M+1-OUT-OF-N:G SYSTEM WITH CORRELATED FAILURE AND SINGLE REPAIR FACILITY

Fumio Ohi and Toshio Nishida, Osaka University

and

Masanori Kodama, Nagoya Institute of Technology

(Received April 26, 1977; Revised August 29, 1977)

ABSTRACT

This paper deals with the M + 1 - out-of-N:G system with correlated failure and single repair facility, where correlated failure means that i (1 ≤ i ≤ N) operating units in the system are possible to fail at the same time. We consider two repair policies, and under each repair policy Laplace transform of point-wise availability and reliability, meantime to the first system failure and stationary availability are derived. Finally some properties of stationary availability for each repair policy are given.

1. Introduction.

In this paper the simultaneous failure of i operating units may be described as SF_i.

Several authors have studied the system with correlated failure. R. HARRIS [1] has studied the two units system with bivariate exponential failure distribution which is defined by A.W.MARSHALL and I.OLKIN [5]. T.ITOI, T. MURAKAMI, M.KODAMA and T.NISHIDA [2] have defined the bivariate Erlang distribution and applied it to the system reliability analysis. About the N-unit system with correlated failure T.ITOI, T.NISHIDA, M.KODAMA and F.OHI [3] have studied, but the studied system is N-unit parallel redundant system and are considered only SF_2 and simultaneous failure of all the operating units. In this paper we deal with the M+1-out-of-N:G system with SF_i (2 ≤ i ≤ N) and single repair facility.
We consider two repair policies. One of which is that the failed unit
is repaired as soon as the unit fails and after completion of repair the unit
becomes to the operating unit, and the other is that the repair is begun after
N-M units fail and after completion of all repairs system begins to operate.
We call the model under the former repair policy as Model 1 and the one
under the latter repair policy as Model 2.

For both models we derive Laplace transforms (L-T) of point-wise avail-
ability and reliability, mean time to the system failure (MTSF) and station-
ary availability. Finally we show that stationary availability of Model 2 is
decreasing in M and N, and tends to zero as N \rightarrow \infty for any fixed M (0 \leq M \leq N-1) when there is no correlated failure.

2. Definition of Models and Notations

The system consists of N units and initially all units are operating. In
the system there are \( q_i = \binom{N}{i} \) groups of size \( i \) (1 \leq i \leq N) which are sym-
bolized as \( G_{i,1}, \ldots, G_{i,q_i} \). Poisson process with parameter \( \lambda_{i,j} \) governs the occur-
rence of shocks to \( G_{i,j} \). These Poisson processes are mutually independent. When
the shocks to \( G_{i,j} \) occurs, the operating units in \( G_{i,j} \) fail. In this paper we
assume that \( \lambda_{i,j} = \lambda_i \) (\( j = 1, \ldots, q_i \)). Then the rate at which occurs the shock
to group of size \( i \) is \( \binom{N}{i} \lambda_i \). The system is considered good when at least M+1 units are operating. When the system is down the residual operating units do
not fail. There is one repair man and distribution of repair time to a failed
unit is general.

Through this paper we use the following notations;

\[
\begin{align*}
\{i\} & \quad \text{state that} \ i \ \text{units are operating} \\
P_\iota(t) & \quad \text{Pr[ the system is in state} \ \{i\} \ \text{at time } t \ \text{]} \ (0 \leq \iota \leq N) \\
P_\iota(t, x) dx & \quad \text{Pr[ the system is in state} \ \{i\} \ \text{at time } t \ \text{and elapsed repair} \\
& \quad \text{time of unit under repair lies between } x \ \text{and} \ x+dx \ ] \\
& \quad (0 \leq \iota \leq N-1 \ \text{for Model 1,} \ 0 \leq \iota \leq M \ \text{for Model 2}) \\
P_\iota(t) & = \int_0^\infty P_\iota(t, x) dx \\
P_\iota(x) dx & \quad \text{Pr[ in equilibrium state the system is in state} \ \{i\} \ \text{and elapsed} \\
& \quad \text{repair time of unit under repair lies between } x \ \text{and} \ x+dx \ ] \\
& \quad (0 \leq \iota \leq N-1 \ \text{for Model 1,} \ 0 \leq \iota \leq M \ \text{for Model 2}) \\
\int_0^\infty P_\iota(x) dx & = \lim_{t \to \infty} P_\iota(t) \ (0 \leq \iota \leq N-1) \\
\lim_{t \to \infty} P_N(t) & \quad \text{point-wise availability of Model} \ j \ (j = 1, 2) \\
\lim_{t \to \infty} P_A(t) & \quad \text{stationary availability of Model} \ j \ (j = 1, 2) \\
\end{align*}
\]

NII-Electronic Library Service
The Operations Research Society of Japan

NII-Electronic Library Service

98

EOhi, M. Kodama and T. Nishida

F. Ohi, M. Kodama and T. Nishida

system reliability of Model $j$ ($j = 1, 2$)

$R^{(j)}(t)$

mean time to the first system failure of Model $j$ ($j = 1, 2$)

$\text{MTSF}^{(j)}(t)$

repair time density of Model $j$ ($j = 1, 2$)

$g_j(t)$

repair rate, i.e., $g_j(t) / \int_0^\infty g_j(x)dx$ ($j = 1, 2$)

$1/\mu_j$

repair rate, i.e., $\int_0^\infty xg_j(x)dx$ ($j = 1, 2$)

$\lambda_j / \mu_j$ ($j = 1, 2$)

$g_j^k(t)$

the $(N-M)$-th convolution of $g_j(t)$

$h_j^k(t)$

$\int_0^\infty s^k g_j^k(s)dx$

$f(s)$

Laplace transform of $f(t)$, i.e., $\hat{f}(s) = \int_0^\infty e^{-st}f(t)dt$

$\delta_{ij}$

Kronecker's delta


3.1. Analysis for $0 \leq M \leq N-2$

Viewing the nature of this system, the following set of integro-differential equations can be set up easily:

\[
\frac{d}{dt} + \sum_{j=0}^{N-1} \lambda_{N-j} P^{(j)}(t) = \int_0^t P_{N-1}(t,x)h_1(x)dx,
\]

(3.1.1)

\[
\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \sum_{j=0}^{N-1} \sum_{n=0}^{N-j} \sum_{\xi=1}^{N-\xi} \lambda_{\xi-j+n} P_{j-\xi}(t,x) = (1-\delta_{i,N-1}) \sum_{j=0}^{N-j} \sum_{n=0}^{N-j} \sum_{\xi=1}^{N-\xi} \lambda_{\xi-j+n} P_{j-\xi}(t,x),
\]

(3.1.2)

\[
\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + h_1(x) P_i(t,x) = \sum_{j=0}^{N-j} \sum_{n=0}^{N-j} \sum_{\xi=1}^{N-\xi} \lambda_{\xi-j+n} P_{j-\xi}(t,x),
\]

(3.1.3)

Equations (3.1.1)-(3.1.3) are to be solved under the following boundary and initial conditions:

\[
P_i(t,0) = \left(\sum_{N-i}^{N} \lambda_{N-i} P_{N-i}^{(j)}(t) + (1-\delta_{i,N}) \int_0^t P_{N-i}(t,x)h_1(x)dx\right) (0 \leq i \leq N-1),
\]

(3.1.4)

\[
P_{i}(0) = \delta_{i,N} (0 \leq i \leq N).
\]

(3.1.5)

Taking Laplace transform of (3.1.1)-(3.1.4) under the initial conditions (3.1.5), we have:

\[
\left[s + \sum_{j=0}^{N-j} \lambda_{N-j} \right] \hat{P}_{N}(s) = 1 + \int_0^\infty P_{N-1}(s,x)h_1(x)dx,
\]

(3.1.6)
\[ M + 1 \text{- out-of-N:G System with Correlation} \]

\[ \frac{d}{dx} \left[ s + \sum_{j=0}^{N-1} \frac{i^j}{j!} \right] \hat{P}_G(s,x) = \int_0^\infty P_{G-1}(s,x) h_1(x) dx \quad (0 \leq \xi \leq M), \]

\[ \hat{P}_G(s,0) = \left( \frac{N}{N-\xi} \right)^{\frac{1}{\chi}} P_N(s) + (1-\delta_{\xi,0}) \int_0^\infty \hat{P}_{G-1}(s,x) h_1(x) dx \quad (0 \leq \xi \leq M). \]

Using the discrete transform [4], i.e.,

\[ \hat{u}_G(s,x) = \sum_{j=0}^{N-1} \frac{(-1)^j}{j!} \hat{u}_G(s,x) \quad (M+1 \leq j \leq N-1), \]

\[ \hat{u}_G(s,0) = \left[ g_1(s+\beta_1) g_{G-1}(s+\beta_{G-1}) \right] \}

we can rewrite the equations (3.1.7) and (3.1.9) for \( M+2 \leq \xi \leq N-1 \) as follows. (See appendix 1 and 2 respectively.)

\[ \frac{d}{dx} \left[ s + \sum_{j=0}^{N-1} \frac{i^j}{j!} \right] \hat{u}_G(s,x) = 0 \quad (M+1 \leq j \leq N-1), \]

\[ \hat{u}_G(s,0) = \left[ g_1(s+\beta_1) g_{G-1}(s+\beta_{G-1}) \right] \}

Consequently, by solving (3.1.6), (3.1.8), (3.1.9), (3.1.11), (3.1.12) and (3.1.13), we have after simple but tedious calculation:

\[ \hat{P}_G(s) = \int_0^\infty P_{G-1}(s,x) dx = \sum_{j=0}^{N-1} \frac{(-1)^j}{j!} \hat{u}_G(s,0) \{ 1-\gamma(s+\beta_j) \} / (s+\beta_j) \quad (M+1 \leq j \leq N-1), \]

\[ \hat{P}_G(s) = \{ 1+\hat{u}_{G-1}(s,0) \} g(s+\beta_{G-1}) / (s+\beta_N), \]

where

\[ \hat{u}_G(s,0) = \hat{a}_G(s) \hat{u}_{G+1}(s,0) + \hat{b}_G(s) \quad (M+1 \leq j \leq N-1), \]

\[ \hat{a}_G(s) = \left[ g_1(s+\beta_{G+1}) \sum_{n=0}^{N} \frac{k_{n-1,M}(s)}{n!} \{ (s+\beta_n)^{N-n} - a_n \} \right] / \left[ \sum_{n=0}^{N} \sum_{n=M+2}^N \frac{k_{n-1,M}(s)}{n!} \{ (s+\beta_n)^{N-n} - a_n \} \right] \quad (M+1 \leq j \leq N-1), \]

\[ \hat{b}_G(s) = \left[ \sum_{n=0}^{N} \sum_{n=M+2}^N \frac{k_{n-1,M}(s)}{n!} \{ (s+\beta_n)^{N-n} - a_n \} \right] / \left[ g_1(s+\beta_{G+1}) \cdot k_{G+1,M}(s) \right] \quad (M+1 \leq j \leq N-1), \]

\[ \hat{k}_{G+1,M}(s) = \left\{ \begin{array}{ll} 1 \quad & (j=M+1), \\
                   \end{array} \right. \]

\[ \hat{g}_1(s+\beta_{G+1}) \}

NII-Electronic Library Service
The L-T of point-wise availability $\hat{p}_A^{(1)}(s)$ is obtained as follows from (3.1.12) and (3.1.14):

$$
\hat{p}_A^{(1)}(s) = \left\{ \begin{array}{ll}
\hat{g}_1(s+\beta) \left( \gamma_{1(j)}(s) \right) - \left( \gamma_{2(j)}(s) / \beta \right) \left( \gamma_{2(j)}(s) \right) M^{1-j} B(i, j, M) & (0 \leq j \leq M), \\
\hat{g}_1(s+\beta) \left( \gamma_{1(j)}(s) \right) - \left( \gamma_{2(j)}(s) / \beta \right) \left( \gamma_{2(j)}(s) \right) M^{1-j} B(i, j, M) & (M+1 \leq j \leq N-1), \\
\end{array} \right.
$$

and

$\hat{p}_A^{(1)}(s) = \left\{ \begin{array}{ll}
\hat{g}_1(s+\beta) \left( \gamma_{1(j)}(s) \right) - \left( \gamma_{2(j)}(s) / \beta \right) \left( \gamma_{2(j)}(s) \right) M^{1-j} B(i, j, M) & (0 \leq j \leq M), \\
\hat{g}_1(s+\beta) \left( \gamma_{1(j)}(s) \right) - \left( \gamma_{2(j)}(s) / \beta \right) \left( \gamma_{2(j)}(s) \right) M^{1-j} B(i, j, M) & (M+1 \leq j \leq N-1), \\
\end{array} \right.$

The L-T of R(1)(t) can be obtained from $\hat{p}_A^{(1)}(s)$ by making suitable transformations. Putting $\gamma_{1(j)}(s) = 0$ (M+1 \leq j \leq N-1) and $\gamma_{2(j)}(s) = 0$ (0 \leq j \leq M) in (3.1.16) yields $R_1^{(1)}(s)$ since the substitution is equivalent to the assertion that the probability of the system moving from down state to up state is zero. Moreover, if we set s=0 in $\hat{R}_1^{(1)}(s)$, we obtain the MTSF(1).

In order to derive stationary availability we set up the following set of differential-difference equations:

(3.1.17) $\beta N P_N = \int_0^\infty P_{(N-1)}(x) h_1(x) dx,$

(3.1.18) $\left\{ \begin{array}{ll}
\frac{d}{dx} + \sum_{j=0}^{N-1} \left( j \cdot j+1 \right) \left( \frac{N-1}{n-1} \right) \lambda_{j-1} \right\} P_1(x) = (N-1) \left( j \cdot j+1 \right) \left( \frac{N-1}{n-1} \right) \lambda_{j-1} \right\} P_1(x) = \left\{ \begin{array}{ll}
\hat{b}_1(s+\beta) \left( \gamma_{1(j)}(s) \right) - \left( \gamma_{2(j)}(s) / \beta \right) \left( \gamma_{2(j)}(s) \right) M^{1-j} B(i, j, M) & (0 \leq j \leq M), \\
\hat{b}_1(s+\beta) \left( \gamma_{1(j)}(s) \right) - \left( \gamma_{2(j)}(s) / \beta \right) \left( \gamma_{2(j)}(s) \right) M^{1-j} B(i, j, M) & (M+1 \leq j \leq N-1), \\
\end{array} \right.
$$

and

\[ P_1(x) = \left\{ \begin{array}{ll}
\hat{g}_1(s+\beta) \left( \gamma_{1(j)}(s) \right) - \left( \gamma_{2(j)}(s) / \beta \right) \left( \gamma_{2(j)}(s) \right) M^{1-j} B(i, j, M) & (0 \leq j \leq M), \\
\hat{g}_1(s+\beta) \left( \gamma_{1(j)}(s) \right) - \left( \gamma_{2(j)}(s) / \beta \right) \left( \gamma_{2(j)}(s) \right) M^{1-j} B(i, j, M) & (M+1 \leq j \leq N-1), \\
\end{array} \right.
$$

with the following boundary conditions and normalizing conditions:

(3.1.20) $P_1(0) = \left( \frac{N}{N-j} \right) P_1(1) P_1(0) \int_0^\infty P_1(1)(x) h_1(x) dx$ (0 \leq j \leq N-1),
Thus using the similar discrete transform, stationary availability $p^{(1)}_A$ is given as

$$p^{(1)}_A = \sum_{i=M+1}^{N-1} \frac{(-1)^{i-M-1} \hat{a}_i(0) (1-\hat{g}(\beta_i)) / \beta_i + \hat{a}_{N-1}(0) \hat{g}(\beta_{N-1}) / \beta_{N-1}}{(-1)^{i-M-1} \hat{a}_i(0) (1-\hat{g}(\beta_i)) / \beta_i + \hat{a}_{N-1}(0) \hat{g}(\beta_{N-1}) / \beta_{N-1}},$$

where

$$u_{M+1} = \frac{\beta_N}{\sum_{m=M+1}^{N-1} (-1)^{m-1} a_m(0) (1-\hat{g}(\beta_m)) / \beta_m + \hat{a}_{N-1}(0) \hat{g}(\beta_{N-1}) / \beta_{N-1}}.$$

### 3.2. Analysis for $M=N-1$

In this case the equations describing the behavior of the system are (3.1.6), (3.1.8) and (3.1.9), and equations (3.1.7) are cut. These equations can be easily solved and we have

$$p^{(1)}_A(s) = \frac{1}{[s+\beta_N]} - \sum_{j=0}^{N-1} \lambda_{N-j} \hat{g}_1(s)^{N-j},$$

$$\hat{g}_1(s) = \frac{1}{[s+\beta_N]},$$

$$MTSF = 1/\beta_N.$$ 

And stationary availability can be obtained easily as

$$p^{(1)}_A = \mu_1 + \sum_{j=0}^{N-1} \lambda_{N-j} \hat{g}_1(s)^{N-j}.$$

### 3.3. Properties of $p^{(1)}_A$ when there is no correlated failure

We assume that $M=0$ and for convenience we use the notation $p^{(1)}_A(N)$ in place of $p^{(1)}_A$.

If $M=0$ and $\lambda_j = 0$ for $2 \leq j \leq N$, $p^{(1)}_A(N)$ given by (3.1.22) is that

$$p^{(1)}_A(N) = \sum_{n=1}^{N-1} \sum_{j=1}^{n} \frac{(-1)^{j-1} \hat{g}_1(\beta_j) \hat{k}_n(0) + \{1/N\}}{[n \sum_{j=1}^{N-1} \hat{g}_1(\beta_j) \hat{k}_n(0) + \{1/N\}}.$$ 

Intuitively we conceive that $p^{(1)}_A(N)$ is increasing in $N$ for any repair time distribution. But the following example shows that this is not true. Then the problem to determine the class of repair time distribution to assure that $p^{(1)}_A(N)$ is increasing in $N$ will arise, which remains open.

**Example.** Noticing that we may understand $\hat{g}_1(s)$ as the Laplace-Stieltjes
transform of repair time distribution $G_1(t)$, we consider the $G_1$ described in Fig.1. In this case with that $\rho_1=1$, it is easily shown that $P(1)_{A}^{(1)}(3) - P(1)_{A}^{(1)}(2)$ is negative. If $g_1(t)=e^{-t}$, $P(1)_{A}^{(1)}(N)=[\sum_{j=1}^{N} \frac{1}{j!} \frac{1}{(1/\rho_1)^j} \cdot \frac{1}{(1/\rho_1)^j}].$ Then $P(1)_{A}^{(1)}(N) + 1-e^{-1/\rho_1}$ ($N \to \infty$).

4. Model 2

4.1. Analysis

When $M=N-1$ the results of this model are coincident with (3.2.1), (3.2.2), (3.2.3) and (3.2.4) evidently. In the case that $0<M<N-2$ the following set of integro-differential equations can be set up easily:

\[
\begin{align*}
\left[ \frac{d}{dt} + \sum_{j=0}^{N-1} \frac{1}{(N-j)j!} \right] P_{N}(t) &= \int_{0}^{t} P_{M}(t,x) h_{2}^{*}(x) dx \\
\left[ \frac{d}{dt} + \sum_{j=0}^{N-1} \frac{1}{(N-j)j!} \right] P_{N}(t) &= \sum_{i=1}^{N} \frac{1}{i!} \sum_{j=0}^{N-1} \frac{1}{(N-j)!} \lambda_i \delta_{i,j} P_{j}(t) + \int_{0}^{t} P_{-1}(t,x) h_{2}(x) dx \\
\frac{\partial}{\partial x} P_{M}(t,x) + h_{2}(x) P_{M}(t,x) &= 0 \quad (0 \leq t \leq M), \\
\frac{\partial}{\partial x} P_{-1}(t,x) + h_{2}(x) P_{-1}(t,x) &= 0 \quad (0 \leq t \leq M).
\end{align*}
\]

Equations (4.1.1)\textsuperscript{-}(4.1.4) are to be solved under the following boundary and initial conditions:

\[
\begin{align*}
P_{-1}(t,0) &= \sum_{j=0}^{N} \frac{1}{j!} \delta_{i,j} \sum_{n=0}^{N-1} \frac{1}{(N-j)!} \lambda_i \delta_{i,n} P_{j}(t) + (1-\delta_{i,0}) \int_{0}^{t} P_{-1}(t,x) h_{2}(x) dx \\
0 \leq t \leq M, \\
P_{-1}(0) &= \delta_{i,N} \quad (0 \leq t \leq N).
\end{align*}
\]

Taking the L-T of equations (4.1.1)\textsuperscript{-}(4.1.5) under the initial conditions (4.1.6) and using the discrete transform

\[
\begin{align*}
\hat{u}(s) &= \sum_{j=0}^{N-1} \frac{1}{j!} \hat{P}_{j}^{(i)}(s) \quad (M+1 \leq N), \\
\hat{P}_{j}^{(i)}(s) &= \sum_{i=0}^{N-1} (-1)^{i} \hat{\delta}_{i,j} \hat{u}_{N}^{(i)}(s) \quad (M+1 \leq N-1),
\end{align*}
\]

we have

\[
\begin{align*}
[s+\beta_{N}^{1}] \hat{P}_{N}(s) &= 1+\int_{0}^{t} P_{M}(t,x) h_{2}^{*}(x) dx, \\
\end{align*}
\]
\[ (4.1.10) \quad [s^{+\beta_j}] \hat{u}_j(s) = \alpha_j \hat{P}_N(s) \quad (M+1\leq j \leq N-1), \]
\[ (4.1.11) \quad [s^{+\frac{d}{dx}} + h_2(x)] \hat{P}_M(s,x) = 0, \]
\[ (4.1.12) \quad [s^{+\frac{d}{dx}} + h_2(x)] \hat{P}_c(s,x) = 0 \quad (0 \leq i \leq M-1), \]
\[ (4.1.13) \quad \hat{P}_i(s,0) = \sum_{j=N+1}^{N-1} \sum_{n=0}^{N-j} \lambda_j^{\hat{u}_j(s)} \lambda_j^{\hat{p}_j(s)} + (1-\delta_{i,0}) \int_0^\infty \hat{P}_{i-1}(s,x) h_2(x) dx \quad (0 \leq i \leq M). \]

These equations (4.1.8)-(4.1.13) are easily solved and we obtain
\[ (4.1.14) \quad \hat{P}_A^N(s) = [1 + \sum_{i=M+1}^{N-1} (-1)^{M+1+i} \beta_j^{\hat{u}_j(s)} \gamma_{j,i}(s)] / \sum_{i=M+1}^{N-1} (-1)^{M+1+i} \beta_j^{\hat{u}_j(s)} \gamma_{j,i}(s) \]
\[ \hat{P}_A^N(s) = \hat{P}_A^{N-1}(s) \gamma_{N-1}(s). \]

Putting \( \gamma_{N-1}(s) = 0 \) in (4.1.14), we obtain
\[ (4.1.15) \quad \hat{R}_A^N(s) = [1 + \sum_{i=M+1}^{N-1} (-1)^{M+1+i} \beta_j^{\hat{u}_j(s)} \gamma_{j,i}(s)] / \sum_{i=M+1}^{N-1} (-1)^{M+1+i} \beta_j^{\hat{u}_j(s)} \gamma_{j,i}(s). \]

Taking the inverse transform of \( \hat{R}_A^N(s) \), we obtain
\[ (4.1.16) \quad R_A^N(t) = \sum_{i=M+1}^{N-1} (-1)^{M+1+i} \beta_j^{\hat{u}_j(s)} \gamma_{j,i}(s) / \sum_{i=M+1}^{N-1} (-1)^{M+1+i} \beta_j^{\hat{u}_j(s)} \gamma_{j,i}(s) \]
\[ R_A^{N-1}(t) \gamma_{N-1}(t). \]

In order to derive stationary availability we set up the following set of differential-difference equations:
\[ \beta_j^{\hat{u}_j(s)} = \int_0^\infty \hat{P}_M(x) h_2(x) dx, \]
\[ \hat{P}_j(s) = \sum_{i=0}^{N-1-\delta_j} \lambda_j^{\hat{u}_j(s)} \hat{P}_j(s) + (1-\delta_{i,0}) \int_0^\infty \hat{P}_{i-1}(s,x) h_2(x) dx \quad (0 \leq i \leq M), \]
\[ \hat{P}_{i+1}(s) = \sum_{i=0}^{N-1-\delta_j} \lambda_j^{\hat{u}_j(s)} \hat{P}_j(s) + (1-\delta_{i,0}) \int_0^\infty \hat{P}_{i-1}(s,x) h_2(x) dx \quad (0 \leq i \leq M). \]

with the following boundary conditions and normalizing condition:
\[ P(0) = \sum_{j=0}^{N-1} \lambda_j^{\hat{u}_j(s)} P_j(s) + (1-\delta_{i,0}) \int_0^\infty P_{i-1}(s,x) h_2(x) dx \quad (0 \leq i \leq M), \]
\[ N \sum_{j=0}^{N-1} \lambda_j^{\hat{u}_j(s)} P_j(s) = 1. \]

Thus using the similar discrete transform, we obtain
\[ (4.1.18) \quad \hat{P}_A^N(s) = \mu_2 [1 + \sum_{i=0}^{N-1} (-1)^{M+1+i} \beta_j^{\hat{u}_j(s)}] / \sum_{i=0}^{N-1} (-1)^{M+1+i} \beta_j^{\hat{u}_j(s)} \gamma_{j,i}(0) + (N-M) \beta_j^{\hat{u}_j(s)} \mu_2 \]
\[ + \mu_2 \sum_{i=0}^{N-1} (-1)^{M+1+i} \beta_j^{\hat{u}_j(s)} \gamma_{j,i}(0) \]
\[ \hat{P}_A^N(s) = \hat{P}_A^{N-1}(s) \gamma_{N-1}(s), \]
where
The Operations Research Society of Japan

104

F. Ohi, M. Kodama and T. Nishida

\[
\Gamma_{i,j}(s) = \sum_{j=M+1}^{N-1} \frac{(-1)^j}{j!} \sum_{i=0}^{N-j} \frac{\lambda_{j+i}}{\lambda_{j+i+n}} \sum_{n=0}^{N-j} \frac{\lambda_{N+i}}{\lambda_{N+i-n}} \quad (0 \leq i \leq M) .
\]

Since \( \sum_{m=1}^{M} = 0 \) \((j \neq i)\), the results \((4.1.14), (4.1.15), (4.1.16), (4.1.17)\) and \((4.1.18)\) are valid for \( M = N - 1 \).

4.2. Properties of \( P_{A}^{(2)} \) when there is no correlated failure

In this section we show that \( P_{A}^{(2)} \) given by \((4.1.18)\) is decreasing in \( M \) and \( N \) respectively, and tends to zero \( N \to 0 \) for any fixed \( 0 \leq M \leq N - 1 \) when there is no correlated failure, i.e., \( \lambda_{\cdot} = 0 \) \((2 \leq i \leq N)\).

For convenience we use the notation \( P_{A}^{(2)}(N,M) \) in place of \( P_{A}^{(2)} \) throughout this section.

When \( \lambda_{\cdot} = 0 \) \((2 \leq i \leq N)\), \( P_{A}^{(2)}(N,M) \) given by \((4.1.18)\) is that

\[
P_{A}^{(2)}(N,M) = \sum_{j=M+1}^{N} \left( \frac{1}{j} \right) \left( \frac{1}{j} \right) \frac{1}{(N-M)j+1} \sum_{j=M+1}^{N} \left( \frac{1}{j} \right) \left( \frac{1}{j} \right) \frac{1}{(N-M)j+1} \quad (0 \leq i \leq N) .
\]

Theorem 4.2.1. (i) \( P_{A}^{(2)}(N,M) \) is increasing in \( M \) \((0 \leq M \leq N - 1)\) for any fixed \( N \geq 1 \).

(ii) \( P_{A}^{(2)}(N,M) \) is decreasing in \( N \) for any fixed \( M \geq 0 \), where \( N > M \).

(iii) \( P_{A}^{(2)}(N,M) \to 0 \) as \( N \to \infty \) for any fixed \( M \geq 0 \).

Proof: (i) and (ii) are easily proved.

(iii) We may assume \( N > M \).

\[
(4.2.1) \quad \left[ \frac{1}{(N-M)} \right] \sum_{j=M+1}^{N} \left( \frac{1}{j} \right) \to 0 \quad \text{as} \quad N \to \infty .
\]

Since

\[
\left[ \frac{1}{(N-M)} \right] \sum_{j=M+1}^{N} \left( \frac{1}{j} \right) < \left[ \frac{1}{(N-M)} \right] \int_{M+1}^{N} \left( 1/x \right) dx = [\log(N/(M+1))] / (N-M) ,
\]

\[
[\log(N/(M+1))] / (N-M) \to 0 \quad \text{as} \quad N \to \infty ,
\]

and

\[
1 / [(N-M)(M+1)] \to 0 \quad \text{as} \quad N \to \infty .
\]

Then \((4.2.1)\) is valid.

5. Concluding Remarks.

We studied two models and derived several measures for each model. Some properties of stationary availability were discussed in uncorrelated failure case.

The repair policy of Model 1 has been discussed by many authors. In the practical case, however, it is more troublesome and expensive than that of
Model 2. When \( g_1 = g_2 \) it may be conceived that \( P_A^{(1)} \geq P_A^{(2)} \), but F.OHI, M.KODAMA and T.NISHIDA [6] shows that it does not necessarily hold. On the other hand, it is easily shown from (3.1.22) and (4.1.18) that \( P_A^{(2)} \geq P_A^{(1)} \) for sufficiently large \( \mu_2 \) when \( g_1 \neq g_2 \). Thus if \( \mu_2 \) is sufficiently large we had better used the repair policy of Model 2. But we must notice that from theorem 4.2.1. as increases the number of units, \( \mu_2 \) must be increased to assure that \( P_A^{(2)} \geq P_A^{(1)} \).

Acknowledgments

The authors are grateful to the referees for their valuable comments.

References


Fumio OHI and Toshio NISHIDA: Department of Applied Physics, Faculty of Engineering, Osaka University, Yamada-Kami, Suita, Osaka, 565, Japan.
Masanori KODAMA: Department of Fine Measurement, Faculty of Engineering, Nagoya Institute of Technology, Gokiso-Cho, Showa-Ku, Nagoya, 466, Japan.
Appendix 1.

If \( b_{ij} = \sum_{i=0}^{N-1} \binom{n}{i} a_i \) (1 \( \leq j \leq N-1 \))

then

\[
\begin{align*}
\text{(1)} & \quad - \sum_{i=j}^{N-1} \binom{n}{i} a_i \sum_{m=0}^{N-2} \binom{n}{m} \binom{N-2}{m} - \sum_{i=j}^{N-1} \binom{n}{i} a_i \sum_{m=0}^{N-1} \binom{n}{m} \binom{N-1}{m} \\
& \quad + \sum_{i=j}^{N-1} \binom{n}{i} a_i \sum_{m=0}^{N-1} \binom{n}{m} \binom{N-1}{m} \\
& \quad - \beta_j b_{ij}.
\end{align*}
\]

Proof: (the first step)

\[
\begin{align*}
\text{(2)} & \quad \sum_{i=j}^{N-1} \binom{n}{i} a_i \sum_{m=0}^{N-1} \binom{n}{m} \binom{N-1}{m} - \sum_{i=j}^{N-1} \binom{n}{i} a_i \sum_{m=0}^{N-1} \binom{n}{m} \binom{N-1}{m} \\
& \quad + \sum_{i=j}^{N-1} \binom{n}{i} a_i \sum_{m=0}^{N-1} \binom{n}{m} \binom{N-1}{m}.
\end{align*}
\]

Then

\[
\begin{align*}
\text{(3)} & \quad \sum_{i=j}^{N-1} \binom{n}{i} a_i \sum_{m=0}^{N-1} \binom{n}{m} \binom{N-1}{m} - \sum_{i=j}^{N-1} \binom{n}{i} a_i \sum_{m=0}^{N-1} \binom{n}{m} \binom{N-1}{m} \\
& \quad + \sum_{i=j}^{N-1} \binom{n}{i} a_i \sum_{m=0}^{N-1} \binom{n}{m} \binom{N-1}{m}.
\end{align*}
\]

Noticing that

\[
\begin{align*}
\sum_{m=0}^{N-1} \binom{n}{m} \frac{\binom{n}{m}}{\binom{N-1}{m}} &= \frac{N}{n} - \frac{N-\binom{n}{m}}{\binom{N-1}{m}},
\end{align*}
\]

the first term of the left hand side of (3) is

\[
\sum_{n=0}^{N-1} \binom{n}{m} \frac{\binom{n}{m}}{\binom{N-1}{m}}.
\]

the bracketed passage of the left hand side of (3)

\[
\begin{align*}
& \quad = \lambda_1 \sum_{i=j}^{N-1} \binom{n}{i} a_i \sum_{m=0}^{N-1} \binom{n}{m} \binom{N-1}{m} \\
& \quad + \lambda_{N-j} \sum_{i=j}^{N-1} \binom{n}{i} a_i \sum_{m=0}^{N-1} \binom{n}{m} \binom{N-1}{m}.
\end{align*}
\]
\[ M+1 \text{- out-of-} N \text{-} G \text{ System with Correlation} \]

\[ + \sum_{m=N-n+1}^{N-1} \binom{N-n}{j} \binom{m}{N-1} \alpha_{m} \]

\[ = -\lambda_{1}^{j} b_{j} + \lambda_{N-j}^{j} [1-\binom{N}{n}] \beta_{j} + \sum_{n=2}^{N-j-1} \binom{N-j-1}{n} \beta_{n} \binom{N-j}{n} \lambda_{n} \beta_{j} \]

Then

the left hand side of (1)

\[ = -\left[ i \lambda_{1} + \sum_{n=2}^{N} \binom{N}{n} \beta_{j} \right] \binom{N-j}{n} \lambda_{j} \beta_{j} \]

\[ = -\left[ \sum_{n=1}^{N} \binom{N}{n} - \binom{N-j}{n} \right] \lambda_{j} \beta_{j} \]

\[ = -\beta_{j} \beta_{j} \]

Appendix 2.

For \( M+2 \leq i \leq N-1 \) using (3.1.9),

\[ u_{i}(s,0) = \sum_{j=i}^{N-1} \binom{N}{j} P_{j}(s,0) \]

\[ = a_{i}^{N}(s) \int_{0}^{x} u_{i}(s,x) h_{1}(x) dx + \int_{0}^{x} u_{i-1}(s,x) h_{1}(x) dx - \left[ \int_{0}^{x} u_{i-1}(s,x) h_{1}(x) dx \right] \]

Substituting (3.1.15) and

\[ u_{i}(s,x) = u_{i}(s,0) e^{-r(x-s)} \int_{0}^{x} h_{1}(x) dx \]

resulted from (3.1.12),

we have

\[ u_{i}(s,0) = g_{1}(s+\beta_{r}) u_{i}(s,0) + g_{1}(s+\beta_{r-1}) u_{i-1}(s,0) \]

\[ + \left[ (a_{i}^{N} - (s+\beta_{r}) a_{i}) g_{1}(s+\beta_{N-1}) u_{i-1}(s,0) + a_{i}^{N}/(s+\beta_{r}) \right]. \]

Then (3.1.13) is concluded.
アブストラクト

修理人一人で同時故障が存在する場合の

\[ M + 1 - \text{of} - N: G \] システムの解析

大阪大学 大錦 史男
名古屋大学 児玉 正憲
大阪大学 西田 俊夫

我々は以下のシステムを解析し、信頼度、時点アベイラビリティそれぞれの L - 変換、MTS, 定常アベイラビリティを求める。

システムにはサイズ \( i \) のグループが \( q_i = \binom{N}{i} \) 個あり、それらを \( G_{1i}, \ldots, G_{qi} \) とする。各 \( G_{ij} \) へのショックは互いに独立にパラメータ \( \lambda_{ij} \) のポアソン過程に従って発生する。\( G_{ij} \) へのショックが生じた時、\( G_{ij} \) の中で動作中のユニットは必ず故障する。本報告では \( \alpha_{ij} = \lambda_{ij} (i = 1, 2, \ldots, q_i) \) とする。故にサイズ \( i \) のグループへのショックが発生する率は \( \left( \binom{N}{i} \right) \lambda_i \) である。システムは少なくとも \( M + 1 \) 個のユニットが動作していれば正常と見なされる。システムが正常でない時、残りの正常ユニットは故障しない。最初すべてのユニットは動作しているものとする。修理人は一人で、一つの故障ユニットに対する修理時間分布は一般分布であるとする。修理の方策として次の二つを考える。一つはユニット故障と同時に修理を行い、修理完了と同時にシステムに復帰させる。他はシステムがダウンしてから修理を開始し、すべての修理が完了してからシステムを再度動作させるというものである。各修理方策下で定常アベイラビリティの若千の性質が示される。

前者の修理方策（修理方策 1）下での定常アベイラビリティを \( P_A(N) \)、後者の修理方策（修理方策 2）下での定常アベイラビリティを \( P_A^{(2)}(N, M) \) とする。

\[ P_A^{(1)}(N) = N \text{に関して単調増加であるだろうと思われるが、そうではないことが示される。又修理分布が} G_1(t) = 1 - e^{-\mu_1 t} \text{で} \lambda_1 = 0 \quad (i = 2, \ldots, N) \text{の時、} P_A^{(1)}(N) \rightarrow 1 + e^{-1/\rho_1} \quad (\rho_1 = \lambda_1 / \mu_1) \quad (N \rightarrow \infty) \text{であることが示される。} \]

\[ P_A^{(2)}(N, M) \text{に関しては} \lambda_i = 0 \quad (i = 2, \ldots, N) \text{の時、次の性質が示される。任意に固定された} N \geq 1 \text{に対して} P_A^{(2)}(N, M) \uparrow M_0. \text{任意に固定された} M \geq 0 \text{に対して} P_A^{(2)}(N, M) \downarrow N. \text{任意に固定された} M \geq 0 \text{に対して} P_A^{(2)}(N, M) \rightarrow 0 \quad (N \rightarrow \infty). \]

\[ P_A^{(2)}(N, M) \text{は平均修理時間} 1 / \mu_2 \text{にのみ依存し、その修理時間分布には直接影響されない} \]

ことは修理方策の定義より明らかである。任意の \( N, M \leq N \) に対して \( \mu_2 \text{を十分大にすれば、} P_A^{(2)}(N, M) \geq P_A^{(1)}(N) \text{になることは容易に予想されるが、上述の結果より} M \text{が大になるほど} \mu_2 \text{に関する条件がきびしくなることは明らかである。} \]

修理方策 1 に比べて修理方策 2 は簡単であるが、\( N \) が大になるほど、修理方策 2 は悪くなることがわかる。