On Controllability of Neutron Flux Distribution

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As a prerequisite to the optimal control of the neutron flux distribution of large reactors, the controllability of the flux distribution is investigated. The distributed parameter system is reduced to a lumped parameter system by means of truncated modal expansion. In a one dimensional reactor, any number of modes are controllable by a single control rod. To maintain the amplitude of each mode at a prescribed level however, it is necessary to have as many control rods as the number of the modes. Discussions are also presented on the decomposition of the optimal control of flux distribution into those of the individual modes, on the controllability of reactors with feedback, and on the controllability when there is constraint on the control rod speed.

I. INTRODUCTION

The one point approximation of reactor kinetics is not suitable for large power reactors, which require the spatial distribution of the neutron flux to be taken into consideration. It therefore becomes an important problem in the control of such reactors to strive for attainment of optimal control of the flux distribution, that is, to modify the given initial distribution into a prescribed final configuration with an optimal movement of the control rods, such as in minimum time, or with minimum control effort.

The problem becomes that of the optimal control of distributed parameter systems, and is a little more complicated than the optimal control of point reactors\(^\text{(1)(2)}\). While general theories of the optimal control of distributed parameter systems are available\(^\text{(3)(4)}\), a different approach has been adopted in the present study: Use is made of the modal expansion of the flux distribution, and the transient flux distribution is approximated by the superposition of a finite number of modes, so that the optimal control problem is reduced to that of a lumped parameter system. The control input is the reactivity change due to the movement of the control rods, and it is assumed that the reactivity change may be approximated by a \(\delta\)-function at the center of the control rods.

The above approximations permit the time dependent flux distribution to be expressed by

\[
\phi(r, t) = \sum_{k=0}^{\infty} n_k(t) \phi_k(r) = \sum_{k=0}^{K} n_k(t) \phi_k(r), \tag{1}
\]

where \(\phi_k(r) (k = 0, 1, \cdots)\) is the shape function and \(n_k(t)\) the amplitude of the \(k\)-th mode. If one chooses as shape functions an appropriate set of eigenfunctions — generally known as \(\phi_k\) modes, the amplitude \(n_k(t)\) is obtained by solving differential equations similar to the one point kinetics equations\(^\text{(5)}\):\n
\[
\begin{align*}
\frac{dn_k(t)}{dt} &= \rho s - \beta n_k(t) + \sum_{i=1}^{\infty} \lambda_i C_{i\alpha}(t) \\
&\quad + \sum_{m=1}^{M} N_m \delta(r_m) \phi_k(r_m) n_m(t), \tag{2}\n\frac{dC_{i\alpha}(t)}{dt} &= \beta_i n_i(t) - \lambda_{i\alpha} C_{i\alpha}(t) \\
&\quad + \beta_i n_i(t) - \lambda_{i\alpha} C_{i\alpha}(t)
\end{align*}
\]

\(i = 1, 2, \cdots, 6, \quad k = 0, 1, \cdots, K, \tag{3}\)

where \(\rho_s\) is the subcriticality, \(C_{i\alpha}(t)\) the concentration of the delayed neutron precursors of the \(i\)-th variety, and \(\delta(r)\) is the importance function of the \(k\)-th mode, while \(r_m\) represents the position of the \(m\)-th control rod. All other symbols have conventional meaning.

By virtue of the approximation embodied in Eq.(1), the desired flux distribution is represented by a set of corresponding values \(n_k\) for the amplitude \(n_k(t) (k = 0, 1, \cdots, K)\). Hence the optimal control problem can be stated as follows: Find control inputs \(u_m(t) (m=1, 2, \cdots, M)\) that optimally transfer \(n_k(t) (k = 0, 1, 2, \cdots, K)\) in a system represented by Eqs. (2) and (3) from a given set of initial values

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Let $n_k(0)$ to the prescribed values $n_{kd}$.

Before attempting to solve this optimal control problem, the minimum number of control rods necessary to control the $(K+1)$ modes should be obtained first. This is a problem of controllability, and has been previously investigated by Kalman(6) and others(7)(8). The main aim of the present study is to analyze the controllability of the flux distribution, with the view to deriving the minimum number of control rods, as the first step to the optimal control of flux distribution.

**II. BASIC RESULTS ON CONTROLLABILITY**

Equations (2) and (3) are converted to a standard state variable representation: Define the state vector $x_k(t)$ for the $k$-th mode by $x_k(t) = [n_k(t), c_{1k}(t), \ldots, c_{6k}(t)]^T$ and the control input vector $u(t)$ by $u(t) = [u_1(t), u_2(t), \ldots, u_M(t)]^T$, where the superscript $T$ denotes the transpose, then Eqs. (2) and (3) are rewritten

$$\frac{dx_k(t)}{dt} = A_k x_k(t) + B_k u(t), \quad k = 0, 1, \ldots, K,$$

(4)

where the matrices

$$A_k = \begin{bmatrix}
\rho_k & \beta & \lambda_1 & \ldots & \lambda_6 \\
\beta & -\lambda_1 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & -\lambda_6 \\
\end{bmatrix},$$

and $B_k = N_k$

Gathering together the equations for all of the $(K+1)$ modes,

$$\frac{dx(t)}{dt} = A x(t) + B u(t),$$

(6)

where the $7(K+1)$ vectors $x(t), 7(K+1) \times 7(K+1)$ matrices $A$ and $7(K+1) \times M$ matrices $B$ are defined by

$$A = \begin{bmatrix}
A_0 & 0 & \ldots & \ldots & \ldots \\
0 & A_1 & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & A_K & \ldots \\
\end{bmatrix}, \quad B = \begin{bmatrix}
B_0 \\
B_1 \\
\ldots \\
B_K \\
\end{bmatrix}. \quad (7)$$

Choose an arbitrary control rod out of $M$ rods and designate it $u_m(t)$ $(1 \leq m \leq M)$, then the transfer function from $u_m(s)$ to $n_k(s)$ is readily obtained from Eq.(4):

$$\frac{n_k(s)}{u_m(s)} = N_k \phi_k(r_m) \frac{\rho_k}{s^2 + \sum_{i=1}^{6} \beta_i/s + \lambda_k}.$$

(8)

Hence

$$\frac{n_k(s)}{u_m(s)} = N_k \frac{\rho_k}{s^2 + \sum_{i=1}^{6} \beta_i/s + \lambda_k} \frac{\Pi(s + \lambda_k)}{\Pi(s - s_m)}.$$

(9)

where $s_m$'s are the poles of the right hand side of Eq.(8), and are the reciprocals of the roots $T$ of the inhour equation. The following properties of the $s_m$'s are easily verified.

(1) All the roots $s_m$ are real and non-positive.

(2) All the roots $s_m$ are distinct from the zeros of Eq.(8) $\rightarrow (-\lambda_k)$.

(3) All the roots for the same mode are distinct.

(4) All the roots for different modes are distinct, if the subcriticality of those modes are different.

The relationship of these properties to controllability is now examined.

A system is controllable if and only if there is no cancellation of poles and zeros in its transfer function. The property (2) above indicates that there is no cancellation, and therefore the amplitude of the $k$-th mode is controllable with any single control rod, provided that $\phi_k(r_m) \neq 0$. This condition $\phi_k(r_m) = 0$
means that the m-th rod is at a node of the k-th mode shape function, and obviously in this case the k-th mode is not controllable with this rod. Thus it is proved that any individual mode can be controlled with any single rod if the rod is not located at a node. But it is not yet verified whether all the modes are simultaneously controllable with a single rod.

The roots s_k's are the eigenvalues of the matrix A_k. Since they are real and distinct by properties (1) and (3), the state equation can be converted into diagonal form by means of non-singular linear transformation:

$$x_k(t) = T_k z_k(t), \quad (10)$$

that is,

$$\frac{dx_k(t)}{dt} = T_k^1 A_k T_k z_k(t) + T_k^1 B u(t)$$

$$= \Lambda_s z_k(t) + P_k u(t), \quad (11)$$

where $\Lambda_s$ is a diagonal matrix whose entries are $s_k$'s ($l = 1, 2, \ldots, 7$). If the eigenvalues are real and distinct, the necessary and sufficient condition for controllability is that no row of the matrix $P_k$ in Eq. (11) is equal to a zero vector. Since the controllability of the k-th mode has already been proved, therefore no row of $P_k$ is zero. Now consider the linear transformation

$$x(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \\ \vdots \\ x_K(t) \end{bmatrix} = \begin{bmatrix} T_0 z_0(t) \\ T_1 z_1(t) \\ \vdots \\ T_K z_K(t) \end{bmatrix} \equiv T z(t). \quad (12)$$

Then

$$\frac{dz(t)}{dt} = T^{-1} A T z(t) + T^{-1} B u(t)$$

$$= \Lambda z(t) + P u(t). \quad (13)$$

It is readily seen that the matrix $\Lambda$ is a diagonal matrix with entries $s_0, s_1, \ldots, s_K$, and that the matrix

$$P = \begin{bmatrix} P_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_K \end{bmatrix}. \quad (14)$$

Let us consider a simple one dimensional uniform core such as an infinite slab reactor. Then the subcriticality is different for all the modes. On account of the properties (3) and (4) all the eigenvalues are distinct, and from the results of the above discussion none of the rows of $P$ is equal to a zero vector. Hence in this case all the modes are simultaneously controllable if and only if each mode is controllable. This conclusion implies that any number of modes can be controlled with a single control rod if the rod is not at a node of any of the modes.

So far the controllability of all the variables $n_k(t), c_0(t), \ldots, c_K(t)$ has been investigated. In actual application, however, the values of $c_k(t)$ are not of great concern, and it is $n_k(t)$ that requires to be transferred to the desired value. This is a problem of output controllability, $n_k(t)$ being considered as the output. Since however state controllability always implies output controllability, the above analysis provides assurance of the controllability of $n_k(t)$ (k = 0, 1, \ldots, K) as well.

### III. MAINTAINABILITY

In the previous chapter it was concluded that any number of modes are controllable by means of a single control rod. But from a practical point of view mere controllability is not sufficient. Controllability implies that the system can be brought to any prescribed target state. It does not infer however, that the system can be maintained at the target state as long as one wishes. If the target state is the origin of the state space, then it is obvious that the system can be maintained at the origin by setting all the control inputs equal to zero — $u(t) = 0$. Otherwise, the “maintainability” of the system at the target state requires examination.

In the following, it is assumed that the $(K+1)$ outputs, $n_0(t), n_1(t), \ldots, n_K(t)$, must be maintained at any desired value, $n_0d, n_1d, \ldots,
and that it is not necessary to maintain the precursor density \( C_{ki}(t) \) constant.

Suppose that the amplitude of the \( k \)-th mode is maintained at \( n_{kd} \) for \( t \geq T \), then

\[
\frac{dn_{k}(t)}{dt} = 0, \quad t \geq T. \tag{15}
\]

Hence from Eqs. (2) and (3),

\[
\frac{dC_{ki}(t)}{dt} = \beta_{k} n_{kd} - \lambda_{C_{ki}}(t) \quad i=1, 2, \cdots, 6. \tag{16}
\]

If the \( C_{ki}(t) \)'s obtained from Eq.(17) are substituted into Eq.(16),

\[
\sum_{i=1}^{n} \beta_{km} n_{ki} u_{m}(t) = - \beta_{k} n_{kd} + \sum_{i=1}^{5} \left( \beta_{k} n_{kd} - \lambda_{C_{ki}}(T) \right) \tag{18}
\]

where \( \beta_{km} = N_{0} \phi_{m}(r_{m}) \phi_{k}(r_{m}) \). \tag{19}

The right hand side of Eq.(18) is a known function of time, if the desired value \( n_{kd} \) is fixed. It will therefore be designated \( f_{k}(t; n_{kd}) \).

Equation(18) is a set of linear simultaneous equations for the unknown functions \( u_{1}(t), u_{2}(t), \cdots, u_{M}(t) \). The solutions of these equations give the control input that maintains the system at the target state. Unique solutions exist for Eq.(18), provided that,

\[
M = K + 1, \tag{20}
\]

\[
\det \begin{bmatrix} \beta_{1} & \cdots & \beta_{M} \\ \vdots & \ddots & \vdots \\ \beta_{K + 1} & \cdots & \beta_{K + 1} \end{bmatrix} \neq 0. \tag{21}
\]

The first condition implies that \((K+1)\) control rods are necessary to maintain the \((K+1)\) modes at the desired value. The second condition is a restriction on the rod configuration. If Eq.(19) is taken into consideration, the condition is equivalent to

\[
\left| \begin{array}{ccc} \phi_{1}(r_{1}) & \cdots & \phi_{1}(r_{K+1}) \\ \vdots & \ddots & \vdots \\ \phi_{K+1}(r_{1}) & \cdots & \phi_{K+1}(r_{K+1}) \end{array} \right| \neq 0, \tag{22}
\]

which implies that the effectiveness of each rod to the \((K+1)\) modes must be linearly independent. The physical interpretation of this condition is quite obvious. If this condition is not satisfied, the effectiveness of some of the rods becomes a linear combination of those of the other rods. Then this rod does not contribute any additional degree of freedom in synthesizing the necessary control input.

IV. SEPARATION OF MODES

It has been assumed that the eigenfunctions for the modal expansion are so selected that there are no interactions between the amplitudes of the different modes, and that the amplitude of each mode satisfies the one point kinetic equations (2) and (3). In this respect the \((K+1)\) modes behave as if there were \((K+1)\) one point reactors. There is however the difference that whereas with \((K+1)\) one point reactors every control rod only affects the flux of the one reactor in which the rod is installed, in the present case all the \((K+1)\) modes are affected by any single rod. The \((K+1)\) modes are interrelated in that they are affected by the same control input.

Now if the variables \( \mu_{k}(t) \) \((k=0, 1, \cdots, K)\) are defined by the equation

\[ \mu_{k}(t) = \sum_{m=1}^{M} \beta_{km} u_{m}(t), \quad k=0, 1, \cdots, K \tag{23} \]

then \( \mu_{k}(t) \) may be considered the control input to the \( k \)-th mode. If the conditions of maintainability — Eqs.(20) and (21) — are satisfied, then the above equations are solved uniquely for \( u_{m}(t), \quad (m=1, 2, \cdots, K+1) \), given \( \mu_{k}(t), \quad (k=0, 1, \cdots, K) \). Hereby the optimal control problem may be discussed in terms of control input to the \( k \)-th mode \(-\mu_{k}(t)\), rather than of the movement of rods \(-u_{m}(t)\). It is always possible to convert \( \mu_{k}(t) \) into \( u_{m}(t) \) by means of Eq.(23) and obtain the actual movement of the rods.

If the conditions of maintainability are met, and further if the performance index of the optimal control problem is such that it does not introduce couplings between the modes, then the optimal control of the flux distribution is decomposed into the optimal control of \((K+1)\) separate one-point reactors. Time-optimal control and the tracking problem with the performance index

\[
J = \int_{0}^{T} \left[ \frac{\delta}{\delta \mu_{k}} \left( n_{k}(t) - n_{kd} \right)^{2} + \sum_{m=1}^{M} \mu_{m}(t) \right] dt \tag{24}
\]

are examples of such type of optimal control.
V. More Results on Controllability

1. Controllability of Azimuthal Mode

In a cylindrical reactor, the azimuthal mode as well as the radial has to be controlled. The axial mode is not considered for the moment. The general form of the shape functions of such reactors is

\[ \phi_{mn}(r,\theta) = R_n(r) \left( \alpha \cos \frac{m}{2\pi} \theta + \beta \sin \frac{m}{2\pi} \theta \right) \]  

(25)

\[ = R_n(r) A \cos \left( \frac{m}{2\pi} \theta + \varphi \right), \]  

(26)

\[ A^2 = \alpha^2 + \beta^2, \quad \varphi = \tan^{-1} \frac{\alpha}{\beta}. \]

This is a single mode, corresponding to a single eigenvalue. In connection with the azimuthal mode however, there are two variables to be determined - the amplitude A and the phase angle \( \varphi \). The angle \( \varphi \) is measured with respect to any arbitrary diameter of the core.

Now it is true that the amplitude A is controllable with any single rod. But it is not possible to control both the amplitude and the phase angle with a single rod, and at least two rods are necessary.

In this case, one may regard the two terms in Eq.(25) to be two different modes with the same eigenvalue. Then the amplitude of either of the two modes \( \alpha_{nn}(t) \) and \( \beta_{nn}(t) \) is a solution of the same set of one point kinetic equations. The controllability in such cases is analyzed below.

Suppose that there are two modes with the same eigenvalues and two control rods, either of the modes being controllable by either rod. The state vector representation of the two modes are

\[ \frac{dx_1(t)}{dt} = Ax_1(t) + Bu(t) \]

\[ \frac{dx_2(t)}{dt} = Ax_2(t) + Bu(t) \]

(27)

where the matrices

\[ B_1 = \begin{bmatrix} b_1 & b_1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_1 & b_1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}. \]

(28)

Some manipulation of vectors and matrices reveals (See Appendix 1.) that the two modes are simultaneously controllable with the two rods, if and only if

\[ \begin{bmatrix} b_1 & b_1 \\ b_1 & b_1 \end{bmatrix} \neq 0. \]

(29)

It is thus concluded that the azimuthal mode is controllable in angle as well as in amplitude if there are two control inputs that satisfy the condition (29).

2. Reactor with Power Coefficient Feedback

So far the discussion has been restricted to the controllability of zero power dynamics. In power reactors there is always present a variety of power coefficient feedback. The effect of such feedback phenomena is easily incorporated into the present analysis provided the validity of the two assumptions:

(1) The dynamics of the feedback phenomena are linear.

(2) They do not introduce couplings among modes*.

The latter condition may be accepted for an approximate analysis of radial temperature variations or linearized xenon poisoning effect. It is not valid however for axial temperature variations induced by the coolant flow.

Denote the zero power transfer function as \( G_1(s) \) and the feedback transfer function as \( G_2(s) \). The condition that this system with feedback is controllable is that the cascaded system \( G_1(s)G_2(s) \) is controllable. Hence the reactor with feedback is controllable if no cancellation of zeros and poles takes place in \( G_1(s)G_2(s) \). As for maintainability the conditions are the same as in a zero power reactor, only provided that \( n_0(t) \) is maintained. If some other variable — such as temperature — is also to be maintained at a prescribed level, then the \( (K+1) \) rods are not sufficient and other control inputs are needed — such as change in coolant flow rate.

3. Controllability under Constraint in Rod Speed

The conditions of controllability obtained * The eigenfunctions are selected with consideration given to the feedback equation, i.e. "Kaplan modes", and hence they are different from those of the \( \omega_n \) modes.
so far have not referred to the magnitude of the control input. In practice however, excessive control rod speed should be avoided, which introduces the importance of examining the controllability under constraint in rod speed.

Since the constraint is imposed upon the control rod speed $\frac{du_m}{dt}$ (m=1, 2, ..., K+1), these variables are selected for the control inputs, and $u_m(t)$ (m=1, 2, ..., K+1) representing rod displacement are regarded as additional state variables. It is easily proved that the new system of $8(K+1)$ variables

\[
\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t) + B\mathbf{v}(t),
\]

where

\[
\mathbf{x}(t) = \begin{bmatrix}
    x(1)(t) \\
    \vdots \\
    x(K+1)(t) \\
    u(1)(t) \\
    \vdots \\
    u(K+1)(t)
\end{bmatrix}, \quad \mathbf{v}(t) = \begin{bmatrix}
    \frac{du(1)}{dt}(t) \\
    \vdots \\
    \frac{du(K+1)}{dt}(t)
\end{bmatrix},
\]

is always controllable if the original system (8) is controllable (Appendix I).

As explained in Chap. IV, the discussion will evolve in terms of $\mu_k(t)$ rather than $u_k(t)$. If the variables are changed so that

\[
\begin{align*}
    n^*_k(t) &= n_k(t) - n_{0k} \\
    C^*_k(t) &= C_k(t) - \frac{\beta_k}{\lambda_k} n_{0k} \\
    \mu^*_k(t) &= \mu_k(t) + \frac{\beta_k}{\lambda_k} n_{0k}
\end{align*}
\]

the new set of variables satisfy the equations

\[
\frac{d\mathbf{x}^*(t)}{dt} = A\mathbf{x}^*(t) + B\mathbf{v}(t), \quad \mathbf{v}(t) = \frac{d}{dt} \mathbf{u}(t),
\]

and the target state is now the origin $n^*_1(t)=0$, $C^*_k(t)=0$ and $\mu^*_k(t)=0$.

Let us now impose on the control input the constraint

\[
|\mu_k(t)| \leq M, \quad k=0, 1, 2, \ldots, K,
\]

then all the conditions of controllability under constraint are satisfied, that is,

1. The set of admissible controls (34) is compact, convex, time invariant and contains the origin.
2. The target state is the origin.
3. The system is controllable if the constraint is not imposed.
4. All the eigenvalues are real and non-positive (the property (1) of Chap. IV).

And thus the system is controllable under the constraint (34).

VI. CONCLUSIONS

The controllability of the neutron flux distribution was investigated with the use of the truncated modal expansion. The amplitudes of many modes are found to be simultaneously controllable by a single control rod. To maintain the amplitudes at a prescribed level, however, as many control rods as the number of modes are necessary. If (K+1) control rods are installed to control and maintain the amplitudes of (K+1) modes, then the control of flux distribution can be discussed in terms of the input $\mu_k(t)$ to the k-th mode, rather than of the movement $u_m(t)$ of the m-th rod. Then by selecting a suitable performance index, the optimal control problem of flux distribution is decomposed into those of the individual modes. Furthermore, the controllability under constraint in rod speed has been examined.

APPENDIX I

By assumption, $x_1(t)$ in Eq.(27) is controllable by a single input $u_1(t)$. According to the well established theorem

\[
\text{rank } M_0 = \text{rank } (b : Ab : \ldots : A^k b) = 7,
\]

that is,

\[
\begin{pmatrix}
    b \\
    a_1 b \\
    \vdots \\
    a_k b
\end{pmatrix}
\left[\begin{array}{c}
    1 \\
    a_1 \\
    \vdots \\
    a_k
\end{array}\right]
= \text{rank } \begin{pmatrix}
    0 \\
    b a_1 \\
    \vdots \\
    b a_k
\end{pmatrix} = 7.
\]

where $a_{ij}$ is the i-j element of the matrix $A^n$.

In this case, $M_0$ is a $7 \times 7$ square matrix and
therefore rank $M_0 = 7$ implies that $\det|M_0| \neq 0$.

It is noted that $x_1(t)$ and $x_2(t)$ of Eq.(27) are simultaneously controllable by $u(t)$ if and only if

$$\text{rank } M = \text{rank } \begin{bmatrix} B^* & A^*B^* & \cdots & A^{*13}B^* \end{bmatrix} = 14,$$

(A3)

where $A^* = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$

$$B^* = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_7 \\ B_8 \\ B_9 \end{bmatrix} = \begin{bmatrix} b_1^* \\ b_2^* \\ \vdots \\ b_7^* \\ b_8^* \\ b_9^* \end{bmatrix} \equiv [b_1^* : b_9^*]$$

and $M$ is a $14 \times 28$ matrix. Consider a matrix $M_1$ such that

$$\text{rank } M_1 = \text{rank } \begin{bmatrix} B_1 & A_1B_1 & \cdots & A_1^{*13}B_1 \end{bmatrix} = 7.$$  
(A4)

Then, rank $M_1$ is at most 7, since the $j$-th row and the $(j+7)$-th row of $M_1$ are proportional to each other for $j=1, 2, \cdots, 7$, and thus not more than 7 rows of $M_1$ can be linearly independent. Now take the first 7 columns of $M_1$ and form

$$\tilde{M} = \begin{bmatrix} b_1^* & A_1^*b_1^* & \cdots & A_1^{*13}b_1^* \end{bmatrix},$$

(A5)

then it is readily shown that

$$\text{rank } M_1 = \text{rank } \tilde{M} \leq 7.$$  
(A6)

Thus,

$$\text{rank } M = \text{rank } (M_1 : M_2) = \text{rank } (\tilde{M} : \tilde{M}_2) \leq 14,$$

(A7)

It will now be demonstrated that rank $(\tilde{M}_1 : \tilde{M}_2)$ is equal to 14, if and only if the condition of Eq.(29) is satisfied. Assume that for a set of coefficients, $\lambda_1, \lambda_2, \cdots, \lambda_{14}$, the linear combinations of the columns of $\tilde{M}_1$ and $\tilde{M}_2$ are zero, that is,

$$\begin{bmatrix} \lambda_1 b_1^* & \lambda_2 b_1^* & \cdots & \lambda_{14} b_1^* \\ \lambda_1 b_2^* & \lambda_2 b_2^* & \cdots & \lambda_{14} b_2^* \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1 b_9^* & \lambda_2 b_9^* & \cdots & \lambda_{14} b_9^* \end{bmatrix} = 0.$$  
(A8)

The condition of Eq.(29) implies that

$$\lambda_j = 0, \quad j = 1, 2, \cdots, 14.$$  
(A9)

Thus the sufficiency is proved. The necessity is obvious.

**APPENDIX II**

The matrix $M$ in this case is given by

$$M = \begin{bmatrix} B & AB & \cdots & A^{[K(K+1)-1]}B \end{bmatrix},$$

(A10)

Since the system (6) is controllable, the rank of the matrix

$$M_i = \begin{bmatrix} b_i^* & A_i^*b_i^* & \cdots & A_i^{*13}b_i^* \end{bmatrix}, \quad i = 1, 2.$$  
(A11)

$M_i$ is at most 7, since the $j$-th row and the $(j+7)$-th row of $M_i$ are proportional to each other for $j=1, 2, \cdots, 7$, and thus not more than 7 rows of $M_i$ can be linearly independent. Now take the first 7 columns of $M_i$ and form

$$\tilde{M}_i = \begin{bmatrix} b_i^* & A_i^*b_i^* & \cdots & A_i^{*13}b_i^* \end{bmatrix},$$

(A12)

then it is readily shown that

$$\text{rank } M_i = \text{rank } \tilde{M}_i \leq 7.$$  
(A13)

Thus,

$$\text{rank } M = \text{rank } (M_1 : M_2) = \text{rank } (\tilde{M}_1 : \tilde{M}_2) \leq 14,$$

(A14)

It will now be demonstrated that rank $(\tilde{M}_1 : \tilde{M}_2)$ is equal to 14, if and only if the condition of Eq.(29) is satisfied. Assume that for a set of coefficients, $\lambda_1, \lambda_2, \cdots, \lambda_{14}$, the linear combinations of the columns of $\tilde{M}_1$ and $\tilde{M}_2$ are zero, that is,

$$\begin{bmatrix} \lambda_1 b_1^* & \lambda_2 b_1^* & \cdots & \lambda_{14} b_1^* \\ \lambda_1 b_2^* & \lambda_2 b_2^* & \cdots & \lambda_{14} b_2^* \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1 b_9^* & \lambda_2 b_9^* & \cdots & \lambda_{14} b_9^* \end{bmatrix} = 0.$$  
(A15)

Since $\det|M_0| \neq 0$,

$$\begin{bmatrix} \lambda_1 b_1^* + \lambda_2 b_2^* = 0, & \lambda_1 b_2^* + \lambda_2 b_3^* = 0, & \cdots, & \lambda_1 b_7^* + \lambda_2 b_8^* = 0, \\ \lambda_1 b_3^* + \lambda_2 b_4^* = 0, & \lambda_1 b_4^* + \lambda_2 b_5^* = 0, & \cdots, & \lambda_1 b_9^* + \lambda_2 b_{10}^* = 0, \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1 b_9^* + \lambda_2 b_{10}^* = 0, & \lambda_1 b_{10}^* + \lambda_2 b_{11}^* = 0, & \cdots, & \lambda_1 b_{12}^* + \lambda_2 b_{13}^* = 0 \end{bmatrix}$$  
(A16)

The condition of Eq.(29) implies that

$$\lambda_1 = 0, \quad j = 1, 2, \cdots, 14.$$  
(A17)

Thus the sufficiency is proved. The necessity is obvious.

**REFERENCES**


