Numerical Solution to Space-Angle Energy-Dependent Neutron Integral Transport Equation

Kiyoshi TAKEUCHI*

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A numerical approach to the steady-state, space-, angle- and energy-dependent neutron transport equation is presented for neutron shielding calculations. The scattering integral, with anisotropic treatment of elastic scattering and isotropic treatment of inelastic scattering, is evaluated by the use of Gaussian and straightforward quadratures. A system of coupled one-group integral equations for all the energy meshes of interest, converted from the energy-dependent integral transport equation, is calculated by performing a line integration along the neutron path in the direction of motion. For this purpose the direction of neutron motion is represented by discrete-ordinate directions $\Omega_{pq}$ on the unit sphere.

The final presentation of the integral transport equation is derived in a difference form convenient for machine computation. A computation program PALLAS has been written in Fortran IV for IBM 360-75 computer to perform neutron transport calculations based on this approach.

Comparisons are given of the numerical solutions with analytical solutions for unscattered fluxes in various geometries such as plane, spherical and two-dimensional cylindrical, for volume sources with self absorption, and with experimental spectra for angular neutron fluxes in graphite-, polyethylene- and water-shield. Excellent agreement is obtained between the present calculations and analytical or experimental results.

I. INTRODUCTION

Neutron flux density is the most fundamental information for the evaluation of neutron shields. Hence considerable effort has been directed in the past to the calculation of neutron-flux density through solution of the neutron transport equation. The problem however is inherently difficult because of the complicated form taken by the equation, involving space, angle and energy variables, so that exact solutions are limited to simplified cases.

A number of approaches have been proposed in the past for solving the integral transport equation in calculations for reactors of relatively simple geometry and with the adoption of simplified neutron scattering assumptions. These approaches may be classified into two major categories: (a) the collision probability method and (b) the Fourier-Laplace transform method. The former is usually based on volume integration of the Green function representation of neutron transport. It was originally developed as an application of the first flight collision probability method to neutron calculations on lattice systems, and later developed by Takahashi into a generalized first flight collision method. The second approach originated as the multiple collision method. It subsequently developed in two directions — one resulting in the $j_N$ method by Asaoka based on the Fourier-Laplace transform, and the other on the Laplace-Fourier transform of the transport equation, which is the $IT_N$ method of Hembd.

For neutron transport problems in shielding however, no reports have so far appeared of successful calculations based on solution of the integral transport equation. This can be attributed mainly to such difficulties in shielding as, for deep penetration, the extremely concentrated peak distribution at higher neutron energies and the attenuation of neutron flux in an abrupt manner, and in general much thicker media to be treated in shield than in reactor calculations. Thus for shielding calculations, it is required to take

* Ship Research Institute, Mitaka-shi, Tokyo.
into account a more exact representation of the spacewise angular and energy spectra as well as the angular distribution of neutrons, and also to deal with higher order anisotropy of elastic scattering and with inelastic scattering calculations on the neutrons, which are of particular importance. For practical shield calculations, the approaches cited above have not served satisfactorily in treating these particular characteristics of shielding problems. Yet practical requirements of radiation transport problems in shielding make it indispensable to obtain numerical solutions to cover individual cases.

The aim of this paper is to present a numerical approach to the practical solution of neutron shielding calculations, to circumvent the difficulties mentioned above. The essential point of our method is to lay aside the use of such current procedures as volume integration of the Green function representation or Fourier-Laplace transform for the integral transport equation, but to resort to line integration of the equation along the neutron flight path in the direction of neutron motion between successive spatial mesh intervals, with discrete-ordinate representation of space, angle and energy variables. Hence space and energy are divided into finite mesh points, and in addition, the direction, represented by the polar and azimuthal angles, is also divided into finite mesh points with roughly equal weights to satisfy the symmetry conditions approximately in their relation to the invariants of the solutions to arbitrary coordinate rotations. As a result the neutron-flux density can be evaluated for each set of specific points of spatial, directional and energetic meshes. The underlying feature of the method is quite general, and hence it can quite easily be applied to various geometrical problems.

In Chap. II we will formulate the neutron integral transport equation in its general form to predict the steady-state space-angle energy dependent neutron-flux density at any given position in heterogeneous media. The energy-dependent integral equation will be converted into a set of coupled one-group equations for each energy mesh, with account taken of sources due both to the true source and to the elastically or inelastically scattered neutrons. The scattering source may be evaluated in a way manner similar to that successively used in earlier studies based on direct numerical integration of the scattering integral through the application of quadrature schemes such as Gaussian, Neutron-Cotes and mechanical. Anisotropic scattering of neutrons is taken into account for elastic scattering treatment by expanding in Legendre polynomials the angular distribution function of scattering, while for inelastic scattering we assume isotropic scattering in the laboratory system.

In Chap. II we will present several illustrations of calculations in comparison with analytical solutions for the unscattered fluxes in plane, spherical and two-dimensional cylindrical geometries, and with experimental spectra in graphite, polyethylene and watermedium for spherical and finite-cylindrical geometries.

II. FORMULATION OF THE INTEGRAL TRANSPORT EQUATION

1. General Forms of the Equation

Referring to Fig. 1, the steady-state integral transport equation is

\[
\Phi(r, \Omega, E) = \Phi(r - R \Omega, \Omega, E)
\]

\[
 \cdot \exp \left[ - \int_0^R \Sigma_i(r - R \Omega, E) dR \right]
\]

\[
+ \int_0^R dR' \cdot S(r - R' \Omega, \Omega, E)
\]

\[
 \cdot \exp \left[ - \int_0^{R'} \Sigma_i(r - R' \Omega, E) dR' \right]
\]

\[
+ \int_0^R dR' \sum_{j=0}^{\infty} d\Omega' \int_{\Omega'}^{\Omega} dE'
\]

\[
\cdot \Sigma_s \cdot \langle r - R' \Omega; E' \rightarrow E, \Omega' \rightarrow \Omega \rangle
\]

\[
\cdot \Phi(r - R' \Omega, \Omega, E')
\]

\[
\cdot \exp \left[ - \int_0^{R'} \Sigma_i(r - R' \Omega, E) dR' \right]
\]

\[
(1)
\]

The first term on the right-hand side represents those neutrons which have escaped collision with nuclei while propagating in a straight line from \( r' \) to \( r \) in the direction \( \Omega \) with energy \( E \). The second and third terms represent, respectively, those neutrons contributed by the true source and the scattered neutron source with energy \( E \) in the direction
$Q$, with exponential attenuation while propagating in a straight line from their birth point—between $r'$ and $r$—to $r$.

![Fig. 1 Vector system adopted](image)

The suffix $i$ in Eq. (1) designates the type of nucleus involved in each material region.

Equation (1) has the form of a Fredholm integral equation of the second kind, and has already been ascertained\(^{17}\) to possess a unique, positive solution within a given volume of space bounded by a surface on which the incoming angular distribution is specified, with the condition of $\int_0^\infty \frac{d\Omega}{d\omega} < 1.0$.

We rewrite Eq. (1) in the form referred to in Appendix D of Ref.\(^{17}\):

\[
\Phi(r, \Omega, E) = Q(r, \Omega, E) + \lambda \Phi(r, \Omega, E),
\]

where the integral operator

\[
\lambda \Phi(r, \Omega, E) = \int_{r'}^r dR' \left\{ \sum_{i=1}^n d\Omega' \int_{E'}^E dE' \cdot \Sigma_{i}(r', E'; \Omega', \Omega) \cdot \Phi(r', \Omega', E') \exp\left[ -\tau(r, r', E) \right] \right\},
\]

while

\[
Q(r, \Omega, E) = \Phi(r', \Omega, E) \exp\left[ -\tau(r, r', E) \right] + \int_{r'}^r dR' \cdot S(r, \Omega, E) \cdot \exp\left[ -\tau(r, r', E) \right],
\]

with $r' = r - R' \Omega$

and $\tau(r, r', E) = \int_{r'}^r dR' \Sigma_{e}(r - R' \Omega, E)dR'$,

$\tau(r, r', E)$ being called the optical thickness between the points $r'$ and $r$ for energy $E$.

2. Evaluation of Eq. (3)

We now define a function $\Psi(r, \Omega, E)$ such that it satisfies the integral form

\[
\Psi(r, \Omega, E) = \sum_{i=1}^n d\Omega' \int_{E'}^E \Sigma_{i}(r, E'; \Omega, \Omega') \cdot \Phi(r, \Omega', E').
\]

With this function Eq. (3) can be rewritten

\[
\lambda \Phi(r, \Omega, E) = \int_{r'}^r dR' \Psi(r - R' \Omega, \Omega, E) \cdot \exp\left[ -\tau(r, r' - R' \Omega, E) \right].
\]

The function $\Psi(r, \Omega, E)$ will hereafter be called the scattering integral term.

(1) Evaluation of the Scattering Integral Term

The scattering integral term $\Psi(r, \Omega, E)$ does not depend upon the choice of geometry, since the term is related only to the transfer of energy and direction of neutrons at a point $r$, and so the integral calculation of Eq. (6) may be performed in a way entirely analogous to the procedure described in a previous paper\(^{16}\). Then the scattering cross section $\Sigma_{i}(r, E'; \Omega, \Omega')$ may be represented by

\[
\Sigma_{i}(r, E'; \Omega, \Omega') = \int_{E'}^{E} f_{s}(E'; \Omega, \Omega') \frac{d\Omega' \cdot d\Omega}{dE}.
\]

To calculate the integral on the right-hand side of Eq. (6) by a formula for numerical integration, we must specify discrete variables for the continuous energy and direction variables of interest keeping in mind the concept of the discrete ordinates method. Let the discrete-ordinate directional points be $\Omega_{pq}(\omega_{p}, \phi_{pq})$, in which the $\omega_{p}$'s are the discrete points on the cosine of the polar angle in the region $(-1, 1)$. We may then adopt the zeros of Legendre polynomials for them for $p=1, 2, \ldots, P$. The $\phi_{pq}$ is in the same expression are the discrete points on the azimuthal angle, and as a choice for these points we may divide the $\phi$-range $(0, \pi)$ into $2p$ equal intervals and choose a center point in each interval for each $p$-level of the polar angles (see Fig. 2). Hence the directional neutron-flux density is approximated by a finite number of discrete neutron rays with certain weights. For the energy variable, since the lethargy variable $u$ is used instead of energy $E$ for the integral calculation over neutron energy, the lethargy is divided into $J$ meshes with uniform lethar-
gy intervals equal to $h$.

For the elastic scattering, substituting Eqs. (8) and (9) into Eq. (6) and expanding the scattering distribution function $f^s(E, \Omega, \Omega')$ in a finite series of Legendre polynomials with $\Omega \cdot \Omega' = \mu$, where $\mu$ is the cosine of the scattering angle in the center of mass system, Eq. (6) becomes, with the neutron lethargy instead of energy,

$$
\Phi^a(r, \Omega, \mu) = \sum_{\nu=1}^{P} \sum_{\nu'=1}^{P} \alpha_{\nu \nu'} \Phi^E_{\nu \nu'}(r, \mu) \sin^{\nu}(\mu) \cdot \delta(\cos r-\alpha) \exp(\mu-\nu') \cdot \Phi^E_{\nu \nu'}(r, \mu).$$

The summation with respect to $i$ is ignored, for simplicity, in the above equation as also in the subsequent discussion. Equation (11) has the same form as Eq. (14) in Ref. (16), which permits us to follow the procedure adopted in that work and carry out the calculation of Eq. (11) assuming (1) a step function approximation of the directional flux density $\Phi(r, \Omega, \phi, \mu)$ with respect to $\Delta \Omega_{\nu \nu'}$ and $\Delta \mu$, and (2) similarly also of the scattering cross section $\Sigma^s(r, \nu)$ and $f^s(nu)$, which is the Legendre expansion coefficient of $f^s(nu, \mu)$, with respect to $\Delta u$. Thus, applying the Gaussian quadrature to the integration over $\mu$ for discrete points $\mu_n (m=1, 2, \ldots, M)$ in reference to the zeros of the Legendre polynomials, and a straightforward quadrature to the integration over $\phi$ for discrete points $\phi_n (n=1, 2, \ldots, P)$ in reference to $\phi_n$, we transform Eq. (11) into a form of finite sums for the specific points of angular and energetic meshes at space $r$:

$$
\Psi^a(r, \Omega_{\nu \nu'}, \mu_n) = \sum_{\nu=1}^{P} \sum_{\nu'=1}^{P} \alpha_{\nu \nu'} \Phi^E_{\nu \nu'}(r, \mu_n) \Phi^E_{\nu \nu'}(r, \mu_n),
$$

where $\alpha_{\nu \nu'} = W_{\nu} W_{\nu'}$, $W_{\nu}$: Weighting coefficients for Gaussian quadrature with respect to the $\mu$ integration $W_{\nu}$: Weighting coefficients for the $\phi$ integration; $\Phi^E_{\nu \nu'}(r) = \Phi^E_{\nu \nu'}(r, \phi_{\nu \nu'}, \mu_n)$. equation have been evaluated in a previous work (16) and hence further discussion is omitted here.

For the inelastic scattering, Eq. (6) can be written in this case also in terms of lethargy $u$ instead of energy $E$:

$$
\Psi^{in}(r, \Omega, u) = \int_{t_{\nu}}^{t_{\nu}'} d\psi \cdot \Phi(r, \Omega, u') \Sigma^{in}(r, u') \sin^{-1}(\mu) \cdot \delta(\cos r-\alpha) \exp(u-u') \cdot \Phi(r, \Omega, u', \mu).
$$

Equation (15) has a form identical with Eq. (22) of the previous paper (16), which has been evaluated numerically in the same paper. Thus

$$
\Psi^{in}(r, \Omega_{\nu \nu'}, u_n) = \sum_{\nu=1}^{P} \sum_{\nu'=1}^{P} b_{\nu \nu'} C_{\nu \nu'}(r) \Phi_{\nu \nu'}(r).
$$

Here the matrix $C_{\nu \nu'}(r)$ and $\Phi_{\nu \nu'}(r)$ are wholly identical in form with those of the previous work (16), the only difference being that, instead of $Q$ in Eq. (23) of the previous work (16), we here have in the summation of Eq. (16)

$$
Q_{\nu \nu'} = \begin{cases} 2p' & 0 \leq p' \leq 2P \\ 2(P-P' + 1) & P < 2P' \leq P. \end{cases}
$$

and consequently the weighting coefficients also are replaced by $b_{\nu'} = (2\pi/Q_{\nu \nu'}) \lambda_{\nu'}$, where $\lambda_{\nu'}$ are the weights of the Gaussian quadrature. Thus the scattering integral term $\Psi(r, \Omega, E)$ can be obtained from Eqs. (12) and (16) for
the specific points of angular and energetic meshes at space r:

\[ \mathbb{F}(r, \alpha, u_i) = \mathbb{F}(r, \alpha, u_i) + \mathbb{G}(r, \alpha, u_i). \]  

(17)

(2) Evaluation of Eq. (7)

As mentioned earlier the scattering integral term \( \mathbb{F}(r, \alpha, E) \) does not depend upon the choice of geometry, and this term has already been calculated, while now Eq. (7) does depend on the choice of geometry, and it will be treated with reference to a generalized geometry and calculated on the basis of the following two assumptions with respect to the spatial variable: (1) The spatial interval \( r' \) is chosen that the total cross section \( \Sigma_i(r' - R, E) \) can be assumed to be constant over that interval, and (2) the scattering integral term \( \mathbb{F}(r - R', \alpha, E) \) can be approximated by a linear function in the interval \( r' \). Consequently the neutron path \( R \) is defined specifically as \( R = |r - r'| \).

We apply the assumption (2) to reformulate

\[ \mathbb{F}(r - R', \alpha, E) = a(r) + b(r)R', \]  

(18)

and we further let \( \Sigma_i(r' - R, E) = \beta \), which make it possible to calculate Eq. (7):

\[ \int_0^\infty (a + bR') \exp(-\beta R') dR' \]

\[ = \frac{a}{\beta} (1 - \exp(-\beta R)) - \frac{b}{\beta^2} \beta R \exp(-\beta R') + \exp(-\beta R) - 1. \]  

(19)

Rewriting the unknown factors \( a(r) \) and \( b(r) \) by using the relations

\[ a(r) = \frac{\mathbb{F}(r)}{r}, \]

\[ b(r) = \frac{\mathbb{F}(r') - \mathbb{F}(r)}{r} \]  

(20)

Eq. (7) is reduced to the form

\[ A\Phi(r, \alpha, E) = (\mathbb{F}(r, \alpha, E) \Sigma_i(r', E) R + \exp(-\Sigma_i(r', E) R) - 1) + \mathbb{G}(r, \alpha, E) (1 - [1 + \Sigma_i(r', E) R] \exp(-\Sigma_i(r', E) R)) \]

\[ / (\Sigma_i(r', E) R)^3 R. \]  

(21)

3. Evaluation of \( Q(r, \alpha, E) \)

The first term on the right-hand side of Eq. (4) can easily be evaluated by applying the assumption (1) in the preceding Sec. 2- (2). On the other hand the second term in Eq. (4) is of an integral form similar to Eq. (7), and numerical calculation thereof should result in an expression similar to Eq. (21). Thus we have

\[ Q(r, \alpha, E) = \Phi(r', \alpha, E) \exp(-\Sigma_i(r', E) R) + [\mathbb{G}(r, \alpha, E) \Sigma_i(r', E) R + \exp(-\Sigma_i(r', E) R) - 1] + \mathbb{F}(r', \alpha, E) (1 - [1 + \Sigma_i(r', E) R] \exp(-\Sigma_i(r', E) R)) \]

\[ / (\Sigma_i(r', E) R)^3 R, \]  

(22)

where \( R = |r - r'| \).

4. Evaluation of \( \Phi(r, \alpha, E) \)

As in the preceding sections, \( Q(r, \alpha, E) \) and \( A\Phi(r, \alpha, E) \) on the right-hand side of Eq. (2) has been evaluated, so that the equation can now be solved. Then substituting Eqs. (21) and (22) into Eq. (2) and defining a new function

\[ G(r, \alpha, E) = \Phi(r, \alpha, E) + S(r, \alpha, E), \]  

(23)

we have the equation

\[ \Phi(r, \alpha, E) = \Phi(r', \alpha, E) \exp(-\Sigma_i(r', E) R) + [G(r, \alpha, E) \Sigma_i(r', E) R + \exp(-\Sigma_i(r', E) R) - 1] + \mathbb{F}(r', \alpha, E) (1 - [1 + \Sigma_i(r', E) R] \exp(-\Sigma_i(r', E) R)) \]

\[ / (\Sigma_i(r', E) R)^3 R. \]  

(24)

Since the derivation of Eq. (24) has been performed in a specific spatial interval \( (r', r) \), it is convenient to proceed with the discussion by adopting the concept of discrete mesh points in the treatment of space coordinates. Thus, we number \( (r_{i-1}, r_i) \) the spatial intervals \( (r', r) \) and \( E_i \) the energy intervals \( E \) to correspond to the lethargy mesh \( u_j \), the above equation can then be expressed in the matrix form

\[ \Phi_i = E_i \Phi_{i-1} + F_i G_i + H_i \Phi_{i-1}, \]  

(25)

or

\[ \Phi_i = A_i^{-1} \Phi_0 + B_i^{-1}, \]  

for \( i \geq 1 \)  

(26)

where \( A_i^{-1} = \left[ \begin{array}{c} I \\ \frac{E_i}{I} \end{array} \right] \) for \( i \leq 1 \)  

(27)

\[ B_i^{-1} = \frac{1}{I} \left[ \begin{array}{c} I \\ \frac{E_i}{I} \end{array} \right] A_i^{-1} \mathbb{F}(G_i + H_i \Phi_{i-1}). \]  

(28)

Further, \( \Phi_i \) and \( G_i \) are \( J \)-element vectors containing, respectively, the multigroup-flux densities \( \phi_{(r_i, \alpha, E_i), \cdots, (r_i, \alpha, E_f)} \), and the multigroup-source densities \( G_{(r_i, \alpha, E_i), \cdots, (r_i, \alpha, E_f)} \), while \( E_i, F_i, \) and \( H_i \) are \( J \times J \) diagonal matrices whose elements will be given later, while \( I \) represents a \( J \times J \) unit matrix and \( \Phi_0 \)
corresponds to a boundary condition.

For shortening machine time required in computation, no iteration technique is applied to compute neutrons scattered within the same lethargy group. Instead, we apply an alternate technique that has successfully been used before\(^{(15)(16)}\). For this purpose the scattering integral term \(\Psi(r, \Theta, E)\) is rewritten in the form of a summation of the two terms

\[
\Psi(r, \Theta, E) = \Psi_u(r, \Theta, E) + \Psi_w(r, \Theta, E),
\]

where \(\Psi_u(r, \Theta, E)\): Down-scattering term

\[
\Psi_w(r, \Theta, E) : \text{Within-scattering term}
\]

and

\[
\Psi_u(r, \Theta, E) = A(r, E)\Phi(r, \Theta, E),
\]

\[
A(r, E) = 2\sqrt{\pi} \omega T_{\text{eff}}(r) \exp\left[-\sum_{E_i}^{E_f} \frac{\omega E_i}{1 + \omega E_i} \right].
\]

Substituting Eqs. (29) and (30) into Eq. (24) yields the final equation:

\[
\Phi(r, \Theta, E) = \Psi(r', \Theta, E) \exp\left[-\sum_{E_i}^{E_f} \frac{\omega E_i}{1 + \omega E_i} \right] R
\]

\[
\cdot \left\{ G(r', \Theta, E) + \Psi_w(r', \Theta, E) R + \exp\left[-\sum_{E_i}^{E_f} \frac{\omega E_i}{1 + \omega E_i} \right] - 1 \right\}
\]

\[
+ G(r', \Theta, E) \left[1 - \sum_{E_i}^{E_f} \frac{\omega E_i}{1 + \omega E_i} \right] R
\]

\[
\cdot \mathbf{K}(r, \Theta, E),
\]

where

\[
G(r', \Theta, E) = G(r', \Theta, E)
\]

\[
- A(r, E)\Phi(r, \Theta, E).
\]

As seen from Eq. (33) the source term \(G(r', \Theta, E)\) has now been replaced by a new source term \(G(r', \Theta, E)\), so that the elements of the vector \(G_i\) in the matrix equations (25) and (28) must be replaced by new elements \(G'(r_i, \Theta_i, E_i)\). The remaining undefined elements for \(E_i, F_i\) and \(H_i\) are given by

\[
E_{ij} = \exp\left[-\sum_{E_k}^{E_f} \frac{\omega E_k}{1 + \omega E_k} \right] R_{ij}
\]

\[
F_{ij} = \sum_{E_k}^{E_f} \frac{\omega E_k}{1 + \omega E_k} R_{ij}
\]

\[
H_{ij} = \sum_{E_k}^{E_f} \frac{\omega E_k}{1 + \omega E_k} R_{ij}.
\]

Thus Eq. (2) — and hence Eq. (1) — has been solved numerically for the directional flux density \(\Phi(r, \Theta, E)\) for every set of specific points in the spatial, angular and energetic meshes of interest and with proper boundary conditions. It should be noted that this permits us to calculate the directional flux density along the specified angular rays in the direction of neutron motion. This characteristic feature in calculation is completely analogous to the case of the method of discrete ordinates in the calculation of the Boltzmann transport equation.

The scalar flux density \(\Phi_0(r, E)\) at the specific points in the spatial and energetic meshes can be calculated as a weighted sum over discrete rays:

\[
\Phi_0(r_i, E_j) = \sum_{i=1}^{N} \sum_{j=1}^{M} h_i \Phi(r_i, \Theta_i, E_j).
\]

It is clear from the above discussions that Eq. (32) is valid for any geometry, since no assumptions or restrictions have been made on the geometry in the process of its derivation. Besides, Eq. (32) can be easily applied to various geometries since the neutron-flight path \(R\) appearing in Eq. (32) can quite readily be represented in actual coordinates.

A computer program PALLAS\(^{(18)(19)}\) has been written in Fortran IV language for IBM 360-75 computer to perform the calculations for plane, spherical and two-dimensional cylindrical geometries based on the present method, with which several numerical results have been obtained, as presented in the next chapter.

### III. Comparisons of Numerical Calculations with Analytical and Experimental Results

Several calculations have been performed for various geometries such as plane, spherical and two-dimensional cylindrical. The results have been compared with analytical solutions and with experimental data to verify the accuracy of the method and of the PALLAS program for unscattered flux and angular neutron flux spectra in graphite, polyethylene and water media, these being the most fundamental quantities, while the relation between angular flux spectrum and position can
provide all the information needed to compute integral quantities in the shield.

Figure 3 represents a comparison of the PALLAS-calculated scalar flux with analytical solutions\(^{(20)(21)}\) for a slab- or a spherical-volume source, with self absorption. A very fair agreement is seen between them.

![Figure 3](image)

1. A spherical source of 26 cm radius with a constant source distribution and with self absorption of \(\mu_s = 0.1 \text{ cm}^{-1}\), with absorption of \(\mu = 0.2 \text{ cm}^{-1}\) in the shield. 2. A slab source of 20 cm thickness with a constant source distribution and with self absorption of \(\mu_s = 0.1 \text{ cm}^{-1}\), with absorption of \(\mu = 0.2 \text{ cm}^{-1}\) in the shield.

**Fig. 3** Comparison of PALLAS numerical solutions with analytical solutions for unscattered fluxes in plane and spherical geometries

Figures 4 and 5 show another comparison of the PALLAS-calculated scalar flux with analytical solutions\(^{(20)(21)}\) for a finite-cylindrical-source geometry, with self absorption. In **Fig. 4** the solid lines represent the numerical solutions in reference to the \(z\)-distance at \(r = 1.5, 3.0, 9.0\) and 12.0 cm, which last distance corresponds to the side surface of the cylindrical (as noted by the remark 'Surface' in the figure). The last curve in the same figure is drawn against \(r\)-distance, for \(z = 12.0\) cm, the top surface of the cylinder, as noted. In **Fig. 5** are shown two comparative calculations for scalar fluxes in the interior and exterior of cylindrical sources\(^{(20)(21)}\). The upper curves represent the results for a 20 cm radius, 36 cm long finite cylindrical source, the curves marked 'A' being those for the scalar flux along an \(r\)-axis passing the origin and those marked 'B' for that along a similar axis on the top surface. The lower curve depicts the flux along the \(z\)-axis, with origin as zero point on the graph, for a 20 cm radius, 26 cm half-length finite cylindrical source. The dotted lines represent the analytical solutions, and in particular the curve marked 'U' and 'L' respectively are those for the upper and lower limits of the analytical solutions.

![Figure 4](image)
Fig. 5 Comparison of the PALLAS numerical solutions with analytical solutions for unscattered fluxes in two-dimensional cylindrical geometry.

Figures 6 and 7 represent the angular neutron flux spectra from a fission source in graphite. The measured spectra are taken from the experimental result\(^{(22)}\) by Profio et al., which was performed by means of the pulsed-source, time-of-flight method. In the PALLAS calculation the source was represented as an outward-directed emitter on a surface of 4.45 cm radius, which was precisely the same condition as that of the 05R calculation\(^{(22)}\), and besides the source spectrum and its angular distribution were made to coincide with those of the same calculation, except for the angular distribution of the radial direction, which was determined inversely from the attenuation of high energy neutrons in the measured spectra. The calculation was performed for 80 cm radius graphite with 31 group P\(_1\) cross section (0.4 to 8.0 MeV) obtained from ENDF/B data, in a computing time of 4 min. The spectra illustrated in Figs. 6 and 7 have not been normalized, and yet excellent agreement is seen between measurement and calculation, the only discrepancies being for the 16.6° — and 60.0° — directions in the higher energy region at \(R = 35.6\) cm.

Figure 8 shows another comparison on angular neutron flux spectra from a fission source in polyethylene. The experiment\(^{(23)}\) was carried out with the use of an uranium target of approximately 110 cc volume, irradiated with a pulsed beam of 45 MeV electrons, and measured with time-of-flight technique. The source and the geometry used in the PALLAS calculation were made to coincide exactly with those of the NIOME calculation\(^{(23)}\), i.e., a constant isotropic source of the
The spectra have not been normalized.

**Fig. 7** Comparison between PALLAS-calculated and measured\(^{(22)}\) spectra in graphite at 35.6 cm radius uranium with identical source spectrum, and 2.08 cm thick air gap, with 5.08 cm thick polyethylene covering the source. The time required for the calculation was about 10 min for 42-group, 0.1 lethargy interval \(\Phi_s\) cross section (0.12 to 7.32 MeV) obtained from ENDF/B data for hydrogen and carbon, and from UNC data\(^{(24)}\) for uranium. The PALLAS calculated spectra have been normalized to a single point at 0.12 MeV on the experimental 0° spectrum. Good agreement is obtained for the 0° direction, but appreciable discrepancy is discernible in the higher energy region for 45°.

In order to ascertain the validity of the two-dimensional cylindrical geometry version of the PALLAS code, another comparison was performed, this time on the angular fast neutron spectra in water shield. **Figures 9** and 10 show the results of the comparison of the PALLAS calculation with the measurements. The experimental data given in Fig. 9 are from Harris et al.\(^{(25)}\) who obtained them on the water pool of the University of Michigan Ford Nuclear Reactor, while those in Fig. 10 are from Verbinski et al.\(^{(26)}\) on the water pool of the Bulk Shielding Reactor 1. The sources assumed in the PALLAS calculations were such that: for the spherical reactor configuration the neutron-source density along the radial distance was made perfectly identical with that used in the NIOBE calculation\(^{(26)}\), while for the finite cylindrical reactor configuration the neutron-source density along the cylinder axis (26 cm long, being half the source cylinder axis length) was made to coincide exactly with that along the radial distance of the sphere figuring in the foregoing case, and the density along a radius of the cylinder (20.4 cm overall radius) was set at a constant value of 1.0, the same power as that of the spherical source being adopted. The calculations were performed with 0.1 lethargy intervals (1.34 to 14.75 MeV, 25 group) for the spherical geometry, and with 0.2 lethargy in-
The calculated spectra in the two-dimensional cylindrical geometry are those relevant to the cylindrical axis. The spectra have not been normalized.

Fig. 9 Comparison of PALLAS-calculated and measured\(^{25}\) spectra in water at \(0^\circ\) intervals (2.0 to 12.08 MeV, 10 group) for the cylindrical. The computing times required for these calculations were respectively 10.8 and 20.1 min. All the nuclear data for the calculations were taken from ENDF/B. The spectra illustrated in Figs. 9 and 10 have not been normalized, but excellent agreement is seen between all corresponding curves and plots for both \(0^\circ\) and \(40^\circ\) spectra, with only slight discrepancies for \(0^\circ\) spectra at 8 and 5.5 MeV of deep penetrations in the calculated spectra by the spherical version, which seems to be due to the narrow dips of the total cross section of oxygen. It is seen from Fig. 10 that the calculated spectra with the cylindrical system represent values slightly higher than with the spherical system, which can be attributed to the effect of the edge of the cylindrical source.

![Graph showing spectral comparison](image)

The calculated spectra in the two-dimensional cylindrical geometry are those relevant to the cylindrical axis. The spectra have not been normalized.

Fig. 10 Comparison of PALLAS-calculated and measured\(^{28}\) spectra in water at \(40^\circ\)

**IV. DISCUSSION AND CONCLUSION**

The present method offers an approach to neutron transport calculations in plane, spherical and two-dimensional cylindrical geometries. The fair agreement obtained between the PALLAS calculations and the analytical solutions, as evidenced in Figs. 3–5 indicate that the numerical solutions by the present method are close to exact for unscattered fluxes in plane, spherical and finite-cylindrical geometries. Comparisons with experimental results seen in Figs. 6–10 further attest to the method as a reliable approach to the neutron transport calculation in spherical and finite-cylindrical geometries. The same should be applicable to plane geometry because the same computing routines are used in PALLAS code as with spherical system except for the calculation of geometrical attenuation. Reliable information is insufficiently available today to permit more detailed comparisons between the results calculated by the present method and other results for two-dimensional cylindrical geometry.

The method has several advantages:

1. It can be easily applied to various geo-
metries such as plane, spherical and two-
dimensional cylindrical geometries.
(2) It can be applied to neutron transport
calculations with a variety of source con-
figurations such as surface- and volume-
distributed sources.
(3) It offers a useful tool for radiation
transport calculations since it can provide
space-, angle- and energy-dependent an-
gular flux density, which is the most
fundamental quantity and from which one
can immediately obtain all the necessary
information for shielding calculation.
(4) It is ideally suited to deep penetration
calculations, since it can treat the com-
plicated neutron phenomena arising in
shield by virtue of its embodying repre-
sentation of anisotropy in elastic scattering
and consideration of inelastic scattering.
(5) It can be readily extended to three-
dimensional geometries such as rectangu-
lar, through replacement of the neutron-
flight path $R$ in Eq. (32) by variables of
actual coordinates.
(6) The execution time required for com-
putation is short enough to allow applica-
tion of the method to practical fast neu-
tron shielding calculations in the case of
plane and spherical geometries, while the
calculations for two-dimensional cylindri-
cal geometry may be used as a reference
to evaluate the results of shielding design
calculations based on more simple tech-
niques.

On the other hand, it should be noted that
the method has the drawback of what is
known as the ray effect, which characterizes
also the method of discrete ordinates. This
is caused by the adoption of a limited number
of discrete directions for the evaluation of
the directional flux density, so that the scalar
flux density at a given position in a medium
is evaluated by these directional flux densities
with their characteristic directions. This de-
fect can largely be overcome by increasing
the number of discrete directions, although
complete elimination of error is impossible.

From the above discussions we may con-
clude that the present method is a reliable
and useful tool for neutron penetration calcu-
lations for a variety of shielding problems in
various geometries.

**Notations**

$r$: Spatial variable
\( \vec{Q} \): Unit vector in the direction of
neutron motion
$E$: Energy variable
\( \Phi(r, \vec{Q}, E) \): Neutron directional flux density —
the number of neutrons with energy
$E$ per unit energy per unit solid
angle, moving in the direction \( \vec{Q} \)
and crossing in unit time a unit
area at $r$ on a plane normal to the
direction of neutron motion
\( \Sigma_i(r, E) \): Position-dependent total macroscop-
ic cross section at energy $E$
\( S(r, \vec{Q}, E) \): True source — the number of neu-
trons with energy $E$ and moving in
the direction \( \vec{Q} \) at a position $r$ per
unit volume in unit time
\( \Sigma_{ss}(r; E', \vec{Q} \rightarrow \vec{Q}) \): Position-dependent differ-
ential scattering cross section of $i$-th
nucleus which describes the proba-
bility that a neutron with energy
$E'$ and direction \( \vec{Q}' \) undergoes a
scattering collision at $r$ which
places it into unit solid angle about
\( \vec{Q} \) with new energy $E$ in unit energy
\( f_{i}(E' \rightarrow E, \vec{Q}' \rightarrow \vec{Q}) \): Differential scattering dis-
tribution function of $i$-th nucleus,
which is defined as
\[
\int f_{i}(E' \rightarrow E, \vec{Q}' \rightarrow \vec{Q}) d\vec{Q}' dE'=1
\]
$u$: Neutron lethargy \((=\ln E_{max}/E)\)
\( \phi \): Angle between the \((z, \vec{Q})\)-and \((\vec{Q},
\vec{Q}')\)-planes
\( \gamma \): Scattering angle in the laboratory
system
\( \alpha \): Cosine of the scattering angle \( \gamma \)
\( \Sigma_{iss}(r, u) \): Position-dependent macroscopic
elastic scattering cross section at
lethargy $u$
\( \Sigma_{in}(r, u) \): Position-dependent macroscopic in-
elastic scattering cross section at
lethargy $u$

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