THE EFFICIENCY OF TWO-STAGE SERIES LINES

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Abstract In this paper, we consider the procedure to estimate the various effects of the design factors in the two-stage series lines in which the first stage has an arbitrary service time. First, the system states are represented by the imbedded Markov chain and the steady state probabilities are solved by the Laplace transform with respect to the arbitrary service time distribution. Next, the efficiency, the idling time distribution, the blocking time distribution and the number of in-process works distribution are considered from the system states. Moreover the relations between the dual models are discussed.

1. Introduction

Many papers have been published concerning the two-stage series lines. Hunt and others estimate the efficiencies and the mean number of in-process works for the lines with exponential or erlang service times by the Markov model [6], [7], [8], [11], [12], [13]. However, most of the papers discuss only the efficiencies for the lines with the arbitrary service times by the approximation methods [1], [3], [10]. On the other hand, the queueing system $M/G/1$ with a finite waiting room which is related to the dual model considered in this paper is discussed by Hashida and others, and the various results are presented [4], [5].

In this paper we consider the procedure to estimate the effects of the design factors in the two-stage series lines in which the first stage has an arbitrary service time and the second stage has an exponential service time. First, we show that the system states can be represented by the imbedded Markov chain like $GI/M/1$ or $M/G/1$ queueing model and that the state probabilities in the steady state condition can be solved by the Laplace transform with respect to the arbitrary service time distribution. And from the state probabilities the efficiency, the idling time distribution, the blocking time distribution and

the number of in-process works distribution are represented. Moreover, we discuss the relations between the dual models and show that the various properties for the dual model can be represented by the other model.

2. Model

The two-stage series line consists of two stages to operate the work and the buffer storage holding in-process works temporarily, as shown in Fig. 2.1.

Fig. 2.1 Two-stage series line

Then the line is defined by the service time in stage i ($i=1,2$) represented by the probability density function (p.d.f.) $f_i(x)$ and the buffer capacity M .

Assume that there are infinite works ready to be operated in the first stage and there is infinite buffer capacity just behind the second stage, therefore idling due to lack of input works is never occurred in the first stage and blocking is never occurred in the second stage because the completed work is always ejected from the stage. And the service times are mutually independent.

Now we formulate the work flow of this model. This flow is defined by the output interval $0_{i,j}$ between the $(j-1)$ -th work and the j -th work in stage i , which is represented by

$$(2.1) \quad 0_{i,j} = I_{i,j-1} + X_{i,j} + B_{i,j} \quad (i=1,2; j=1,2,3,\dots),$$

where $X_{i,j}$, $I_{i,j-1}$ and $B_{i,j}$ denote the service time for the j -th work in stage i , the idling time occurred when stage i waits the j -th work in stage i after ejecting the $(j-1)$ -th work from the stage and the blocking time occurred when the service for the j -th work in stage i is completed but the work finds the buffer full and is held in the stage, respectively. In this equation $I_{i,j-1}$ and $B_{i,j}$ are unknown random variables, but if we have the initial conditions that the numbers of work are independently counted in each stage and that stages 1 and 2 begin to operate the first work at the same time $t=0$ and then there is no in-process work in the buffer, i.e. the j -th work in stage 1 is the $(j+1)$ -th work in stage 2 and $I_{i,0}=0$, $I_{i,j-1}$ and $B_{i,j}$ are given by

$$(2.2) \quad I_{2,j-1} = \left[\left(\sum_{n=1}^{j-2} 0_{1,n} + X_{1,j-1} \right) - \sum_{n=1}^{j-1} 0_{2,n} \right]^+$$

$$B_{1,j} = \left[\sum_{n=1}^{j-M} 0_{2,n} - \left(\sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} \right) \right]^+$$

$$I_{1,j-1} = 0$$

$$B_{2,j} = 0,$$

where the notation $[Y]^+$ denotes

$$[Y]^+ = \text{Max} (0, Y)$$

and $I_{1,j-1} = B_{2,j} = 0$ is given by the above assumption.

3. The imbedded Markov chain

We consider the system states at the time $t = \sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} - 0$, i.e. the time just before the completion of service for the j -th work in stage 1. These states are defined by the state of the each stage and the number of in-process works in the buffer and are represented as follows;

$S_j(W_1, W_2 | m)$: Stages 1 and 2 are in operating and m in-process works are in the buffer ($0 \leq m \leq M$)

$S_j(W_1, I_2 | 0)$: Stage 1 is in operating and Stage 2 is in idling and there is no in-process work in the buffer.

Furthermore the time in which the each state is occurred is given by the following range of time. The state $S_j(W_1, I_2 | 0)$ is occurred in

$$\sum_{n=1}^j 0_{2,n} \leq \sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} \leq \sum_{n=1}^j 0_{2,n} + [X_{1,j} - X_{2,j}]^+,$$

the state $S_j(W_1, W_2 | m)$ is occurred in

$$\sum_{n=1}^{j-1-m} 0_{2,n} \leq \sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} < \sum_{n=1}^{j-m} 0_{2,n} \quad (0 \leq m \leq M-1),$$

and the state $S_j(W_1, W_2 | M)$ is occurred in

$$\sum_{n=1}^{j-M} 0_{2,n} - [X_{2,j-M} - X_{1,j}]^+ \leq \sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} < \sum_{n=1}^{j-M} 0_{2,n}.$$

In Fig. 3.1 the above relations are described.

Fig. 3.2 The system state

In this figure $I_{2,j}$, $B_{1,j}$ and $\delta_{2,j-m}$ ($0 \leq m \leq M-1$) denote the idling time occurred immediately after the ejection of the j -th work in stage 2, the blocking time occurred immediately after the completion of service for the j -th work in stage 1 and the rest part of service time for the $(j-m)$ -th work which exists in stage 2 when the service for the j -th work in stage 1 is completed and also there are m in-process works in the buffer, and they are given by

$$I_{2,j} = \left[\left(\sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} \right) - \sum_{n=1}^j 0_{2,n} \right]^+ \leq [X_{1,j} - X_{2,j}]^+$$

$$B_{1,j} = \left[\sum_{n=1}^{j-M} 0_{2,n} - \left(\sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} \right) \right]^+ \leq [X_{2,j-M} - X_{1,j}]^+$$

$$\delta_{2,j-m} = \left[\sum_{n=1}^{j-m} 0_{2,n} - \left(\sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} \right) \right]^+ < X_{2,j-m}.$$

In this model the system state at any time does not form the Markov chain but the state at $\sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} - 0$ forms the imbedded Markov chain, therefore this transition probabilities are determined by the number of works for which the services are completed in stage 2 during the service time $X_{1,j+1}$. In Table 3.1 the transition probabilities are represented. In this table $\delta_{2,j-m}$ is the unknown random variable, but the p.d.f. $h_m(\delta)$ can be calculated as follows.

We define the completed part of the service time $\Delta_{2,j-m}$ for the $(j-m)$ -th work which exists in stage 2 when the $(j+1)$ -th work in stage 1 is ejected, and denote the p.d.f. by $g_m(\Delta)$. Then $\Delta_{2,j-m}$ and $X_{2,j-m}$ are mutually independent and

Table 3.1 Transition probability

State at $\sum_{n=1}^{j-1} 0_{1,n} + X_{1,j}^{-0}$	State at $\sum_{n=1}^j 0_{1,n} + X_{1,j+1}^{-0}$	Transition Probability
$S_j(W_1, W_2 M)$	$S_{j+1}(W_1, W_2 M)$	Prob. $(X_{1,j+1} \leq X_{2,j+1-M})$
	$S_{j+1}(W_1, W_2 m)$	Prob. $(\sum_{n=m+1}^M X_{2,j+1-n} < X_{1,j+1} \leq \sum_{n=m}^M X_{2,j+1-n})$
	$S_{j+1}(W_1, I_2 0)$	Prob. $(\sum_{n=0}^M X_{2,j+1-n} < X_{1,j+1})$
$S_j(W_1, W_2 m)$	$S_{j+1}(W_1, W_2 m+1)$	Prob. $(X_{1,j+1} \leq \delta_{2,j-m})$
$(0 \leq m \leq M-1)$	$S_{j+1}(W_1, W_2 m)$	Prob. $(\delta_{2,j-m} < X_{1,j+1} \leq \delta_{2,j-m} + X_{2,j-m+1})$
$(0 \leq k \leq m-1)$	$S_{j+1}(W_1, W_2 k)$	Prob. $(\delta_{2,j-m} + \sum_{n=k+1}^m X_{2,j+1-n} < X_{1,j+1} \leq \delta_{2,j-m} + \sum_{n=k}^m X_{2,j+1-n})$
	$S_{j+1}(W_1, I_2 0)$	Prob. $(\delta_{2,j-m} + \sum_{n=0}^m X_{2,j+1-n} < X_{1,j+1})$
$S_j(W_1, I_2 0)$	$S_{j+1}(W_1, W_2 0)$	Prob. $(X_{1,j+1} \leq X_{2,j+1})$
	$S_{j+1}(W_1, I_2 0)$	Prob. $(X_{2,j+1} < X_{1,j+1})$

$\delta_{2,j-m}$ is given by

$$(3.1) \quad \delta_{2,j-m} = X_{2,j-m} - \Delta_{2,j-m},$$

therefore $h_m(\delta)$ is represented by

$$\begin{aligned}
 (3.2) \quad h_m(\delta) &= \frac{\int_0^\infty g_m(\Delta) f_2(\Delta + \delta) d\Delta}{\int_0^\infty \int_0^\infty g_m(\Delta) f_2(\Delta + \delta) d\Delta d\delta} \\
 &= \lambda_2 \exp(-\lambda_2 \delta),
 \end{aligned}$$

where the p.d.f. $f_2(x)$ of the service time in stage 2 is

$$f_2(x) = \lambda_2 \exp(-\lambda_2 x).$$

Consequently it is appeared that $\delta_{2,j-m}$ has the same p.d.f. with that of $X_{2,j-m}$ and all the transition probabilities can be calculated and finally we can estimate the system state probabilities in the steady state condition.

Now we estimate the steady state probabilities. These transition probabilities can be rewritten by k_i

$$(3.3) \quad k_i = \int_0^\infty \frac{(\lambda_2 x)^i}{i!} \exp(-\lambda_2 x) f_1(x) dx$$

and the system state equations in the steady state condition are given by

$$(3.4) \quad \begin{aligned} P(W_1, W_2 | M) &= \{P(W_1, W_2 | M) + P(W_1, W_2 | M-1)\} k_0 \\ P(W_1, W_2 | M-1) &= \{P(W_1, W_2 | M) + P(W_1, W_2 | M-1)\} k_1 + P(W_1, W_2 | M-2) k_0 \\ P(W_1, W_2 | M-2) &= \{P(W_1, W_2 | M) + P(W_1, W_2 | M-1)\} k_2 + P(W_1, W_2 | M-2) k_1 \\ &\quad + P(W_1, W_2 | M-3) k_0 \\ &\vdots \\ P(W_1, W_2 | 0) &= \{P(W_1, W_2 | M) + P(W_1, W_2 | M-1)\} k_M + P(W_1, W_2 | M-2) k_{M-1} \\ &\quad + \dots + P(W_1, W_2 | 0) k_1 + P(W_1, I_2 | 0) k_0 \\ P(W_1, I_2 | 0) &= \{P(W_1, W_2 | M) + P(W_1, W_2 | M-1)\} \sum_{i=M+1}^{\infty} k_i + P(W_1, W_2 | M-2) \sum_{i=M}^{\infty} k_i \\ &\quad + \dots + P(W_1, W_2 | 0) \sum_{i=2}^{\infty} k_i + P(W_1, I_2 | 0) \sum_{i=1}^{\infty} k_i \end{aligned}$$

$P(\cdot)$: The steady state probability that the system state is $S(\cdot)$.

Therefore, if P_i denotes $P(W_1, W_2 | M-i)$ and P_{M+1} denotes $P(W_1, I_2 | 0)$, these equations are rewritten by using $K(Z)$ and $F(Z)$

$$(3.5) \quad K(Z) = \sum_{i=0}^{\infty} k_i Z^i$$

$$F(Z) = \sum_{i=0}^{\infty} P_i Z^i,$$

where $P_i = 0$ ($i \geq M+2$), and we have

$$(3.6) \quad F(Z) = P_0 \cdot \frac{1-Z}{1-Z/K(Z)} + Z^{M+2} H(Z),$$

where $H(Z)$ is the power series of Z .

If the Laplace transform of $f_1(x)$ is represented by $U_1(s)$, we have

$$K(Z) = U_1(\lambda_2(1-Z)),$$

so the above equation is

$$(3.7) \quad F(Z) = P_0 \cdot \frac{1-Z}{1-Z/U_1(\lambda_2(1-Z))} + Z^{M+2} H(Z).$$

In this equation the steady state probabilities P_i ($0 \leq i \leq M+1$) are given by the coefficient of Z^i and expressed in terms of P_0 if $U_1(\cdot)$ can be expanded in a power series. Moreover the root of the equation $1-Z/U_1(\lambda_2(1-Z))=0$ has always $Z=1$ and if we denote the other roots by $1/\alpha_i$, the equation (3.7) is

$$(3.8) \quad F(Z) = P_0 \cdot \frac{1}{(1-\alpha_1 Z)(1-\alpha_2 Z) \dots} + Z^{M+2} H(Z).$$

4. Various line characters

First we estimate the idling time $I_{2,j+1}$ and the blocking time $B_{1,j+1}$. In this model $B_{1,j+1}$ is occurred when the system state at $\sum_{n=1}^{j-1} 0_{1,n} + X_{1,j}^{-0}$ is $S_j(W_1, W_2 | M)$ and changes to $S_{j+1}(W_1, W_2 | M)$ at $\sum_{n=1}^j 0_{1,n} + X_{1,j+1}^{-0}$ or when the state is $S_j(W_1, W_2 | M-1)$ and changes to $S_{j+1}(W_1, W_2 | M)$, and it is not occurred in the other cases. Therefore $B_{1,j+1}$ is represented by

$$(4.1) \quad B_{1,j+1} = [X_{2,j-M+1} - X_{1,j+1}]^+ \text{ or}$$

$$= [\delta_{2,j-M+1} - X_{1,j+1}]^+ \text{ or}$$

$$= 0.$$

Consequently the p.d.f. $w_B(x)$ ($x > 0$) for the blocking time $B_{1,j+1}$ is given by

$$(4.2) \quad w_B(x) = P_j(W_1, W_2 | M) \cdot P_T[S_j(W_1, W_2 | M) \rightarrow S_{j+1}(W_1, W_2 | M)].$$

$$\frac{\bar{f}(x)}{\text{Prob. } (X_{2,j-M+1} \geq X_{1,j+1})} \\ + P_j(W_1, W_2 | M-1) \cdot P_T[S_j(W_1, W_2 | M-1) \rightarrow S_{j+1}(W_1, W_2 | M)]. \\ \frac{\bar{f}(x)}{\text{Prob. } (\delta_{2,j-M+1} \geq X_{1,j+1})},$$

$P_j(\cdot)$: The probability that the system state at $\sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} - 0$ is $S_j(\cdot)$

$P_T[S_j(\cdot) \rightarrow S_{j+1}(\cdot)]$: The transition probability that the system state changes from $S_j(\cdot)$ to $S_{j+1}(\cdot)$

$\bar{f}(x)$: $\int_0^\infty f_1(y) f_2(y+x) dy$.

On the other hand $I_{2,j+1}$ is occurred when the system state at $\sum_{n=1}^{j-1} 0_{1,n} + X_{1,j} - 0$ is $S_j(W_1, W_2 | M)$ and changes to $S_{j+1}(W_1, I_2 | 0)$ at $\sum_{n=1}^j 0_{1,n} + X_{1,j+1} - 0$, when the state is $S_j(W_1, W_2 | m)$ ($0 \leq m \leq M-1$) and changes to $S_{j+1}(W_1, I_2 | 0)$ or when the state is $S_j(W_1, I_2 | 0)$ and changes to $S_{j+1}(W_1, I_2 | 0)$, and it is not occurred in the other cases. Therefore it is represented by

$$(4.3) \quad I_{2,j+1} = [X_{1,j+1} - \sum_{n=0}^M X_{2,j+1-n}]^+ \text{ or} \\ = [X_{1,j+1} - \delta_{2,j-m} - \sum_{n=0}^m X_{2,j+1-n}]^+ \text{ or} \\ = [X_{1,j+1} - X_{2,j+1}]^+ \text{ or} \\ = 0.$$

Consequently the p.d.f. $w_I(x)$ ($x > 0$) for the idling time $I_{2,j+1}$ is given by

$$(4.4) \quad w_I(x) = P_j(W_1, W_2 | M) \cdot P_T[S_j(W_1, W_2 | M) \rightarrow S_{j+1}(W_1, I_2 | 0)]. \\ \frac{\bar{f}_{M+1}(x)}{\text{Prob. } (X_{1,j+1} > \sum_{n=0}^M X_{2,j+1-n})} \\ + \sum_{m=0}^{M-1} P_j(W_1, W_2 | m) \cdot P_T[S_j(W_1, W_2 | m) \rightarrow S_{j+1}(W_1, I_2 | 0)].$$

$$\frac{\bar{f}_{m+2}(x)}{\text{Prob. } (X_{1,j+1} > \delta_{2,j-m} + \sum_{n=0}^m X_{2,j+1-n})} \\ + P_j(W_1, I_2 | 0) \cdot P_T[S_j(W_1, I_2 | 0) \rightarrow S_{j+1}(W_1, I_2 | 0)] \\ \frac{\bar{f}_1(x)}{\text{Prob. } (X_{1,j+1} > X_{2,j+1})},$$

$$\bar{f}_m(x) = \int_0^\infty f_2^{(m)}(y) f_1(y+x) dy,$$

where $f_2^{(m)}(x)$ denotes the m times convolution function of $f_2(x)$. From the Table 3.1 we have the next equalities, too

$$(4.5) \quad \begin{aligned} P_T[S_j(W_1, W_2 | M) \rightarrow S_{j+1}(W_1, W_2 | M)] &= \text{Prob. } (X_{2,j-M+1} \geq X_{1,j+1}) \\ P_T[S_j(W_1, W_2 | M-1) \rightarrow S_{j+1}(W_1, W_2 | M)] &= \text{Prob. } (\delta_{2,j-M+1} \geq X_{1,j+1}) \\ P_T[S_j(W_1, W_2 | M) \rightarrow S_{j+1}(W_1, I_2 | 0)] &= \text{Prob. } (X_{1,j+1} > \sum_{n=0}^M X_{2,j+1-n}) \\ P_T[S_j(W_1, W_2 | m) \rightarrow S_{j+1}(W_1, I_2 | 0)] &= \text{Prob. } (X_{1,j+1} > \delta_{2,j-m} + \sum_{n=0}^m X_{2,j+1-n}) \\ P_T[S_j(W_1, I_2 | 0) \rightarrow S_{j+1}(W_1, I_2 | 0)] &= \text{Prob. } (X_{1,j+1} > X_{2,j+1}), \end{aligned}$$

so that in the steady state condition (4.2) and (4.4) are rewritten by

$$(4.6) \quad \begin{aligned} w_B(x) &= (P_0 + P_1) \cdot \bar{f}(x) \\ w_I(x) &= (P_0 + P_1) \cdot \bar{f}_{M+1}(x) + \sum_{m=2}^{M+1} P_m \cdot \bar{f}_{M+2-m}(x). \end{aligned}$$

In this model the efficiency ρ is represented by

$$(4.7) \quad \rho = \frac{1}{\bar{X}_1 + \bar{B}_1} = \frac{1}{\bar{X}_2 + \bar{I}_2}$$

where $\bar{X}_1$, $\bar{B}_1$ and $\bar{I}_2$ denote the mean operation time in stage 1, the mean blocking time in stage 1 and the mean idling time in stage 2. And the number of in-process works distribution at the time just before the completion of service in stage 1 is given by $P_i (0 \leq i \leq M+1)$ and the mean number $\bar{N}$ of the in-process works is given by

$$(4.8) \quad \bar{N} = \sum_{m=0}^M m \cdot P(W_1, W_2 | m) = \sum_{m=0}^M m \cdot P_{M-m}.$$

5. The dual model

In this section we discuss the dual model in which the service time in each stage is exchanged.

At first we consider the system states occurred at the time $t = \sum_{n=1}^j 0_{2,n} - 0$, i.e. the time just before the ejection of the j -th work in stage 2. These states are defined by

$S_j^*(W_1, W_2 | m)$: Stages 1 and 2 are in operating and m in-process works are in the buffer ($0 \leq m \leq M$)

$S_j^*(B_1, W_2 | M)$: Stage 1 is in blocking, Stage 2 is in operating and M in-process works are in the buffer,

and the relation between the system states and the range of time in which each state is occurred are described in Fig. 5.1. In this figure $I_{2,j}^*$, $B_{1,j+M}^*$ and $\delta_{1,j+m}^*$ ($1 \leq m \leq M$) denote the idling time occurred immediately after the ejection of the j -th work in stage 2, the blocking time occurred immediately after the completion of service for the $(j+M)$ -th work in stage 1 and the rest part of service time for the $(j+m)$ -th work which exists in stage 1 when the service for the j -th work in stage 2 is completed and also there are m in-process works in the buffer.

Fig. 5.1 The system state

In this model the state at $\sum_{n=1}^j 0_{2,n} - 0$ forms the imbedded Markov chain and these transition probabilities are determined by the number of works for which the services are completed in stage 1 during the service time $X_{2,j+1}$. In Table 5.1 these probabilities are represented, where $\delta_{1,j+m}^*$ has the same p.d.f. with the p.d.f. for the service time in stage 1 and it is represented by

$$(5.1) \quad f_1^*(x) = \lambda_1 \exp(-\lambda_1 x).$$

Table 5.1 Transition probability

State at $\sum_{n=1}^j 0_{2,n} - 0$	State at $\sum_{n=1}^{j+1} 0_{2,n} - 0$	Transition Probability
$S_j^*(W_1, W_2 0)$	$S_{j+1}^*(W_1, W_2 0)$	Prob. $(X_{2,j+1} \leq X_{1,j+1})$
	$S_{j+1}^*(W_1, W_2 m)$	Prob. $(\sum_{n=1}^m X_{1,j+n} < X_{2,j+1} \leq \sum_{n=1}^{m+1} X_{1,j+n})$
	$S_{j+1}^*(B_1, W_2 M)$	Prob. $(\sum_{n=1}^{M+1} X_{1,j+n} < X_{2,j+1})$
$S_j^*(W_1, W_2 m)$	$S_{j+1}^*(W_1, W_2 m-1)$	Prob. $(X_{2,j+1} \leq \delta_{1,j+m}^*)$
$(1 \leq m \leq M)$	$S_{j+1}^*(W_1, W_2 m)$	Prob. $(\delta_{1,j+m}^* < X_{2,j+1} \leq \delta_{1,j+m}^* + X_{1,j+m+1})$
$(m+1 \leq k \leq M)$	$S_{j+1}^*(W_1, W_2 k)$	Prob. $(\delta_{1,j+m}^* + \sum_{n=m+1}^k X_{1,j+n} < X_{2,j+1} \leq \delta_{1,j+m}^* + \sum_{n=m+1}^{k+1} X_{1,j+n})$
	$S_{j+1}^*(B_1, W_2 M)$	Prob. $(\delta_{1,j+m}^* + \sum_{n=m+1}^{M+1} X_{1,j+n} < X_{2,j+1})$
$S_j^*(B_1, W_2 M)$	$S_{j+1}^*(W_1, W_2 M)$	Prob. $(X_{2,j+1} \leq X_{1,j+M+1})$
	$S_{j+1}^*(B_1, W_2 M)$	Prob. $(X_{1,j+M+1} < X_{2,j+1})$

From this Table we can estimate the steady state probabilities $P^*(W_1, W_2 | m)$ and $P^*(B_1, W_2 | M)$, and if we denote

$$(5.2) \quad P_i^* = P^*(W_1, W_2 | i) \quad (0 \leq i \leq M)$$

$$P_{M+1}^* = P^*(B_1, W_2 | M)$$

and

$$(5.3) \quad F^*(Z) = \sum_{i=0}^{\infty} P_i^* Z^i,$$

we have the following system equation, where $P_i^* = 0$ ($i \geq M+2$),

$$(5.4) \quad F^*(Z) = P_0^* \cdot \frac{1-Z}{1-Z/U_2^*(\lambda_1(1-Z))} + Z^{M+2} H(Z)$$

$U_2^*(s)$: Laplace transform in terms of $f_2^*(x)$.

In the dual models we have

$$U_2^*(\lambda_1(1-Z)) = U_1^*(\lambda_2(1-Z))$$

$$\lambda_1 = \lambda_2,$$

so the following equalities are obtained

$$(5.5) \quad P_i^* = P_i$$

and

$$(5.6) \quad P^*(W_1, W_2 | m) = P(W_1, W_2 | M-m) \quad (0 \leq m \leq M)$$

$$P^*(B_1, W_2 | M) = P(W_1, I_2 | 0).$$

The idling time $I_{2,j+1}^*$ and the blocking time $B_{1,j+M+1}^*$ are estimated by using the system state at $\sum_{n=1}^j 0_{2,n} - 0$ as follows. The idling time is occurred when the system state is $S_j^*(W_1, W_2 | 0)$ and changes to $S_{j+1}^*(W_1, W_2 | 0)$ or when the system state is $S_j^*(W_1, W_2 | 1)$ and changes to $S_{j+1}^*(W_1, W_2 | 0)$, and it is not occurred in the other cases. Therefore $I_{2,j+1}^*$ is represented by

$$\begin{aligned}
 (5.7) \quad I_{2,j+1}^* &= [X_{1,j+1} - X_{2,j+1}]^+ \text{ or} \\
 &= [\delta_{1,j+1}^* - X_{2,j+1}]^+ \text{ or} \\
 &= 0.
 \end{aligned}$$

And the blocking time $B_{1,j+M+1}^*$ is occurred when the system state is $S_j^*(W_1, W_2 | 0)$ and changes to $S_{j+1}^*(B_1, W_2 | M)$, when the state is $S_j^*(W_1, W_2 | m)$ ($1 \leq m \leq M$) and changes to $S_{j+1}^*(B_1, W_2 | M)$ or when the state is $S_j^*(B_1, W_2 | M)$ and changes to $S_{j+1}^*(B_1, W_2 | M)$, and it is not occurred in the other cases, therefore it is represented by

$$\begin{aligned}
 (5.8) \quad B_{1,j+M+1}^* &= [X_{2,j+1} - \sum_{n=1}^{M+1} X_{1,j+n}]^+ \text{ or} \\
 &= [X_{2,j+1} - \delta_{1,j+m}^* - \sum_{n=m+1}^{M+1} X_{1,j+n}]^+ \text{ or} \\
 &= [X_{2,j+1} - X_{1,j+M+1}]^+ \text{ or} \\
 &= 0.
 \end{aligned}$$

Therefore by using the same procedure in Section 4 the idling time distribution $w_I^*(x)$ ($x > 0$) and the blocking time distribution $w_B^*(x)$ ($x > 0$) in the steady state condition are given by

$$\begin{aligned}
 (5.9) \quad W_I^*(x) &= (P_0^* + P_1^*) \cdot \bar{f}^*(x) \\
 w_B^*(x) &= (P_0^* + P_1^*) \cdot \bar{f}_{M+1}^*(x) + \sum_{m=2}^{M+1} P_m^* \cdot \bar{f}_{M+2-m}^*(x),
 \end{aligned}$$

$$\begin{aligned}
 \bar{f}^*(x) &: \int_0^\infty f_2^*(y) f_1^*(y+x) dy \\
 \bar{f}_m^*(x) &: \int_0^\infty f_1^{*(m)}(y) f_2^*(y+x) dy,
 \end{aligned}$$

where $f_1^{*(m)}(x)$ denotes the m times convolution function of $f_1^*(x)$. From (4.6), (5.5) and (5.9) we have the following equalities,

$$\begin{aligned}
 (5.10) \quad w_I^*(x) &= w_B(x) \\
 w_B^*(x) &= w_I(x).
 \end{aligned}$$

Finally the mean number $\bar{N}^*$ of the in-process works is represented by

$$(5.11) \quad \bar{N}^* = \sum_{m=0}^M m \cdot P^*(W_1, W_2 | m) + M \cdot P^*(B_1, W_2 | M)$$

and from (4.8), (5.6) and (5.11) we have

$$(5.12) \quad \bar{N}^* = M - \bar{N}.$$

6. Conclusion

We have discussed the procedure to estimate the effects of the design factors in the two-stage series lines in which the first stage has the arbitrary service time. And the various line characters are obtained and the relations between the dual models are presented. For example, we represent the mean idling times for the various two-stage series lines in Table 6.1 and 2.

Table 6.1 Numerical results of $\bar{B}_1$ for the balancing line ($\lambda_2=1.0$)

M	Service Time Distribution $f_1(x)$					
	K-Erlang				Uniform	Constant
	K=1	K=2	K=4	K=5	$0.5 \leq x \leq 1.5$	$x=1.0$
0	0.5000	0.4444	0.4096	0.4019	0.3834	0.3679
1	0.3333	0.2807	0.2497	0.2428	0.2268	0.2141
2	0.2500	0.2045	0.1785	0.1731	0.1600	0.1500
3	0.2000	0.1607	0.1389	0.1343	0.1235	0.1154
4	0.1667	0.1324	0.1136	0.1098	0.1006	0.0938
5	0.1429	0.1125	0.0962	0.0928	0.0848	0.0790
6	0.1250	0.0978	0.0833	0.0804	0.0734	0.0682
7	0.1111	0.0865	0.0735	0.0709	0.0646	0.0600
8	0.1000	0.0776	0.0658	0.0634	0.0577	0.0536

Table 6.2 Numerical results of $\bar{B}_1$ for the unbalancing line ($\lambda_1=0.5$, $\lambda_2=1.0$)

M	Service Time Distribution $f_1(x)$					
	K=1	K-Erlang		K=5	Uniform $1.0 \leq x \leq 3.0$	Constant $x=2.0$
		K=2	K=4			
0	0.3333	0.2500	0.1975	0.1859	0.1590	0.1353
1	0.1429	0.0833	0.0530	0.0471	0.0346	0.0251
2	0.0667	0.0303	0.0154	0.0129	0.0081	0.0050
3	0.0323	0.0114	0.0046	0.0036	0.0019	0.0010
4	0.0159	0.0043	0.0014	0.0010	0.0005	0.0002
5	0.0079	0.0016	0.0004	0.0003	0.0001	
6	0.0039	0.0006	0.0001	0.0001		
7	0.0020	0.0002				
8	0.0010	0.0001				

λ_i : The operation rate in stage i

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アブストラクト

2 工程直列型生産ラインの効率

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有限 Buffer 容量を持つ 2 工程直列型生産ラインに関しては多くの研究があるが、工程の作業時間が任意分布を持つ場合に対しては、その設計因子のライン特性に与える影響が十分議論されていない。

そこで本研究は、第一工程が任意の作業時間分布を持つ 2 工程直列型生産ラインの設計因子が種々のライン特性（効率、Idling 時間分布、Blocking 時間分布さらに Buffer 内仕掛品数分布）に与える影響を解析する方法を示し、さらに考えているモデルとその双対モデルの関係を明らかにし、一方のモデルの結果からその双対モデルのライン特性が得られることを示す。

この方法は、工程 1 における個々の製品の作業完了直前のライン状態が隠れマルコフ連鎖となっている点に着目したもので、この状態推移から Idling や Blocking 時間の発生状況が明らかにされその分布が求められる。又、双対モデルに対しても同様の方法を用いて解析し、双対モデル間では効率が等しく、一方の Idling 時間分布と他方の Blocking 時間分布が同一でしかも互いの Buffer 内仕掛品数の平均の和が Buffer 容量と等しくなることが示される。