ON THE JOINT DISTRIBUTION OF THE DEPARTURE INTERVALS IN AN $M/G/1/N$ QUEUE

Akihiko Ishikawa
Iwate University

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Abstract This paper discusses a stationary departure process from the $M/G/1/N$ queue. Using a Markov renewal process, we examine the joint density function f_k of the k -successive departure intervals. In Section 2, we discuss the covariance of departure intervals. The departure intervals are statistically independent in case of $N = 0$ or $N = 1$, but not in case of $N = 2$ or $N = 3$. In Section 3, f_k in the $M/M/1/N$ is shown to be a symmetric function of arrival and service rates, and we find that $cov(d_1, d_k)$ is not dependent on lag k , for $k \leq N + 1$. Further, we prove that the covariance of departure intervals in the dual (reversed) system is equal to one in the original system, for any lag k .

1. Introduction

In this paper, we discuss the departure process of a queueing system. In order to examine the covariance of departure process, we consider the joint density function of the k -successive departure intervals in the $M/G/1/N$ queue.

Many papers have been published on this subject. Burke [1] and Finch [6] have proved that the departure process in the $M/M/1$ queue is again a Poisson process. Jenkins [9] has discussed the covariance of departure process in the $M/E_\lambda/1$ queue. For the $M/G/1/N$ queue, Daley [2] and Daley & Shanbhag [3] have analyzed the departure process. Disney et al. [5], Magalhaes & Disney [11], and Simon & Disney [15] have studied the joint distribution of departure intervals by using a Markov process. Moreover, King [10] has shown that: (1) $cov(d_1, d_k) = 0$ for $k \geq 2$, in the $M/G/1/0$ and $M/D/1/1$ queue; (2) $cov(d_1, d_k) = 0$ for $k \geq 3$, in the $M/G/1/1$ queue. For the $M/M/1$ queue, Hubbard et al. [8] have noted that a probability $P(j, t)$ is a symmetrical expression with regard to an arrival rate λ and a service rate μ , where $P(j, t) = \Pr\{\text{exactly } j \text{ customers depart from the system during a time interval } [0, t]\}$. Saito [14] has analyzed the departure process in an $M/G/s/0$ queue. Makino [12] has discussed a loss probability for the $M/M/1/N \rightarrow /M/1/1$ tandem queue. Furthermore, Daley [4] and Reynolds [13] have surveyed the departure process.

Using a Markov process, we examine the joint density function $f_k(x_1, x_2, \dots, x_k)$ of the k -successive departure intervals in the $M/G/1/N$ queue. In Section 2, we discuss the covariance of departure intervals of lag k . The departure intervals are statistically independent in case of $N = 0$ or $N = 1$, but not in case of $N = 2$ or $N = 3$. Especially, in the $M/G/1/2$ queue, we find that the covariance of departure intervals of lag k are represented as a geometric progression: $cov(d_1, d_k) = \beta_1^{k-3} cov(d_1, d_3)$, for $k \geq 3$, where $\beta_1 = \Pr\{\text{exactly one customer arrives at the system during a service time}\}$.

Furthermore in the $M/M/1/N$ queue, for $k \leq N + 1$, we can see that: (1) f_k is symmetrical with regard to an arrival rate λ and a service rate μ ; (2) $cov(d_1, d_k)$ does not depend on lag k . Lastly, for any lag k , we find that the covariance in the dual (reversed) system is equal to one in the original system.

2. The Stationary $M/G/1/N$ Queue

2.1 The joint density function

We consider a single server queueing system with a Poisson arrival process with rate λ , i.e. the density function $a(x) = \lambda e^{-\lambda x}$. The service times are independently and identically distributed random variables with an arbitrary probability distribution $B(x)$. Suppose that the distribution $B(x)$ has density function $b(x)$; $b(x) = dB(x)/dx$. Let us denote the mean service time by: $\frac{1}{\mu} = \int_0^\infty x b(x) dx$. Further, let $\rho = \frac{\lambda}{\mu}$ be the traffic intensity, and the queue has a capacity N (excluding in service), that is, $M/G/1/N$. If an arriving customer finds the queue is full, then he does not enter into the system. The steady state exists either for $N < \infty (0 < \rho < \infty)$ or for $N = \infty (0 < \rho < 1)$.

Let τ_ξ denote the epoch of departure of the ξ -th customer and let d_ξ denote the departure interval: $d_\xi = \tau_\xi - \tau_{\xi-1} (\xi = \dots, -1, 0, 1, \dots)$. Further, let Q_ξ denote the number of customers in the system just after the ξ -th customer departs.

Let us introduce some notations as follows:

$$\begin{aligned} \Pr\{x \leq d_\xi < x + dx, Q_\xi = Q_{\xi-1} + j - 2 < N \mid Q_{\xi-1} > 0\} / dx &= \frac{1}{(j-1)!} (\lambda x)^{j-1} e^{-\lambda x} b(x) \\ &= b_j(x) \quad (j = 1, 2, \dots, N), \end{aligned}$$

$$\begin{aligned} \Pr\{x \leq d_\xi < x + dx, Q_\xi = N \mid Q_{\xi-1} = N + 2 - j\} / dx &= b(x) - \sum_{k=1}^{j-1} b_k(x) \\ &= \bar{b}_j(x) \quad (j = 2, 3, \dots, N+1), \end{aligned}$$

$$\begin{aligned} \Pr\{x \leq d_\xi < x + dx \mid Q_{\xi-1} = 0\} / dx &= \int_0^x a(x-t) b(t) dt \\ &= c(x) \end{aligned}$$

$$\begin{aligned} \Pr\{x \leq d_\xi < x + dx, Q_\xi = j - 1 \mid Q_{\xi-1} = 0\} / dx &= \int_0^x a(x-t) b_j(t) dt \\ &= c_j(x) \quad (j = 1, 2, \dots, N), \end{aligned}$$

and

$$\begin{aligned} \Pr\{x \leq d_\xi < x + dx, Q_\xi = N \mid Q_{\xi-1} = 0\} / dx &= c(x) - \sum_{k=1}^N c_k(x) \\ &= \bar{c}_{N+1}(x). \end{aligned}$$

Here, it is well known that the bivariate process $\{(d_\xi, Q_\xi)\}$ is a Markov renewal process with a kernel: $U(x) = [u_{ij}(x)] \quad (1 \leq i \leq N+1 \text{ and } 1 \leq j \leq N+1)$,

where for $i = 1$;

$$\begin{aligned} u_{1j}(x) &= c_j(x) & (j = 1, 2, \dots, N), \\ &= \bar{c}_{N+1}(x) & (j = N+1), \end{aligned}$$

for $i = 2, 3, \dots, N+1$;

$$\begin{aligned} u_{ij}(x) &= b_{j-i+2}(x) & (j = i-1, i, \dots, N), \\ &= \bar{b}_{N+3-i}(x) & (j = N+1), \end{aligned}$$

and for $i = 3, 4, \dots, N+1$;

$$u_{ij}(x) = 0 \quad (j = 1, 2, \dots, i-2).$$

So we have a transition probability matrix: $U = [u_{ij}](1 \leq i \leq N+1 \text{ and } 1 \leq j \leq N+1)$,

$$\text{where } u_{ij} = \int_0^\infty u_{ij}(x)dx.$$

Let q_n denote the probability of n customers in the system at the departure epoch in the steady state, i.e., $q_n = \lim_{\xi \rightarrow \infty} \Pr\{Q_\xi = n\}$. That is, the imbedded probability distribution $\{q_n\}$ (the stationary system-size of the departure process) is the stationary distribution of $\{Q_\xi\}$. The imbedded distribution $\{q_n\}$ is a solution of the equilibrium equation:

$${}^T \underline{q} U = {}^T \underline{q}, \quad (\text{where } {}^T \underline{q} = [q_0, q_1, \dots, q_N]).$$

In general, the above imbedded distribution $\{q_n\}$ is not equal to the stationary distribution $\{p_n\}$ at an arbitrary point of time. The relation between q_n and p_n is given by:

$$p_n = C q_n \quad (n = 0, 1, \dots, N),$$

$$1 - C \quad (n = N+1),$$

where $C = \frac{1}{q_0 + \rho}$, (see §5.1.8 in Gross & Harris [7]).

Moreover, the matrices $U(x)$ and U have the following properties:

$$\begin{aligned} (2.1) \quad U(x)\underline{e} &= \underline{e}_1 c(x) + \sum_{i=2}^{N+1} \underline{e}_i b(x) \\ &= \underline{e}_1 [c(x) - b(x)] + \underline{e} b(x), \\ U\underline{e} &= \underline{e}, \end{aligned}$$

$$(2.2) \quad U(x)\underline{e}_j = \underline{e}_1 c_j(x) + \sum_{i=2}^{j+1} \underline{e}_i b_{j+2-i}(x) \quad (j = 1, 2, \dots, N),$$

and

$$(2.3) \quad U(x)\underline{e}_{N+1} = \underline{e}_1 \bar{c}_{N+1}(x) + \sum_{i=2}^{N+1} \underline{e}_i \bar{b}_{N+3-i}(x),$$

where $\underline{e}$ is a column vector each of whose elements are unity; $\underline{e} = {}^T [1, 1, \dots, 1]$ and $\underline{e}_j$ is the j -th fundamental vector; $\underline{e}_j = {}^T [0, 0, \dots, 0, \underset{(j)}{1}, 0, \dots, 0]$.

Let $f_k(x_1, x_2, \dots, x_k)$ be the joint density function of the k -successive departure intervals, at the stationary:

$$\begin{aligned} &f_k(x_1, x_2, \dots, x_k) dx_1 dx_2 \dots dx_k \\ &= \lim_{\xi \rightarrow \infty} \Pr\{x_1 \leq d_{\xi+1} < x_1 + dx_1, x_2 \leq d_{\xi+2} < x_2 + dx_2, \dots, x_k \leq d_{\xi+k} < x_k + dx_k\}. \end{aligned}$$

In the stationary $M/G/1/N$ queue, f_k can be expressed using a matrix form:

$$f_k(x_1, x_2, \dots, x_k) = {}^T \underline{q} U(x_1) U(x_2) \dots U(x_k) \underline{e},$$

(see Disney et al. [5] and Simon & Disney [15]).

2.2 The Covariance of lag k

Let us denote the marginal density functions by:

$$f_{k+1}(\bullet, x_1, x_2, \dots, x_k) = \int_0^\infty f_{k+1}(y, x_1, x_2, \dots, x_k) dy$$

and

$$f_{k+1}(x_1, x_2, \dots, x_k, \bullet) = \int_0^\infty f_{k+1}(x_1, x_2, \dots, x_k, y) dy$$

By the stationary assumption we can write:

$$f_{k+n}(\bullet \bullet \bullet, x_1, x_2, \dots, x_k, \bullet \bullet \bullet) = f_k(x_1, x_2, \dots, x_k).$$

In the stationary, we abbreviate $d_\xi, d_{\xi+1}, d_{\xi+2}$ and $d_{\xi+k}$, to d, d_1, d_2 and d_k , respectively. Let $f_1(x), E(d)$ and $V(d)$ denote the density function, the expectation and the variance of d , respectively:

$$\begin{aligned} f_1(x) &= {}^T \underline{q} U(x) \underline{e} \\ &= b(x) + q_0 \alpha(x), \\ E(d) &= \frac{1}{\mu} + q_0 \frac{1}{\lambda} \text{ and } V(d) = V_b + \frac{1}{\lambda^2} q_0 (2 - q_0) \end{aligned}$$

where $\alpha(x) = c(x) - b(x)$ and $V_b = \int_0^\infty x^2 b(x) dx - \frac{1}{\mu^2}$.

Let us denote the degenerative density function by:

$$\begin{aligned} f_{1,k}(x_1, x_k) &= \int_0^\infty \int_0^\infty \dots \int_0^\infty f_k(x_1, x_2, \dots, x_k) dx_2 dx_3 \dots dx_{k-1} \\ &= {}^T \underline{q} U(x_1) U^{k-2} U(x_k) \underline{e} \quad (k \geq 2). \end{aligned}$$

Furthermore, let us denote the covariance of lag k by:

$$\begin{aligned} cov(d_1, d_k) &= E(d_1, d_k) - E(d_1)E(d_k) \\ &= \int_0^\infty \int_0^\infty x_1 x_k d\gamma_{1,k}(x_1, x_k) \quad (k \geq 2), \end{aligned}$$

where $d\gamma_{1,k}(x_1, x_k) = f_{1,k}(x_1, x_k) dx_1 dx_k - f_1(x_1) f_1(x_k) dx_1 dx_k$.

In this paper, we have assumed $B(x)$ has a density. However, this assumption is not essential. If $B(x)$ has not density, then we can derive similar results for the joint cumulative distribution function F_k :

$$F_k(x_1, x_2, \dots, x_k) = \lim_{\xi \rightarrow \infty} \Pr\{0 \leq d_{\xi+1} \leq x_1, 0 \leq d_{\xi+2} \leq x_2, \dots, 0 \leq d_{\xi+k} \leq x_k\}.$$

2.3 Examples

(1) $N = 0$. Clearly we have: $q_0 = 1$, $U(x) = c(x)$, $f_1(x) = c(x)$,

$$f_k(x_1, x_2, \dots, x_k) = \prod_{i=1}^k f_1(x_i) \text{ and } d\gamma_{1,k}(x_1, x_k) = 0 \quad (k \geq 2).$$

In this $M/G/1/0$ case, the departure intervals are statistically independent and the interdeparture process is a renewal process.

(2) $N = 1$. It is easily to see that:

$$q_0 = \beta_0, \quad q_1 = 1 - \beta_0, \quad U(x) = \begin{bmatrix} c_1(x) & \bar{c}_2(x) \\ b_1(x) & \bar{b}_2(x) \end{bmatrix}, \quad U = \begin{bmatrix} \beta_0 & 1 - \beta_0 \\ \beta_0 & 1 - \beta_0 \end{bmatrix},$$

$$d\gamma_{1,2}(x_1, x_2) = [q_0 c_1(x_1) + q_1 b_1(x_1) - q_0^2 \alpha(x_1) - q_0 b(x_1)] \alpha(x_2) dx_1 dx_2,$$

and

$$\text{cov}(d_1, d_2) = -\frac{1}{\lambda^2}(\rho\beta_0 - \beta_1),$$

where $\beta_j = \Pr\{\text{exactly } j \text{ customers arrive at the system during a service time}\}$

$$\begin{aligned} &= \int_0^\infty b_{j+1}(x) dx \\ &= \int_0^\infty c_{j+1}(x) dx \quad (j = 0, 1, \dots). \end{aligned}$$

In this case the matrix U satisfies $U = \underline{e}^T \underline{q}$, so that we have:

$$\begin{aligned} f_{1,3}(x_1, x_3) &= {}^T \underline{q} U(x_1) U U(x_3) \underline{e} \\ &= {}^T \underline{q} U(x_1) (\underline{e}^T \underline{q}) U(x_3) \underline{e} \\ &= f_1(x_1) f_1(x_3) \end{aligned}$$

and

$$d\gamma_{1,3}(x_1, x_3) = 0.$$

In the same manner we have:

$$(2.4) \quad f_{1,k}(x_1, x_k) = f_1(x_1) f_1(x_k) \quad (k \geq 3),$$

and

$$(2.5) \quad d\gamma_{1,k}(x_1, x_k) = 0 \quad (k \geq 3).$$

The above (2.4) and (2.5) imply that the separate intervals are statistically independent in the $M/G/1/1$ queue.

If the service times are constant, then $\beta_0 = e^{-\rho}$, $\beta_1 = \rho e^{-\rho}$ and $\text{cov}(d_1, d_2) = 0$. So the departure intervals are statistically independent and the interdeparture process is a renewal process in the $M/D/1/1$. The above conclusion of $N = 0$ and $N = 1$ have been already shown by King [10].

(3) $N = 2$. In this part we have:

$$q_0 = \frac{1}{1 - \beta_1} \beta_0^2, \quad q_1 = \frac{1}{1 - \beta_1} \beta_0(1 - \beta_0), \quad q_2 = \frac{1}{1 - \beta_1} (1 - \beta_0 - \beta_1),$$

$$U(x) = \begin{bmatrix} c_1(x) & c_2(x) & \bar{c}_3(x) \\ b_1(x) & b_2(x) & \bar{b}_3(x) \\ 0 & b_1(x) & \bar{b}_2(x) \end{bmatrix}, \quad U = \begin{bmatrix} \beta_0 & \beta_1 & 1 - \beta_0 - \beta_1 \\ \beta_0 & \beta_1 & 1 - \beta_0 - \beta_1 \\ 0 & \beta_0 & 1 - \beta_0 \end{bmatrix},$$

$$d\gamma_{1,2}(x_1, x_2) = -[q_0^2 \alpha(x_1) + q_0 b(x_1) - q_0 c_1(x_1) - q_1 b_1(x_1)] \alpha(x_2) dx_1 dx_2,$$

and

$$\text{cov}(d_1, d_2) = -\frac{1}{\lambda^2} [q_0^2 + \rho q_0 - q_0(\beta_0 + \beta_1) - q_1 \beta_1].$$

The matrix U can be represented as $U = R\Lambda R^{-1}$ where

$$R = \begin{bmatrix} 1 - \beta_0 - \beta_1 & 1 & \beta_1 - \beta_0 + \beta_0^2 \\ 1 - \beta_0 - \beta_1 & 1 & -\beta_0 + \beta_0^2 \\ -\beta_0 & 1 & \beta_0^2 \end{bmatrix} \text{ and } \Lambda = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So that we obtain:

$$\begin{aligned} f_{1,k}(x_1, x_k) &= {}^T \underline{q} U(x_1) U^{k-2} U(x_k) \underline{e} \\ &= {}^T \underline{q} U(x_1) R \Lambda^{k-2} R^{-1} U(x_k) \underline{e} \quad (k \geq 3), \\ d\gamma_{1,k}(x_1, x_k) &= -\beta_1^{k-3} dK(x_1) \alpha(x_k) dx_k \quad (k \geq 3), \end{aligned}$$

and

$$\text{cov}(d_1, d_k) = \beta_1^{k-3} \text{cov}(d_1, d_3) \quad (k \geq 3),$$

where

$$dK(x) = \{q_0^2 \alpha(x) + q_0 b(x) - \beta_0 [q_0 (c_1(x) + c_2(x)) + q_1 (b_1(x) + b_2(x)) + q_2 b_1(x)]\} dx,$$

and

$$\text{cov}(d_1, d_3) = -\frac{1}{\lambda^2} \{q_0^2 + \rho q_0 - \beta_0 [q_0 (\beta_0 + 2\beta_1 + 2\beta_2) + q_1 (\beta_1 + 2\beta_2) + q_2 \beta_1]\}.$$

In general, we see that the departure intervals are generally not independent. The results are summarized as the following theorem.

THEOREM 1. In the stationary $M/G/1/2$ queue, the covariances of lag k are given as follows:

$$\begin{aligned} \text{cov}(d_1, d_2) &= -\frac{1}{\lambda^2} [q_0^2 + \rho q_0 - q_0 (\beta_0 + \beta_1) - q_1 \beta_1], \\ \text{cov}(d_1, d_3) &= -\frac{1}{\lambda^2} \{q_0^2 + \rho q_0 - \beta_0 [q_0 (\beta_0 + 2\beta_1 + 2\beta_2) + q_1 (\beta_1 + 2\beta_2) + q_2 \beta_1]\} \\ \text{and } \text{cov}(d_1, d_k) &= \beta_1^{k-3} \text{cov}(d_1, d_3) \quad (k \geq 4). \end{aligned}$$

(4) $N = 3$. In this case we have:

$$\delta_1 = \beta_1 + \sqrt{\beta_0 \beta_2}, \quad \delta_2 = \beta_1 - \sqrt{\beta_0 \beta_2}, \quad H = \frac{1}{(\delta_1 - 1)(\delta_2 - 1)}$$

$$q_0 = H\beta_0^3, \quad q_1 = H\beta_0^2(1 - \beta_0), \quad q_2 = H\beta_0(1 - \beta_0 - \beta_1), \quad q_3 = 1 - H\beta_0(1 - \beta_1),$$

$$d\gamma_{1,2}(x_1, x_2) = [q_0 c_1(x_1) + q_1 b_1(x_1) - q_0 f_1(x_1)] \alpha(x_2) dx_1 dx_2,$$

$$\text{cov}(d_1, d_2) = \frac{1}{\lambda^2} [q_0 (\beta_0 + \beta_1) + q_1 \beta_1 - q_0 (q_0 + \rho)]$$

and for $k \geq 3$,

$$d\gamma_{1,k}(x_1, x_k) = [\delta_1^{k-2} H_1 v_1(x_1) + \delta_2^{k-2} H_2 v_2(x_1)] \alpha(x_k) dx_1 dx_k,$$

$$\text{cov}(d_1, d_k) = \frac{1}{\lambda} [\delta_1^{k-2} H_1 E(v_1) + \delta_2^{k-2} H_2 E(v_2)]$$

where

$$H_i = \frac{1}{2(\delta_i - 1)(\delta_i - \beta_1)\delta_i}$$

$$\begin{aligned} v_i(x) &= \beta_0^3 f_1(x) + q_0[h_{i,1}c_1(x) + h_{i,2}c_2(x) + h_{i,3}c_3(x)] \\ &\quad + b_1(x)[q_1h_{i,1} + q_2h_{i,2} + q_3h_{i,3}] + b_2(x)[q_1h_{i,2} + q_2h_{i,3}] + b_3(x)q_1h_{i,3}, \\ E(v_i) &= \frac{1}{\lambda} \{ \beta_0^3(q_0 + \rho) + q_0[h_{i,1}(\beta_0 + \beta_1) + h_{i,2}(\beta_1 + 2\beta_2) + h_{i,3}(\beta_2 + 3\beta_3)] \\ &\quad + \beta_1[q_1h_{i,1} + q_2h_{i,2} + q_3h_{i,3}] + \beta_2[q_1h_{i,2} + q_2h_{i,3}] + \beta_3q_1h_{i,3} \} \\ h_{i,1} &= h_{i,2} = \beta_0(\delta_i - 1)(\beta_0 - \beta_1 + \delta_i), \quad h_{i,3} = \beta_0^2(\delta_i - 1) \quad (\text{for } i = 1, 2). \end{aligned}$$

3. The Stationary $M/M/1/N$ Queue

3.1 The joint density function

In this section we derive a closed form for $f_k(x_1, x_2, \dots, x_k)$ in the case of exponential service times i.e. $b(x) = \mu e^{-\mu x}$. We temporally suppose that $\rho \neq 1$, but this assumption is not essential. Here, we have:

$$\begin{aligned} (3.1) \quad q_n &= (1 - \rho)H_N \rho^n & (n = 0, 1, \dots, N), \\ b_j(x) &= \frac{1}{(j-1)!} (\lambda x)^{j-1} \mu e^{-(\lambda+\mu)x} & (j = 1, 2, \dots, N), \end{aligned}$$

$$(3.2) \quad \bar{b}_j(x) = b(x) - \sum_{i=1}^{j-1} b_i(x) \quad (j = 2, 3, \dots, N+1),$$

$$(3.3) \quad c(x) = \frac{1}{1-\rho} [a(x) - \rho b(x)],$$

$$(3.4) \quad c_j(x) = a(x)\rho^{j-1} - \sum_{i=1}^j b_i(x)\rho^{j+1-i} \quad (j = 1, 2, \dots, N),$$

$$\begin{aligned} (3.5) \quad \bar{c}_{N+1}(x) &= c(x) - \sum_{j=1}^N c_j(x) \\ &= c(x) - \frac{a(x)}{1-\rho} (1 - \rho^N) + \sum_{j=1}^N \sum_{k=1}^j \rho^k b_{j+1-k}(x), \end{aligned}$$

$$\alpha(x) = \frac{1}{1-\rho} [a(x) - b(x)],$$

$$\begin{aligned} (3.6) \quad f_1(x) &= {}^T q U(x) \underline{e} \\ &= H_N [a(x) - \rho^{N+1} b(x)], \end{aligned}$$

$$E(d) = \frac{1}{\lambda} H_N (1 - \rho^{N+2}),$$

$$V(d) = \frac{1}{\lambda^2} H_N^2 [(1 - \rho^{N+2})^2 - 2\rho^{N+1}(1 - \rho)^2]$$

and

$$\begin{aligned} (3.7) \quad f_2(x_1, x_2) &= {}^T q U(x_1) U(x_2) \underline{e} \\ &= H_N [a(x_1)a(x_2) - \rho^{N+1} b(x_1)b(x_2)], \end{aligned}$$

where

$$\begin{aligned} H_N &= \frac{1}{1 - \rho^{N+1}} \\ &= \frac{\mu^{N+1}}{\mu^{N+1} - \lambda^{N+1}}. \end{aligned}$$

Now, we prepare two lemmas.

Lemma 1. For $j = 1, 2, \dots, N$, we have:

$$(3.8) \quad {}^T \underline{q} U(x) \underline{e}_j = q_{j-1} a(x).$$

Proof: Using (2.2), (3.1) and (3.4), it is easy to show:

$$\begin{aligned} {}^T \underline{q} U(x) \underline{e}_j &= {}^T \underline{q} [\underline{e}_1 c_j(x) + \sum_{i=2}^{j+1} \underline{e}_i b_{j+2-i}(x)] \\ &= q_0 c_j(x) + \sum_{i=2}^{j+1} q_{i-1} b_{j+2-i}(x) \\ &= (1 - \rho) H_N [a(x) \rho^{j-1} - \sum_{i=2}^{j+1} \rho^{i-1} b_{j+2-i}(x)] + \sum_{i=2}^{j+1} q_{i-1} b_{j+2-i}(x) \\ &= q_{j-1} a(x) \quad (j = 1, 2, \dots, N). \end{aligned}$$

Lemma 2. For $j = N + 1$, we have:

$$(3.9) \quad {}^T \underline{q} U(x) \underline{e}_{N+1} = q_N c(x).$$

Proof: Using (2.2), (2.3) and (3.1) – (3.5) we have:

$$\begin{aligned} {}^T \underline{q} U(x) \underline{e}_{N+1} &= {}^T \underline{q} [\underline{e}_1 \bar{c}_{N+1}(x) + \sum_{i=2}^{N+1} \underline{e}_i \bar{b}_{N+3-i}(x)] \\ &= q_0 \bar{c}_{N+1}(x) + \sum_{i=2}^{N+1} q_{i-1} \bar{b}_{N+3-i}(x) \\ &= (1 - \rho) H_N \left\{ [c(x) - \frac{1}{1 - \rho} (1 - \rho^N) a(x) + \sum_{j=1}^N \sum_{k=1}^j b_{j+1-k}(x) \rho^k] \right. \\ &\quad \left. + \sum_{i=2}^{N+1} \rho^{i-1} [b(x) - \sum_{j=1}^{N+2-i} b_j(x)] \right\} \\ &= (1 - \rho) H_N \left\{ c(x) - \frac{1}{1 - \rho} (1 - \rho^N) [a(x) - \rho b(x)] \right. \\ &\quad \left. + \sum_{k=1}^N \rho^k [\sum_{j=k}^N b_{j+1-k}(x) - \sum_{j=1}^{N+1-k} b_j(x)] \right\} \\ &= (1 - \rho) H_N [c(x) - (1 - \rho^N) c(x)] \\ &= q_N c(x). \end{aligned}$$

From the above lemmas we obtain the following theorem.

THEOREM 2. In the stationary $M/M/1/N$ queue, for $k \leq N+1$, the joint density function can be expressed as follows:

$$(3.10) \quad \begin{aligned} f_k(x_1, x_2, \dots, x_k) &= H_N \left[\prod_{i=1}^k a(x_i) - \rho^{N+1} \prod_{i=1}^k b(x_i) \right] \\ &= \frac{1}{\mu^{N+1} - \lambda^{N+1}} [\mu^{N+1} \prod_{i=1}^k a(x_i) - \lambda^{N+1} \prod_{i=1}^k b(x_i)]. \end{aligned}$$

Proof: We prove (3.10) by the induction on k . From (3.6) and (3.7), for $k=1$ and $k=2$, (3.10) holds. Assuming its validity for $k=h (\leq N)$, we shall show (3.10) for $k=h+1$. From (2.1) and (2.2) we have:

$$U(y)\underline{e} = \underline{e}b(y) + \underline{e}_1\alpha(y)$$

and

$$U(y_1)U(y_2)\underline{e} = \underline{e}b(y_1)b(y_2) + \underline{e}_1[\alpha(y_1)b(y_2) + c_1(y_1)\alpha(y_2)] + \underline{e}_2b_1(y_1)\alpha(y_2).$$

Thus we can write recursively,

$$U(y_1)U(y_2)\cdots U(y_h)\underline{e} = \underline{e}b(y_1)b(y_2)\cdots b(y_h) + \sum_{j=1}^h \underline{e}_j L_{h,j}$$

and

$$\begin{aligned} f_h(y_1, y_2, \dots, y_h) &= {}^T \underline{q} U(y_1)u(y_2)\cdots U(y_h)\underline{e} \\ &= {}^T \underline{q} \underline{e} b(y_1)b(y_2)\cdots b(y_h) + {}^T \underline{q} \sum_{j=1}^h \underline{e}_j L_{h,j} \\ &= \prod_{i=1}^h b(y_i) + \sum_{j=1}^h q_{j-1} L_{h,j}, \end{aligned}$$

where each $L_{h,j}$ is a certain unknown function.

For $h \leq N$, using the inductive hypothesis we have:

$$(3.11) \quad \begin{aligned} \sum_{j=1}^h q_{j-1} L_{h,j} &= f_h(y_1, y_2, \dots, y_h) - \prod_{i=1}^h b(y_i) \\ &= H_N \left[\prod_{i=1}^h a(x_i) - \rho^{N+1} \prod_{i=1}^h b(x_i) \right] - \prod_{i=1}^h b(x_i) \\ &= H_N \left[\prod_{i=1}^h a(x_i) - \prod_{i=1}^h b(x_i) \right] \quad (h \leq N). \end{aligned}$$

Considering the function of the $(h+1)$ -successive departure intervals $x, y_1, y_2, \dots, y_h$, only $h \leq N$, from (2.1), (3.8) and (3.11) we obtain:

$$(3.12) \quad \begin{aligned} f_{h+1}(x, y_1, y_2, \dots, y_h) &= {}^T \underline{q} U(x)U(y_1)U(y_2)\cdots U(y_h)\underline{e} \\ &= {}^T \underline{q} U(x) \left\{ \underline{e} \prod_{i=1}^h b(y_i) + \sum_{j=1}^h \underline{e}_j L_{h,j} \right\} \end{aligned}$$

$$\begin{aligned}
&= {}^T q[\underline{e}b(x) + \underline{e}_1\alpha(x)] \prod_{i=1}^h b(y_i) + \sum_{j=1}^h q_{j-1}a(x)L_{h,j} \\
&= [b(x) + q_0\alpha(x)] \prod_{i=1}^h b(y_i) + a(x)H_N[\prod_{i=1}^h a(y_i) - \prod_{i=1}^h b(y_i)] \\
&= H_N\{a(x) \prod_{i=1}^h a(y_i) + [(1 - \rho^{N+1})b(x) - b(x)] \prod_{i=1}^h b(y_i)\} \\
&= H_N[a(x) \prod_{i=1}^h a(y_i) - \rho^{N+1}b(x) \prod_{i=1}^h b(y_i)].
\end{aligned}$$

Obviously, (3.12) is just the same as (3.10) for $k = h + 1$.

In the above calculations, we need not refer to (3.9) because of $k \leq N + 1$. In case of $k \geq N + 2$, we must use (3.9), and for $k = N + 2$ we have:

$$\begin{aligned}
f_{N+2}(x_1, x_2, \dots, x_{N+2}) &= {}^T qU(x_1)U(x_2) \cdots U(x_{N+2})\underline{e} \\
&= {}^T qU(x_1)[\underline{e} \prod_{i=2}^{N+2} (x_i) + \sum_{j=1}^{N+1} \underline{e}_j L_{N+1,j}(x_2, x_3, \dots, x_{N+2})] \\
&= {}^T q[\underline{e}b(x_1) + \underline{e}_1\alpha(x_1)] \prod_{i=2}^{N+2} b(x_i) + \sum_{j=1}^N q_{j-1}a(x_1)L_{N+1,j} \\
&\quad + q_N c(x_1)L_{N+1,N+1} \\
&= [b(x_1) + q_0\alpha(x_1)] \prod_{i=2}^{N+2} b(x_i) + a(x_1)H_N[\prod_{i=2}^{N+2} a(x_i) - \prod_{i=2}^{N+2} b(x_i)] \\
&\quad + q_N[c(x_1) - a(x_1)]L_{N+1,N+1} \\
&= H_N[\prod_{i=1}^{N+2} a(x_i) - \rho^{N+1} \prod_{i=1}^{N+2} b(x_i)] + R_{N+2},
\end{aligned}$$

where

$$R_{N+2}(x_1, x_2, \dots, x_{N+2}) = q_N[c(x_1) - a(x_1)]L_{N+1,N+1}(x_2, x_3, \dots, x_{N+2}).$$

REMARK 1. In case of $k \geq N + 2$, we have:

$$f_k(x_1, x_2, \dots, x_k) = H_N[\prod_{i=1}^k a(x_i) - \rho^{N+1} \prod_{i=1}^k b(x_i)] + R_k(x_1, x_2, \dots, x_k)$$

where R_k is some unknown function.

3.2 The covariance of the departure intervals

On the basis of Theorem 2, we obtain the following results.

Proposition 1. In the stationary $M/M/1/N$ queue, for $k \leq N + 1$ we have:

$$f_{1,k}(x_1, x_k) = H_N[a(x_1)a(x_k) - \rho^{N+1}b(x_1)b(x_k)]$$

and

$$\gamma_{1,k}(x_1, x_k) = -H_N^2 \rho^{N+1} [a(x_1) - b(x_1)][a(x_k) - b(x_k)].$$

Proposition 2. In the stationary $M/M/1/N$ queue, for $k \leq N+1$ the covariance $\text{cov}(d_1, d_k)$ is independent of lag k , as follows:

$$\begin{aligned} \text{cov}(d_1, d_k) &= -\frac{1}{\lambda^2} H_N^2 (1-\rho)^2 \rho^{N+1} \\ &= -\left[\frac{\lambda - \mu}{\lambda^{N+1} - \mu^{N+1}}\right]^2 (\lambda\mu)^{N-1}. \end{aligned}$$

REMARK 2. Supposing that $\rho < 1$. Then for arbitrary k , the limit exists:

$$\begin{aligned} \lim_{N \rightarrow \infty} f_k(x_1, x_2, \dots, x_k) &= \lim_{N \rightarrow \infty} \frac{1}{1 - \rho^{N+1}} \left[\prod_{i=1}^k a(x_i) - \rho^{N+1} \prod_{i=1}^k b(x_i) \right] \\ &= \prod_{i=1}^k a(x_i). \end{aligned}$$

This conclusion has been already discussed in [1], [6] and [10].

Noting that (3.10) is a symmetrical expression with regard to λ and μ , we consider $M(\lambda)/M(\mu)/1/N$ and its dual $M(\mu)/M(\lambda)/1/N$. In the dual (reversed) system, let $f_k^*(x_1, x_2, \dots, x_k)$ denote the joint density function corresponding to $f_k(x_1, x_2, \dots, x_k)$. Similarly, let $\text{cov}(d_1^*, d_k^*)$ denote the covariance of lag k in the dual system. The relations between the dual systems are given as follows:

THEOREM 3. In the dual systems, for any k we have:

$$f_k^*(x_1, x_2, \dots, x_k) = f_k(x_k, x_{k-1}, \dots, x_1)$$

and

$$\text{cov}(d_1^*, d_k^*) = \text{cov}(d_1, d_k).$$

Proof: See Appendix.

For $k \leq N+1$, from (3.10) we obtain the following relation.

Proposition 3. For $k \leq N+1$, the two functions f_k and f_k^* equal to one another: $f_k(x_1, x_2, \dots, x_k) = f_k^*(x_1, x_2, \dots, x_k)$.

3.3 In case of $\rho = 1$

In this case, the results are given as follows:

$$\begin{aligned} q_n &= \frac{1}{N+1} \quad (n = 0, 1, \dots, N), \\ c(x) &= \lambda^2 x e^{-\lambda x}, \quad \alpha(x) = \lambda(\lambda x - 1)e^{-\lambda x}, \\ f_1(x) &= q_0 \lambda(\lambda x + N) e^{-\lambda x}, \\ E(d) &= \frac{1}{\lambda} q_0 (N+2), \quad V(d) = \frac{1}{\lambda^2} q_0^2 (N^2 + 4N + 2). \end{aligned}$$

And for $k \leq N + 1$:

$$f_k(x_1, x_2, \dots, x_k) = q_0[N + 1 - k + \lambda(x_1 + x_2 + \dots + x_k)] \prod_{i=1}^k a(x_i),$$

$$d\gamma_{1,k}(x_1, x_k) = -q_0^2[\lambda x_1 - 1]a(x_1)[\lambda x_k - 1]a(x_k)dx_1dx_k,$$

and

$$\text{cov}(d_1, d_k) = -\frac{1}{\lambda^2}q_0^2.$$

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Appendix

(proof of Theorem 3)

For the dual system i.e. $M(\mu)/M(\lambda)/1/N$ queue, let $\underline{q}^*$ denote the vector of the imbedded probability, and $U^*(x)$ denote the kernel, corresponding to $\underline{q}$ and $U(x)$, respectively. Naturally, we suppose $\rho \neq 1$, then we have:

$${}^T \underline{q}^* = H_N(1 - \rho)[\rho^N, \rho^{N-1}, \dots, \rho, 1]$$

and

$$U^*(x) = [u_{ij}^*(x)] \quad (i = 1, 2, \dots, N+1 \text{ and } j = 1, 2, \dots, N+1),$$

where for $i = 1$;

$$\begin{aligned} u_{1j}^*(x) &= \rho^{1-j} \bar{b}_{j+1}(x) & (j = 1, 2, \dots, N), \\ & \rho^{-N} \bar{c}_{N+1}(x) & (j = N+1), \end{aligned}$$

for $i = 2, 3, \dots, N+1$;

$$\begin{aligned} u_{ij}^*(x) &= \rho^{i-j} b_{j-i+2}(x) & (j = i-1, i, \dots, N), \\ & \rho^{i-N-1} c_{N+2-i}(x) & (j = N+1), \end{aligned}$$

and for $i = 3, 4, \dots, N+1$;

$$u_{ij}^*(x) = 0 \quad (j = 1, 2, \dots, i-2).$$

Let us denote a transform matrix Z by:

$$Z = [z_{ij}] \quad (i = 1, 2, \dots, N+1 \text{ and } j = 1, 2, \dots, N+1),$$

where

$$\begin{aligned} z_{ij} &= \rho^{i-1} & (i+j = N+2), \\ & 0 & (\text{otherwise}). \end{aligned}$$

It is easy to see that:

$$Z^{-1} = \rho^{-N} Z, \quad ({}^T \underline{q}^*) Z = \{\rho^N H_N(1 - \rho)\}^T \underline{e}, \text{ and } \rho^N H_N(1 - \rho) Z^{-1} \underline{e} = \underline{q}.$$

Here, we note that:

$$\begin{aligned} [Z^T U(x) Z^{-1}]_{ij} &= \rho^{-N} [Z^T U(x) Z]_{ij} \\ &= \rho^{-N} z_{i, N+2-i} u_{N+2-j, N+2-i}(x) z_{N+2-j, j} \\ &= \rho^{i-j} u_{N+2-j, N+2-i}(x). \end{aligned}$$

Each element becomes as follows.

$$\begin{aligned} \text{For } i = 1 \text{ and } j = N+1 : [Z^T U(x) Z^{-1}]_{1, N+1} &= \rho^{-N} u_{1, N+1}(x) \\ &= \rho^{-N} \bar{c}_{N+1}(x) \\ &= u_{1, N+1}^*(x). \end{aligned}$$

$$\begin{aligned} \text{For } i = 1 \text{ and } 1 \leq j \leq N : [Z^T U(x) Z^{-1}]_{1j} &= \rho^{1-j} u_{N+2-j, N+1}(x) \\ &= \rho^{1-j} \bar{b}_{j+1}(x) \\ &= u_{1j}^*(x). \end{aligned}$$

$$\begin{aligned} \text{For } 2 \leq i \leq N+1 \text{ and } i-1 \leq j \leq N : [Z^T U(x) Z^{-1}]_{ij} &= \rho^{i-j} u_{N+2-j, N+2-i}(x) \\ &= \rho^{i-j} b_{j+2-i}(x) \\ &= u_{ij}^*(x). \end{aligned}$$

$$\begin{aligned} \text{For } 2 \leq i \leq N+1 \text{ and } j = N+1 : [Z^T U(x) Z^{-1}]_{i, N+1} &= \rho^{i-N-1} u_{1, N+2-i}(x) \\ &= \rho^{i-N-1} c_{N+2-i}(x) \\ &= u_{i, N+1}^*(x). \end{aligned}$$

$$\begin{aligned} \text{For } 3 \leq i \leq N+1 \text{ and } 1 \leq j \leq i-2 : [Z^T U(x) Z^{-1}]_{ij} &= \rho^{i-j} u_{N+2-j, N+2-i}(x) \\ &= 0 \\ &= u_{ij}^*(x). \end{aligned}$$

Therefore, we get:

$$U^*(x) = Z^T U(x) Z^{-1}.$$

for any k , we obtain:

$$\begin{aligned} f_k^*(x_1, x_2, \dots, x_k) &= {}^T \underline{q}^* U^*(x_1) U^*(x_2) \cdots U^*(x_k) \underline{e} \\ &= {}^T \underline{q}^* (Z^T U(x_1) Z^{-1}) (Z^T U(x_2) Z^{-1} \cdots (Z^T U(x_k) Z^{-1} \underline{e} \\ &= {}^T \underline{q}^* Z \{ {}^T U(x_1) {}^T U(x_2) \cdots {}^T U(x_k) \} Z^{-1} \underline{e} \\ &= {}^T \underline{e} \{ {}^T U(x_1) {}^T U(x_2) \cdots {}^T U(x_k) \} \underline{q} \\ &= {}^T \{ {}^T \underline{q} U(x_k) U(x_{k-1}) \cdots U(x_2) U(x_1) \underline{e} \} \\ &= f_k(x_k, x_{k-1}, \dots, x_2, x_1). \end{aligned}$$

Using another expression, we have:

$$\Pr\{d_1^* \leq x_1, d_2^* \leq x_2, \dots, d_k^* \leq x_k\} = \Pr\{d_1 \leq x_k, d_2 \leq x_{k-1}, \dots, d_k \leq x_1\}$$

and

$$\begin{aligned} \text{cov}(d_1^*, d_k^*) &= \text{cov}(d_k, d_1) \\ &= \text{cov}(d_1, d_k). \end{aligned}$$

So the theorem is proved.

Akihiko Ishikawa
 Fundamentals of Natural Science
 College of Humanities
 and Social Sciences
 Iwate University
 Ueda 3-18-34, Morioka-shi,
 Iwate 020, JAPAN