{n_{11}} m_{11} y_i^2 - \frac{1}{n_{11}^2} \left( \sum_{i=1}^{n_{11}} m_{11} y_i \right)^2 \right],
$$

where $N_1$: Number of batches in second run
$n_{11}$: Total number of independent histories in second run
$m_{11}$: Number of independent histories in each batch of second run
$y_i$: Accumulated estimate in $i$-th batch of second run.

As in the case of the once-through calculation we have the following relations:

$$
n_{11} = N_1 m_1,
$$

$$
y_i = \frac{1}{m_{11}} \sum_{j=1}^{m_{11}} y_{ij},
$$

$$
\bar{y} = \frac{1}{n_{11}} \sum_{i=1}^{n_{11}} m_{11} y_i,
$$

where $y_{ij}$: Estimate from $j$-th history in $i$-th batch of second run
$\bar{y}$: Mean averaged over $n_{11}$ histories in second run.

The fractional standard deviation $F_1$ obtained by the second run is

$$
F_1 = \sigma_1 / \bar{y}.
$$

In this note we will show that under a certain condition which is fulfilled in a usual case the expectations of $F_0$ and $F_1$ are the same, that is, $F_1$ is also the unbiased estimate of the fractional standard deviation. We introduce another notation for each estimate $S_i$, which means the estimate from the $i$-th history in the total number of histories.

Figure 1 shows the difference of batch con-
structructions of the once-through and staged calculation and their relations with \( S_t \). In the staged calculation when particle does not reach the port mouth in the \( i \)-th history of the first run, particle data is not stored on the magnetic tape for that history and this means \( S_t \) is set to be zero. In the case of the example shown in Fig. 1, \( S_2, S_3, S_n(1)+1 \) are such cases and neglected in the second run.

The value of \( n(j) \) is distributed according to a certain distribution function. The probability of obtaining non zero \( S_t \) is
\[
P = \frac{n_{t+1}}{n_t}.
\] (11)

When \( n(j) \) is equal to \( l \) we have \( m_1 \) values which are not zero and \((l - m_1) \) zeros for \( S_t \) in \( j \)-th batch. The last \( S_t \) in each batch must be non-zero. The distribution function is approximately given by the following:
\[
f(l) = \frac{m_1 (1 - P)^{l - m_1}}{(l - m_1)!} \frac{(l-1)!}{(m_1 - 1)!}.
\] (12)

Naturally
\[
\sum_{l=m_1}^{n_1} f(l) = 1
\] and since \( n_t \) is large enough in practical calculations
\[
\sum_{l=m_1}^{n_1} f(l) \approx 1.
\] (13)

Let \( \sigma^2 \) and \( \mu \) be the variance and mean of the set \([S_i]\), respectively. The expectations of the mean and variance of sample (batch) means are given as follows:
\[
E\left[ \frac{1}{l_0} \sum_{i=k(j)}^{l_0 + k(j)-1} S_i \right] = \mu,
\] (15)
\[
E\left[ \left( \frac{1}{l_0} \sum_{i=k(j)}^{l_0 + k(j)-1} S_i - \mu \right)^2 \right] = \frac{n_t - l_0}{n_t - 1} \cdot \frac{\sigma^2}{l_0},
\] (16)

where \( E[A] \) : Expectation of \( A \)
\( l_0 \) : Sample size
\( k(j) \) : \( i \) value of first \( S_i \) in \( j \)-th batch.

Transforming Eq. (16), we obtain
\[
E\left[ \left( \frac{1}{l_0} \sum_{i=k(j)}^{l_0 + k(j)-1} S_i \right)^2 \right] = \mu^2 + \frac{n_t - l_0}{n_t - 1} \cdot \frac{\sigma^2}{l_0}.
\] (17)

Following expectations are calculated by using Eq. (17) and definition of \( \mu \).
\[
E \left[ \frac{1}{n_1} \sum_{i=1}^{n_1} x_i^2 \right] = \frac{1}{n_t} N m E[x_t^2] = E \left[ \left( \frac{1}{m} \sum_{i=1}^{m} x_{ij} \right)^2 \right] = E \left[ \left( \frac{1}{m} \sum_{i=1}^{m} S_i \right)^2 \right] = \mu^2 + \frac{n_t - m}{n_t - 1} \cdot \frac{\sigma^2}{m},
\] (18)

where \( i_0 \) is an arbitrary number between unity and \( n_t - m + 1 \).
\[
E \left[ \frac{1}{n_1^2} \sum_{i=1}^{n_1} y_i^2 \right] = E \left[ \left( \frac{1}{n_1} \sum_{i=1}^{n_1} S_i \right) \right] = \frac{n_1}{n_1} \sum_{i=m_1}^{n_1} P_{m_1} (1-P)^{l - m_1} \frac{(l-1)!}{(l - m_1)!(m_1 - 1)!} \cdot \frac{(l-m_1)}{(l+m_1)} \cdot \frac{\sigma^2}{n_t - 1} \cdot \frac{n_t - l}{n_t - 1}.
\] (19)

\[
E \left[ \left( \frac{1}{n_1} \sum_{i=1}^{n_1} S_i \right)^2 \right] = \frac{n_1}{n_1} \sum_{i=m_1}^{n_1} P_{m_1} (1-P)^{l - m_1} \frac{(l-1)!}{(l - m_1)!(m_1 - 1)!} \cdot \frac{(l+m_1)}{(l+m_1)} \cdot \frac{\sigma^2}{n_t - 1} \cdot \frac{n_t - l}{n_t - 1}.
\] (20)
In usual case $\sigma \gg \mu$ and
\[ \left( \frac{n_{t-1}}{n_{t}} \right)^2 \left( \frac{\sigma^2}{\mu^2 + \sigma^2} \right) = \sigma^2, \]
then
\[ E[\sigma^2] = \left( \frac{n_{t}}{n_{t-1}} \right)^2 \sigma^2 \frac{\sigma^2}{n_{t}}. \tag{25} \]

By comparing Eqs. (22) and (25), the value of variance obtained by the final run is expected to be the right value multiplied by $(n_{t-1}/n_{t})^2$.

Expectations of the fractional standard deviations calculated by Eqs. (5) and (10) are
\[ E[F_a] = \frac{E[\sigma^2]}{\bar{X}} = \frac{\sigma}{\mu n_{t}^{-1/2}}, \tag{26} \]
\[ E[F_1] = \frac{E[\sigma^2]}{\bar{\eta}} = \frac{\sigma}{\mu n_{t}^{-1/2}}. \tag{27} \]

Note that
\[ \bar{\eta} = E[\eta] = \frac{n_{t}}{n_{t-1}} E[S_{t}] = \frac{n_{t}}{n_{t-1}} \cdot \mu. \tag{28} \]
Thus the expectations of $F_a$ and $F_1$ are shown to be practically the same. When $\sigma \gg \mu$ is not true, obtaining small values of statistical error is not difficult as shown in Eq. (26). In that simple case"staged Monte Carlo calculation will not be required.

— References —