SHORT NOTE

Scaling Laws of Tokamak Type Devices

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Received May 6, 1972

KEYWORDS: scaling, Tokamak devices, plasma columns, thermal conductivity, heat transfer, electrons, ions, Joule heating, plateau model, ion temperature, plasma density, energy confinement, thermonuclear devices, high temperature

It has been recently reported by Furth, Rosenbluth, Rutherford & Stodiek (1) that the main characteristics of the temperature and density profiles of current-carrying plasma columns such as embodied in Tokamak type devices can be described for practical purposes by equations of thermal equilibrium, that is, in cylindrical coordinates \((r, \theta, z)\), a detector and its connecting line, especially in leaky cores.

The authors wish to express their appreciation to Messrs. T. Nakamura and M. Obu, and to other colleagues of FCA for their co-operation in the experiment.

—REFERENCES—


\[
\begin{align*}
\eta J_z E_z &= Q_{ie}, \quad 1 > \eta > 0, \tag{1} \\
\frac{1}{r} \frac{d}{dr} \left( r X \frac{dT_i}{dr} + Q_{ie} \right) &= 0, \tag{2} \\
Q_{ie} &= C_0 \frac{n(T_e - T_i)}{\tau_e}, \tag{3} \\
C_0 &= \frac{3 m_e k}{m_i}, \tag{4} \\
\tau_e &= \frac{C_\tau}{n \log A}, \tag{5}
\end{align*}
\]

where \(\chi\) is the ionic thermal conductivity, \(Q_{ie}\) the heat transfer from electrons to ions, and \(\tau_e\) the collision time between ions and electrons. In Eq. (1) a new constant \(\eta\) is tentatively applied by the present authors to estimate the fraction of Joule heat that is transferred from electrons to ions, considering the remainder \((1 - \eta)\) to be lost directly through the electrons themselves. The other quantities have their conventional meanings.

The plateau model of thermal conductivity was used for analytical convenience and toroidal effects.

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were embodied in the major radius \( R \) in Eq. (5). It should be of interest to derive from these equilibrium equations the ion temperature \( T_i \), the plasma density \( n \) and the energy confinement time \( \tau_e \), which can be used as scaling laws for devices of this type, as will be explained later.

By using Spitzer’s electrical conductivity formula,

\[
\sigma = C_x \frac{T_e^{3/2}}{\log A} \frac{3 \times 1.96 e^2}{m_e},
\]

we can derive from Eqs. (1), (3), (4) and (6) the axial electric field

\[
E = \frac{C_x^{1/2} (1 - \varepsilon)^{1/2} \pi R_0 \log A}{C_x^{1/2} C_i A^{1/2} T_e^{1/2}},
\]

and the current density on the axis

\[
j_0 = C_x \frac{B}{q_s R}, \quad C_x = 2/\mu_0,
\]

given by

\[
j_0 = \frac{(C_x C_i)^{1/2} (1 - \varepsilon)^{1/2} \pi R_0 T_e^{1/2} A^{1/2} \rho_i^{1/2}}{A^{1/2} \rho_i^{1/2}},
\]

where \( \varepsilon = T_i / T_e \), while \( A \) is the ion mass number and \( B \) the toroidal magnetic field. The suffix 0 indicates the value on the axis. In Ref. (1) the equilibrium was shown to be thermally stable only if \( \varepsilon > 2/3 \), and also, according to Ref. (2) the application of Suydam’s criterion on such a current-carrying plasma column requires that \( 0.75 > \varepsilon > 0.33 \). Thus the latter will be assumed hereafter as a necessary condition of plasma stability. Equation (8) is the definition of the safety factor \( q_s \) on the axis. Here it should be noted that the electric field in equilibrium state, given by Eq. (7), has the same dependency on plasma density \( n_0 \) and electron temperature \( T_e \), as that of Dreicer’s critical field of electron runaway, but the former is smaller than the latter by a factor of about \( (m_e/m_i)^{1/2} \), and therefore the current is carried, as it were, by particles with ion mass \( m_i \).

As for Eq. (2) we assume that the ion temperature profile can be approximated by

\[
T_i / T_{i0} = 1 - (r/a)^2,
\]

where \( a \) is the radius of the plasma column. With this assumption the heat conduction can be made finite on the axis. From the foregoing we obtain as the scaling laws for the physical quantities in M.K.S. units on the axis in Tokamak type devices

\[
\begin{align*}
T_i &= C_\gamma \frac{\eta_{1/3} (1 - \varepsilon)^{1/3} \rho_i T_e^{1/2}}{A^{1/2} \rho_i^{1/2}} (K), \\
C_T &= C_x \frac{C_i}{4 k C_x \chi} \frac{3}{2} (1 - \varepsilon) \frac{4 \pi \rho_i A^{1/4} T_e^{1/2}}{\log A} \approx 4.7 \times 10^8 (K), \\
\tau_e &= \frac{3 m k (T_{i0} + T_{e0})}{2 j_0 E_0} \\
&= C_x \frac{\eta_{1/3} (1 - \varepsilon)^{1/3} \rho_i T_e^{1/2}}{A^{1/2} \rho_i^{1/2}} (sec), \\
C_x &= \frac{3 k C_x \chi^{1/3} C_i^{1/3} T_e^{1/4}}{2 A^{1/4} \rho_i^{1/2} \log A} \approx 1.21 \times 10^{-1}, \\
C_x &= C_x^{1/3} C_i^{1/3} T_e^{1/4} (m^3 T^{-1}) \\
&= \frac{3 m k (2 m \mu k B)^{1/2}}{2 e^2}.
\end{align*}
\]

While the above scaling laws have been derived under the assumption of the plateau model, we can also obtain with ease similar formulae for the banana model using similar methods.

Substituting Eq. (8) into Eqs. (11) and (13) and eliminating \( a \) from both equations, we find the familiar formula

\[
T_{i0} = 1.01 \times 10^{-2} A^{-1/2} \left[ (1 - \varepsilon) \right] B n_0 R_0^2 q J \log A \right]^{1/2} (K)
\]

where \( J \) is the total discharge current, and \( q(>1) \) is the form factor defined by the current distribution over the cross section of the plasma column, and given by \( q J = \pi a^2 j_s \). Another interesting formula which can be obtained from Eqs. (11), (12) and (13) is

\[
\tau_{e0} = 2.54 \times 10^4 \eta_{1/3} (1 - \varepsilon) A T_{e0}^{1/2} / e^{2} \log A (sec).
\]

Thus we might expect that the Tokamak plasma is located close to a straight line with a gradient of 2/3 in Lawson’s diagram (\( \log T_e \), \( \varepsilon \), \( \log n_e \)), and this is a fact which can be seen roughly satisfied on the diagram.

In these scaling laws, it is worth remarking that the plasma density cannot be chosen arbitrarily for the experiments but must satisfy Eq. (13) with \( 0.75 > \varepsilon > 0.33 \). In other words, it may be possible that, when for example experiments are done with a very small value of \( n \), then the value of \( \varepsilon \) given
by Eq. (13), or equivalently, by

\[ \frac{e^{5/14}}{(1-\varepsilon)^{5/7}} = 2.92 \times 10^{-15} \frac{\eta^{n/7}(\log A)^{1/7}a^{1/7}R^{2/7}B^{1/7}n_0}{\eta^{3/7}A^{1/3}q_{i1}^{2/7}J^{1/7}} \]  

(17)

will become smaller than 0.33, and the plasma column would become unstable, to give values of \( \tau_{E0} \) smaller than given by Eq. (12).

We next consider the time scale for operating the device. In order to ensure quasi-stability of the plasma column during the build-up of discharge current without much loss of input energy from violent instabilities, it may be safe always to maintain the plasma column under conditions not far removed from thermal equilibrium as defined by Eqs. (11), (12) and (13). Of course it is not the unique solution for device operation but it is still one of such solutions that provide optimum plasma parameters \( T_i \) and \( n_0 \). To obtain the build-up time for the discharge, we replace \( f_{\lambda} \) in Eq. (1) by \( f_{\lambda}'(\lambda-1) \) and \( \dot{Q}_{ie} \) in Eq. (2) by \( \eta^*Q_{ie}(\eta^*<1) \), considering that the parts, \( (1-\eta^*)j_{E_0} \) and \( (1-\eta^*)Q_{ie} \), represent the heating-up of electrons and ions respectively. Then we have, instead of Eq. (17),

\[ \frac{e^{5/14}}{(1-\varepsilon)^{5/7}} = 2.92 \times 10^{-15} \frac{\eta^{n/7}(\log A)^{1/7}a^{1/7}R^{2/7}B^{1/7}n_0}{\eta^{3/7}A^{1/3}q_{i1}^{2/7}J^{1/7}} \]  

(18)

In order to avoid \( \varepsilon>0.75 \) (or \( \varepsilon<0.33 \)), we propose to control the plasma density as function of time, with the toroidal field remaining constant, so that,

\[ n_0(t)/J(t)^{5/7} = \text{nearly const.} \]  

(19)

or to control the toroidal field, with the plasma density remaining constant, so that,

\[ J(t)^{5/7}B(t) = \text{nearly const.} \]  

(20)

In the former case \( T_i \) and \( \tau_E \) vary as \( J(t)^{4/7} \) and \( J(t)^{1/7} \) respectively. And in the latter case we have

\[ (3/2)nkdT_i/dt = (1-\eta^*)Q_{ie}, \]  

(21)

and therefrom,

\[ T_i(t) = T_{i, \text{max.}}(t/t_0)^{3/8} \]  

(22)

\[ \tau_E(t) = \tau_{E, \text{max.}}(t/t_0)^{3/8} \]  

(23)

where \( T_{i, \text{max.}} \) and \( \tau_{E, \text{max.}} \) are the ion temperature and energy confinement time given by Eqs. (11) and (12) respectively and

\[ t_0 = 1.84 \times 10^{18} \frac{\eta^{3/4}A^{11/4}a^{1/7}B^{2/7}}{(1-\eta^*)^{2/7}(1-\varepsilon)^{11/4}(\log A)^{6/7}n_0} \]  

(24)

With use made of Eqs. (11), (12), (13) and (24) we can now design a Tokamak type device on theoretical basis. At present it may be difficult to envisage the validity of these scaling laws, since we have few data from experiments in the regime of 0.75>\( \varepsilon \)>0.33, and such data can only be expected to become available in the future. However since Eqs. (15) and (16) have been shown to agree with experimental results to some extent and since, as far as concerns the assumptions, adopted in their derivation, Eqs. (11), (12) and (13) are equivalent to these two equations, the present scaling laws should be of the same validity as Eqs. (15) and (16). In Table 1, the measured values of plasma parameters in Tokamak T-3 are compared with those calculated with our present scaling laws. The agreement seen between the two values are very satisfactory, when it is considered that the calculated values relate to the values applicable to the axis while the measured values are averaged over the entire cross section of the column, which depend upon their radial profiles and are smaller by a factor of something

\begin{table}[h]
\centering
\caption{Comparison of plasma parameters in Tokamak T-3 between measurements and calculations based on scaling laws}
\begin{tabular}{|c|c|c|}
\hline
Device parameters & Measured & Used for calculations\
\hline
Major radius \( R \) (m) & 1.0 & 1.0 & 1.0 \\
Toroidal field \( B \) (Wb/m\(^2\)) & 2.5~3.6 & 2.5 & 3.6 \\
Plasma radius \( a \) (m) & 0.12~0.15 & 0.12 & 0.15 \\
Safety factor & \( q_0 \) at \( r=a \) & 3.6~4.0 & 1.5 \\
& \( q_0 \) at \( r=0 \) & 1.5 \\
& \( q_0 \) at \( r=0 \) & \( q=5/2 \) & \( q=5/2 \) \\\n\hline
Characteristic time & \( t_0 \) (msec) & ... & 3.5 & 6.1 \\
\hline
\end{tabular}
\end{table}

\[ T_e (K) \]  

(1.6~10\(^7\)) 7.9~10\(^6\) 1.23~10\(^7\)

\[ T_i (K) \]  

(3.9~10\(^6\)) 6.2~10\(^6\)

\[ \tau_E \text{ (msec)} \]  

20~25 28 46

\[ n \text{ (m\(^{-3}\))} \]  

(3.8~10\(^{19}\)) 

\[ \tau_E \text{ (msec)} \]  

1.04~10\(^{18}\) 1.96~10\(^{18}\)

\[ J \text{ (kA)} \]  

60~120 48 108

\( \dagger \) is not strictly speaking a device parameter but a plasma quantity which however is practically determined by aperture of current limiter of the device.

\( \ddagger \) under the assumptions \( \eta=1, \varepsilon=0.5, \log A=15 \) and \( A=1 \)

\( \S \) under the assumptions \( \eta^*=\eta^*=0.5 \)
SHORT NOTE

Compression and Decompression Waves in Steam-Water System

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Received May 25, 1972

KEYWORDS: steam-water system, rupturing diaphragm, propagation velocity, transient response, compression wave, decompression wave, condensation shock, flashing, void fraction, saturation pressure

1. Introduction

No essential difference between compression and decompression waves has been discovered in experiments on two-component two-phase media performed by the authors(1,2) using air-water and nitrogen-mercury mixture. In case of one-component two-phase media, however, the propagation characteristics of pressure waves are expected to present significant nonlinearity due to mass transfer between the two phases. Data now available are still limited, and much more information is required on the subject. The results obtained in the present experiment should contribute useful knowledge in analysing the phenomena involved in sodium boiling, which is one of the most important problems at issue in fast reactor safety design.

2. Experimental Apparatus and Procedure

A diagram of the experimental setup is shown in Fig. 1. The test section is about 2m long, with a 40×40 mm² cross section, made of stainless steel. Visual observation of the fluid flowing inside is possible through the two parallel walls of Pyrex glass. The apparatus is of the general same design as that used for the air-water experiment(1) except that the gas void injector has been replaced by a 3 kW electric pipe heater and that a circulation loop has been installed for mixing the water to obtain uniform temperature. The void fractions were measured by level difference technique and the temperature by mercury thermometers. Three strain gage pressure transducers of water cooled type for engine use were employed as detectors of transient pressure. The frequency characteristics of the overall instrumentation system for pressure were restricted to about 1 kHz with use made of a dynamic strain meter of A.C. bridge type.

The fluid was heated under pressure produced by nitrogen cover gas and the generation of steam void was controlled by a needle valve on the gas leak line. Step-up or -down pressure disturbances were applied by rupturing Lumnirror (Mylar) dia-

* This work was partially supported by Grant-in-Aid for Scientific Research from Ministry of Education.
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