GAME THEORETIC ANALYSIS FOR AN OPTIMAL STOPPING PROBLEM BY MEANS OF MOMENTS OF A DISTRIBUTION FUNCTION

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Abstract Let $X_1, X_2, \dots, X_n, \dots$ be mutually independent random variables with a common cdf F , which is unknown but belongs to some class $\mathcal{F}$ of cdf's. The class $\mathcal{F} = \mathcal{F}(\mu, \sigma^2, M)$ is the set of all cdf's whose mean, variance and domain are $-\infty < \mu < \infty, 0 < \sigma^2 < \infty$, and $[\mu - M, \mu + M]$ respectively. It is assumed that they are known. Under an observation cost $c, 0 < c < \infty$, we consider a stopping problem $\phi(x, F)$ as a two-person zero-sum game in which the player 1 decides his stopping set $\{X > x\}, x \in R$, and the player 2 chooses her cdf F in $\mathcal{F}$. We analyze the upper bound problem $\phi^u = \sup_{x \in R} \sup_{F \in \mathcal{F}} \phi(x, F)$ and the game problem $\phi^s = \text{value}_{x \in R, F \in \mathcal{F}} \phi(x, F)$ to derive a simple and meaningful solution with the parameters c, μ, σ and M .

1. Introduction

1.1. Preliminaries

Let $X_1, X_2, \dots, X_n, \dots$ be mutually independent and identically distributed random variables with a common cdf $F(t) = P\{X \leq t\}$ such that $E[X^+] = \int_{R_+} t dF(t) < \infty$, where $R = (-\infty, \infty), R_+ = [0, \infty)$. A positive observation cost $c (\in R_{++} = (0, \infty))$ is incurred to the observation of each $X_n, n \geq 1$. If the observation process is stopped after X_n is observed, a reward $X_n - nc$ is received.

When the cdf F is known precisely, an optimal stopping problem is to find a stopping time N which maximizes the expected reward $E[X_N - cN]$. An integer valued random variable N is called a stopping time if, for each n , the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \dots$. This is a typical example of the stationary stopping problem (cf. Chow, Robbins and Siegmund [1, p.56]), in which the optimal stopping time N is necessarily of the form: to stop at $N = \min\{n \mid X_n \in S\}$ for some stopping set $S \subset R$, and S is stationary and of a control-limit-type $\{X \geq x\}$ or $\{X > x\}$ for some $x \in R$, where x is called a stopping level. For this, we define that a stopping level x (or $x - 0$) means a stopping set $\{X > x\}$ (or $\{X \geq x\}$) respectively.

For any stopping level x and for any cdf F , we define an expected reward $\phi(x, F) = E[X_N - cN]$ of the stopping problem by

$$(1.1) \quad \phi(x, F) = \frac{\int_{(x, \infty)} t dF(t) - c}{\bar{F}(x)} = x + \frac{\int_{(x, \infty)} (t - x) dF(t) - c}{\bar{F}(x)},$$

where $\bar{F}(x) = 1 - F(x)$. Note that $\bar{F}(x) \rightarrow 0$ and $\phi \rightarrow -\infty$ as $x \rightarrow \infty$ and that $\bar{F}(x) \rightarrow 1$ and $\phi \rightarrow \mu_F - c$ as $x \rightarrow -\infty$ where $\mu_F = E[X] = \int_R t dF(t)$. It means that to continue the observation process indefinitely gives us the reward $-\infty$ and to stop the process immediately gives us the reward $\mu_F - c$.

By the assumption $E[X^+] < \infty$, using integration by parts, we have

$$\int_{(x,\infty)} (t-x)dF(t) = - \int_{(x,\infty)} (t-x)d\bar{F}(t) = \int_x^\infty \bar{F}(t)dt = \int_{[x,\infty)} (t-x)dF(t),$$

which is continuous in x . To handle this expectation, it is convenient to define $T_F(x)$,

$$(1.2) \quad T_F(x) = \int_x^\infty (t-x)dF(t) = \int_x^\infty \bar{F}(t)dt.$$

It has some useful properties, which can be obtained by the elementary analytical technique. Then we give Lemma 1 without proof.

Lemma 1. $T_F(x)$ is continuous, non-negative, convex and non-increasing function of x . It satisfies that $T_F(x) \geq (\mu_F - x)^+$ for any $x \in R$ and that $T_F(x) \rightarrow +\infty$ as $x \rightarrow -\infty$ and $T_F(x) \rightarrow 0$ as $x \rightarrow +\infty$. T_F has a derivative a.e.. Moreover, if $T_F(x)$ is positive at any point x , it is strictly decreasing at x .

Now, redefining the expected reward $\phi(x, F)$ by (1.1') for any stopping level x and for any cdf F , we will have the optimal expected reward $\phi^o(F)$ for any cdf F :

$$(1.1') \quad \phi(x, F) = x + \frac{T_F(x) - c}{\bar{F}(x)}.$$

$$(1.3) \quad \phi^o(F) \stackrel{\text{def}}{=} \sup_{x \in R} \phi(x, F).$$

$$(1.4) \quad \frac{d\phi(x, F)}{dF(x)} = \frac{T_F(x) - c}{\bar{F}^2(x)}.$$

The right hand side of (1.4) changes the sign from $+$ to $-$ at most one time as x goes from $-\infty$ to $+\infty$. From Lemma 1, the equation $T_F(x) = c$ for any fixed $c (c > 0)$ has a unique solution $x^o(F) \stackrel{\text{def}}{=} (T_F)^{-1}(c)$, so that the set of optimal stopping levels $\mathbf{x}^o(F)$ (which must contain the point $x^o(F)$) of (1.3) is given by

$$(1.5) \quad \mathbf{x}^o(F) = \{x \mid F(x) = F(x^o(F))\}.$$

Since the cdf F is right-continuous, this set is an interval of the form $[a, b)$. If the cdf F has a positive density in the neighborhood of $x^o(F)$, the set $\mathbf{x}^o(F)$ contains only one point $x^o(F)$ i.e., $\mathbf{x}^o(F) = \{x^o(F)\}$.

We have the optimal expected reward $\phi^o(F)$,

$$(1.3') \quad \phi^o(F) = x^o(F) = \phi(\mathbf{x}^o(F), F),$$

where $\phi(A, F)$ means $\phi(x, F)$ for any x in a set A .

Let $a = \min\{x \mid x \in \mathbf{x}^o(F)\}$ and $b = \sup\{x \mid x \in \mathbf{x}^o(F)\}$, then the set $\mathbf{x}^o(F)$ of (1.5) is represented as the interval $[a, b)$. Since F has no probability mass in (a, b)

$$\bar{F}(a) = P\{X > a\} = P\{X \geq b\} = \bar{F}(b-0) \stackrel{\text{def}}{=} \lim_{x \uparrow b} \bar{F}(x), \quad \int_{(a,\infty)} t dF(t) = \int_{[b,\infty)} t dF(t),$$

and by (1.1) it holds that,

$$(1.3'') \quad x^o(F) = \phi(a, F) = \lim_{x \uparrow b} \phi(x, F) \stackrel{\text{def}}{=} \phi(b-0, F).$$

Then, we have Lemma 2.

Lemma 2. For any given cdf F , the following stopping sets or stopping levels (i)(ii)(iii) are optimal, and the optimal expected reward is given by (1.3'');

- (i) the set $\{X > a\}$ or level a where $a = \min\{x \mid x \in x^o(F)\}$,
- (ii) the set $\{X \geq b\}$ or level $b-0$ where $b = \sup\{x \mid x \in x^o(F)\}$,
- (iii) the set $\{X > x\}$ ($\{X \geq x\}$) or level x ($x-0$) where $\forall x \in (a, b)$.

1.2. Statement of the Problem

The underlying premise is that some information is available about certain moments of an unknown cdf F . This determines some class $\mathcal{F}$ of cdf's. This limited information is to be used in order to obtain an upper (or/and lower) bound for our stopping problem $\phi(x, F)$ of (1.1'). In this paper, we consider the classes $\mathcal{F}$ of all cdf's F which are unknown except for their first two moments and domain.

First, we shall consider the maximal bound ϕ^u for $\phi(x, F)$ on $R \times \mathcal{F}$

$$(1.6) \quad \phi^u = \sup_{x \in R} \sup_{F \in \mathcal{F}} \phi(x, F) = \sup_{F \in \mathcal{F}} \phi^o(F) = \phi(x^o(F^u), F^u) = \phi(x^u, F^u),$$

where (x^u, F^u) is a joint maximizing point of $\phi(x, F)$.

Second, we shall consider $\phi(x, F)$ as a two-person zero-sum game in which the player 1 (gambler) decides his level x in R and the player 2 (nature) chooses her cdf F in $\mathcal{F}$, before the observation of $\{X_n; n \geq 1\}$. Then the minimax value ϕ^* and the maximin value ϕ_* on $R \times \mathcal{F}$,

$$(1.7) \quad \phi^* = \inf_{F \in \mathcal{F}} \sup_{x \in R} \phi(x, F) = \inf_{F \in \mathcal{F}} \phi^o(F) = \phi(x^o(F^*), F^*) = \phi(x^*, F^*),$$

$$(1.8) \quad \phi_* = \sup_{x \in R} \inf_{F \in \mathcal{F}} \phi(x, F) = \phi(x_*, F_*),$$

and the saddle value ϕ^s , the saddle point (x^s, F^s) in $R \times \mathcal{F}$,

$$(1.9) \quad \phi^s = \text{value}_{x \in R, F \in \mathcal{F}} \phi(x, F) = \phi(x^s, F^s),$$

will be derived for the following two classes $\mathcal{F}(\mu, \sigma^2)$ and $\mathcal{F}(\mu, \sigma^2, M)$ of cdf's.

The class $\mathcal{F}(\mu, \sigma^2, M)$ is the set of cdf's whose mean μ , variance σ^2 and domain $[\mu - M, \mu + M]$ are assumed to be known.

$$(1.10) \quad \mathcal{F}(\mu, \sigma^2, M) = \left\{ F \mid \int_A dF(t) = 1, \int_A t dF(t) = \mu, \right. \\ \left. \int_A t^2 dF(t) = \mu^2 + \sigma^2 \text{ where } A = [\mu - M, \mu + M], M \geq \sigma \right\}$$

The class $\mathcal{F}(\mu, \sigma^2)$ is $\mathcal{F}(\mu, \sigma^2, M)$ where M is arbitrary in $[\sigma, \infty)$, and $\mathcal{F}(\mu)$ is $\mathcal{F}(\mu, \sigma^2)$ where σ^2 is arbitrary in R_{++} .

The organization of this paper is as follows. In Section 2, we study some basic results for our stopping problem $\phi(x, F)$ and $T_F(x)$ and properties of cdf's F in the class $\mathcal{F}$, which play fundamental roles in all that follows. Section 3 is a discussion about the existence and the

derivation of the maximizing point and the saddle point for $\phi(x, F)$ in the class $\mathcal{F} = \mathcal{F}(\mu, \sigma^2)$. In fact, the class $\mathcal{F}$ is so rich for the player 2 that the player 1 must stop immediately and may receive the reward of $\mu - c$ as the saddle value. In this case, the information of the value σ^2 is useless for the player 1.

To remove the non-interesting feature appearing in the game theoretic treatment in Section 3, we shall continue the discussion about the more interesting class $\mathcal{F} = \mathcal{F}(\mu, \sigma^2, M)$ in Section 4. In this case, the information of the value σ^2 and M is useful for the player 1. The player 1 can receive the reward of $\mu + (\sigma^2/M - c)^+ - c$ as the saddle value which is greater than $\mu - c$ in Section 3.

2. Some Fundamental Lemmas

The general theory of a duality theorem is presented in the textbook by Ekeland and Teman [2, Chapter 4]. It is hardly difficult to obtain the condition of the existence of a saddle point for our stopping problem $\phi(x, F)$. For an inventory problem where this condition is satisfied, we have derived a saddle point [4]. The following two propositions are used to check a candidate for a saddle point of $\phi(x, F)$ on $R \times \mathcal{F}$.

Proposition 1. [2, p.166] *If ϕ is a function defined on $R \times \mathcal{F}$ with real values, then*

$$(2.1) \quad \phi_* = \sup_{x \in R} \inf_{F \in \mathcal{F}} \phi(x, F) \leq \phi^* = \inf_{F \in \mathcal{F}} \sup_{x \in R} \phi(x, F).$$

Proposition 2. [2, p.168] *If there exist $x^s \in R, F^s \in \mathcal{F}$ and a real value ϕ^s such that*

$$(2.2) \quad \begin{aligned} \phi(x^s, F) &\geq \phi^s \text{ for any } F \in \mathcal{F}, \\ \phi(x, F^s) &\leq \phi^s \text{ for any } x \in R, \end{aligned}$$

then (x^s, F^s) is a saddle point of ϕ and it holds that

$$(2.3) \quad \phi^s = \text{value}_{x \in R, F \in \mathcal{F}} \phi(x, F) = \sup_{x \in R} \inf_{F \in \mathcal{F}} \phi(x, F) = \inf_{F \in \mathcal{F}} \sup_{x \in R} \phi(x, F).$$

Moreover, the set of saddle points of ϕ is of the form $(\mathbf{x}^s, \mathcal{F}^s)$, where $\mathbf{x}^s \subset R$ and $\mathcal{F}^s \subset \mathcal{F}$.

Before beginning our discussion, we shall establish some basic results for $\phi(x, F), T_F(x)$ and the class $\mathcal{F}$.

Let a random variable X has a mean μ with a cdf $F_\mu(t)$, then the new random variable $X - \mu$ has the mean 0 with the cdf $F_0(t) = F_\mu(t + \mu)$. The following Lemma 3 follows immediately from the definition (1.1) of $\phi(x, F)$.

Lemma 3.

$$(2.4) \quad \phi(x, F_\mu) = \mu + \phi(x - \mu, F_0) \text{ for any } x \in R.$$

Therefore, we may assume without loss of generality that all the cdf's in $\mathcal{F}$ have the mean 0. So that, we shall analyze the stopping problem in only two classes $\mathcal{F}(0, \sigma^2)$ and $\mathcal{F}(0, \sigma^2, M)$.

Lemma 4. *For cdf's F_i and non-negative numbers $\lambda_i, i = 1, 2, \dots, n$, such that $\sum_{i=1}^n \lambda_i = 1$, let $F = \sum_{i=1}^n \lambda_i F_i$. Then*

$$(2.5) \quad \phi(x, F) = \sum_{j=1}^n \lambda_j(x) \phi(x, F_j) \text{ for any } x \in R, \text{ where } \lambda_j(x) = \frac{\lambda_j \bar{F}_j(x)}{\sum_{i=1}^n \lambda_i \bar{F}_i(x)}.$$

Proof. It follows at once from the definition (1.1) of ϕ . $\square$

Note, from Lemma 4, that we find $\phi(x, F)$ is not necessarily convex in F for any fixed $x \in R$, and that the well-known sufficient condition for the existence of a saddle point is not satisfied.

Let G_n be a discrete cdf which has n probability masses p_i , $p_i > 0$, at n points t_i , $i = 1, 2, \dots, n$, respectively ($\sum_{i=1}^n p_i = 1$), i.e., it is represented as

$$(2.6) \quad G_n(t) = (< t_1, \dots, t_n > < p_1, \dots, p_n >),$$

and $\mathcal{G}_n(\mu, \sigma^2)$ be all discrete cdf's G_n in $\mathcal{F}(\mu, \sigma^2)$. Let

$$(2.7) \quad G_2(t; q) = (< -\frac{\sigma}{q}, \sigma q > < \frac{q^2}{1+q^2}, \frac{1}{1+q^2} >)$$

for any $q, 0 < q < \infty$. Then $G_2(t; q)$ is the only two-point cdf which has the mean 0 and the variance σ^2 .

Lemma 5. The class $\mathcal{G}_2(0, \sigma^2)$ of two-point cdf's is represented with a parameter q , $0 < q < \infty$, as follows,

$$\mathcal{G}_2(0, \sigma^2) = \{G_2(\cdot; q) \mid 0 < q < \infty\}.$$

Let us define

$$(2.8) \quad T_{\mathcal{F}}^u(x) = \sup_{F \in \mathcal{F}} T_F(x), \quad T_{\mathcal{F}}^l(x) = \inf_{F \in \mathcal{F}} T_F(x).$$

Then, from Lemma 1, we have the following lemma.

Lemma 6. Suppose $\mathcal{F} = \mathcal{F}(0)$ so that $\mu_F = 0$ for all $F \in \mathcal{F}$, then $T_{\mathcal{F}}^u(x)$ and $T_{\mathcal{F}}^l(x)$ have the same property as $T_F(x)$ in Lemma 1 with μ_F replaced by 0, except that $T_{\mathcal{F}}^l(x)$ is not always convex.

From above Lemma 6, $T_{\mathcal{F}}^u(x)$ and $T_{\mathcal{F}}^l(x)$ have inverse functions $(T_{\mathcal{F}}^u)^{-1}(c)$ and $(T_{\mathcal{F}}^l)^{-1}(c)$ for all $c, c > 0$, respectively. Thus we have shown the existence of the values of ϕ^u and ϕ^* :

$$(2.9) \quad \phi^u = \sup_{F \in \mathcal{F}} \{x \mid T_F(x) = c\} = (T_{\mathcal{F}}^u)^{-1}(c),$$

$$(2.10) \quad \phi^* = \inf_{F \in \mathcal{F}} \{x \mid T_F(x) = c\} = (T_{\mathcal{F}}^l)^{-1}(c).$$

3. The Class $\mathcal{F}(\mu, \sigma^2)$

In this section, we shall derive the maximal bound ϕ^u and the saddle value ϕ^s for $\phi(x, F)$ in the class $\mathcal{F} = \mathcal{F}(0, \sigma^2)$.

$$(3.1) \quad \mathcal{F}(0, \sigma^2) = \{F \mid \int_R dF(t) = 1, \int_R t dF(t) = 0, \int_R t^2 dF(t) = \sigma^2\}.$$

3.1 The maximal bound ϕ^u

First, we prepare a useful proposition related to Schwartz inequalities.

Proposition 3. (Feller [3, p.151]) If F is an arbitrary cdf, then

$$(3.2) \quad \left(\int_A u(t)v(t)dF(t) \right)^2 \leq \left(\int_A u^2(t)dF(t) \right) \left(\int_A v^2(t)dF(t) \right)$$

for any set A and any functions u, v for which the integrals on the right exist. Furthermore, the equality sign holds if and only if

$$(3.3) \quad \int_A (au(t) + bv(t))^2 dF(t) = 0 \text{ for some } a, b \in R.$$

Remark on Proposition 3. If u and v are linearly dependent, i.e., for some $a, b \in R$, $au(t) + bv(t) = 0$, the condition (3.3) is satisfied for all $F \in \mathcal{F}$, and that if u and v are linearly independent, the condition (3.3) is satisfied only when the cdf F is degenerated at one point in a set A .

We shall calculate ϕ^u and the maximizing point (x^u, F^u) of the problem (2.9) by Proposition 3.

$$(3.4) \quad \left(\int_{(x, \infty)} (t-x)dF(t) \right)^2 \leq \left(\int_{(x, \infty)} dF(t) \right) \left(\int_{(x, \infty)} (t-x)^2 dF(t) \right),$$

$$(3.4') \quad \left(\int_{(-\infty, x]} (t-x)dF(t) \right)^2 \leq \left(\int_{(-\infty, x]} dF(t) \right) \left(\int_{(-\infty, x]} (t-x)^2 dF(t) \right).$$

For all $F \in \mathcal{F}$, $T_F(x) = \int_{(x, \infty)} (t-x)dF(t) = c$ from (2.9) and $\int_R dF(t) = 1$, $\int_R t dF(t) = 0$, $\int_R t^2 dF(t) = \sigma^2$ from (3.1). Then, putting $p = \int_{(x, \infty)} dF(t)$, it holds that by adding (3.4) to (3.4'),

$$(3.5) \quad \frac{c^2}{p} + \frac{(x+c)^2}{1-p} \leq \sigma^2 + x^2, \quad 0 < p < 1,$$

$$-\frac{c}{p} - \sigma \sqrt{\frac{1-p}{p}} \leq x \leq -\frac{c}{p} + \sigma \sqrt{\frac{1-p}{p}}, \quad 0 < p < 1.$$

Putting $q = \sqrt{(1-p)/p}$, $0 < q < \infty$, the upper bound of x (the right-hand side of (3.5)) is represented by the parameter q .

$$(3.6) \quad \sup_{0 < p < 1} \left(\sigma \sqrt{\frac{1-p}{p}} - \frac{c}{p} \right) = \sup_{0 < q < \infty} (\sigma q - c(1+q^2)) = \sup_{0 < q < \infty} \left[-c \left(q - \frac{\sigma}{2c} \right)^2 + \frac{\sigma^2}{4c} - c \right] = \frac{\sigma^2}{4c} - c,$$

where the supremum is attained at $q = \sigma/2c$ i.e., $p = 4c^2/(\sigma^2 + 4c^2)$.

Then, we obtain the maximal bound ϕ^u .

$$(3.7) \quad \phi^u = \sup_{F \in \mathcal{F}} \{x \mid T_F(x) = c\} = \frac{\sigma^2}{4c} - c.$$

Since the equality holds in two Schwartz inequalities (3.4) and (3.4'), from the remark of Proposition 3, the maximizing cdf F^u should be the two-point cdf. Then, we have

$$(3.8) \quad F^u(t) = G_2(t; \frac{\sigma}{2c}) = (< -2c, \frac{\sigma^2}{2c} > < \frac{\sigma^2}{\sigma^2 + 4c^2}, \frac{4c^2}{\sigma^2 + 4c^2} >),$$

$$(3.9) \quad x^u \in \mathbf{x}^u = \mathbf{x}^o(F^u) = [-2c, \frac{\sigma^2}{2c}) .$$

Theorem 1. For a class $\mathcal{F}(0, \sigma^2)$ of cdf's, the maximal bound ϕ^u is $\sigma^2/4c - c$ by (3.7) and all the maximizing points $(x^u, F^u) \in (\mathbf{x}^u, \{F^u\})$ are given by $F^u(t) = G_2(t; \sigma/2c)$ in (3.8) and $\mathbf{x}^u = [-2c, \sigma^2/2c)$ in (3.9).

Remark on Theorem 1. From Lemma 2, the equation (3.9) means that the player 1 may decide a stopping level x^u for some $x^u \in [-2c, \sigma^2/2c)$ or $\sigma^2/2c - 0$. If the player 1 decides any of the above stopping levels, he stops the process whenever $X_n = \sigma^2/2c$ is observed because the player 2 chooses only one cdf given by (3.8).

Corollary 1. For a class $\mathcal{F}(\mu, \sigma^2)$ of cdf's, the maximal bound ϕ^u is $\mu + \sigma^2/4c - c$ and all the maximizing points $(x^u, F^u) \in (\mathbf{x}^u, \{F^u\})$ are given by $F^u(t - \mu) = G_2(t; \sigma/2c)$ and $\mathbf{x}^u = [\mu - 2c, \mu + \sigma^2/2c)$.

In the theory of extreme order statistics, by letting $Y = \max_{1 \leq n \leq \infty} [X_n - cn]$, it holds that $E[Y^+] < \infty$ if and only if $E[X^+] < \infty$ (See Pickands III [5, Cor. 2.2]). Corollary 1 also says that the maximal bound of $E[Y^+]$ is calculated by using μ and σ^2 if the cdf F is restricted in the class $\mathcal{F}(\mu, \sigma^2)$.

3.2 The minimax value ϕ^*

Second, we shall calculate the minimax value ϕ^* of (2.10) and the minimax-mizing point $(x^*, F^*) \in (\mathbf{x}^*, \mathcal{F}^*)$.

From Lemma 6, $T_{\mathcal{F}}^l(x) \geq (-x)^+$ for all $x \in R$. Then it holds that $T_{F^*}(x) = (-x)^+ \leq T_{\mathcal{F}}^l(x)$ for $x \in (-\infty, -c]$ if a cdf F^* , which has all the mass on $[-c, \infty)$, is contained in $\mathcal{F}$. Since $T_{F^*}(x) = (-x)^+$ is strictly decreasing on $(-\infty, -c]$, we have

$$(3.10) \quad \phi^* = \inf_{F \in \mathcal{F}} \{x \mid T_F(x) = c\} = \{x \mid T_{F^*}(x) = c\} = -c .$$

Such a class $\mathcal{F}^*$ of cdf's F^* always exists in $\mathcal{F}$ for all $c, c > 0$.

$$(3.11) \quad \mathcal{F}^* = \{F \mid \int_{[-c, \infty)} dF(t) = 1, F \in \mathcal{F}\} .$$

In particular, we can find the class $\mathcal{G}_2^* = \mathcal{G}_2^*(0, \sigma^2)$ of two-point cdf's in $\mathcal{F}^*$ from Lemma 5.

$$(3.11') \quad \mathcal{G}_2^* = \{G_2(\cdot; q) \mid q \geq \frac{\sigma}{c}\} .$$

It is easily shown that for any $F^* \in \mathcal{F}^*$ it is optimal for the player 1 to stop the process immediately. That is,

$$(3.12) \quad \mathbf{x}^* = \mathbf{x}^o(F^*) = (-\infty, -c) \text{ for all } F^* \in \mathcal{F}^* .$$

Theorem 2. For a class $\mathcal{F}(0, \sigma^2)$ of cdf's, the minimax value ϕ^* is $-c$ by (3.10) and all the minimax-mizing points $(x^*, F^*) \in (\mathbf{x}^*, \mathcal{F}^*)$ are given by (3.11) and (3.12). In particular, there exists the class $\mathcal{G}_2^*$ of two-point cdf's in $\mathcal{F}^*$ by (3.11').

Corollary 2. For a class $\mathcal{F}(\mu, \sigma^2)$ of cdf's, the minimax value ϕ^* is $\mu - c$, and all the minimax-mizing points $(x^*, F^*) \in (\mathbf{x}^*, \mathcal{F}^*)$ are given by

$$\mathcal{F}^* = \{F \mid \int_{[\mu-c, \infty)} dF(t) = 1, F \in \mathcal{F}(\mu, \sigma^2)\}, \mathbf{x}^* = (-\infty, \mu - c) .$$

3.3 The saddle value ϕ^s

Now, we shall derive the saddle value ϕ^s for $\phi(x, F)$ in $\mathcal{F} = \mathcal{F}(\mu, \sigma^2)$. We have a candidate $(\mathbf{x}^*, \mathcal{F}^*)$ for a set of saddle points $(\mathbf{x}^s, \mathcal{F}^s)$ to be checked by Proposition 2. Since $(\mathbf{x}^*, \mathcal{F}^*)$ includes $(\mathbf{x}^s, \mathcal{F}^s)$ in general, we shall make an attempt to prove the inverse. From Theorem 2 it holds that $\phi(x, F) = -c$ for all $x \in \mathbf{x}^*$ and all $F \in \mathcal{F}^*$. Moreover, from (3.12), it holds that for any fixed $x \notin \mathbf{x}^*$, $\phi(x, F) \leq -c$ for all $F \in \mathcal{F}^*$. It suffices to prove for any fixed $F \in \mathcal{F}$,

$$(3.13) \quad \phi(x, F) \geq -c \text{ for all } x \in \mathbf{x}^*.$$

Proof of (3.13). For any fixed F , it holds that $\phi(-\infty, F) = -c$ and $\phi(x, F)$ is a unimodal function of x and maximized at the point $x^o(F)$ from the remark of (1.1). When (i) $x^o(F) \geq -c$, $\phi(x, F) \geq -c$ for $x \in (-\infty, -c]$, i.e. (3.13) is true in the case (i). When (ii) $x^o(F) \leq -c$, $\phi(x, F) \geq \phi(-c, F)$ for $x \in [x^o(F), -c]$, so that (3.13) is true in the case (ii) if we can prove $\phi(-c, F) \geq -c$.

For $F \in \mathcal{F}$, F has the mean 0, so that $\bar{F}(-c) > 0$ and

$$\int_{(-c, \infty)} t dF(t) = - \int_{(-\infty, -c]} t dF(t) \geq c(1 - \bar{F}(-c)).$$

Thus, we have from (1.1),

$$(3.14) \quad \phi(-c, F) = \frac{\int_{(-c, \infty)} t dF(t) - c}{\bar{F}(-c)} \geq \frac{c(1 - \bar{F}(-c)) - c}{\bar{F}(-c)} = -c.$$

This completes the proof. $\square$

Hence, we have the main theorem in this chapter.

Theorem 3. For a class $\mathcal{F}(0, \sigma^2)$ of cdf's, the saddle value ϕ^s is $-c$ and all the saddle points $(x^s, F^s) \in (\mathbf{x}^s, \mathcal{F}^s)$ are given by $\mathbf{x}^s = \mathbf{x}^*$, $\mathcal{F}^s = \mathcal{F}^*$ and $\mathcal{G}_2^s = \mathcal{G}_2^* \subset \mathcal{F}^s$ defined in Theorem 2.

Corollary 3. For a class $\mathcal{F}(\mu, \sigma^2)$ of cdf's, the saddle value ϕ^s is $\mu - c$ and all the saddle points $(x^s, F^s) \in (\mathbf{x}^s, \mathcal{F}^s)$ are given by $\mathbf{x}^s = \mathbf{x}^*$, $\mathcal{F}^s = \mathcal{F}^*$ defined in Corollary 2.

Theorem 3 says the class $\mathcal{F}(\mu, \sigma^2)$ is so rich for the player 2 that the player 1 must stop immediately. In this case, the information of the value σ^2 is useless for the player 1.

If we consider the problem in the class

$$(3.15) \quad \mathcal{F}(\mu, \cdot, M) = \{F \mid \int_A dF(t) = 1, \int_A t dF(t) = \mu, A = [\mu - M, \mu + M]\},$$

the player 2 may choose the cdf which is degenerated at the point μ and the player 1 receives the reward of $\mu - c$, in which the information of the value M is useless for the player 1.

To remove the non-interesting strategies for the player 1 in the above two cases $\mathcal{F}(\mu, \sigma^2)$ and $\mathcal{F}(\mu, \cdot, M)$, we shall analyze the problem in the more restrictive class $\mathcal{F}(\mu, \sigma^2, M)$ of (1.10) in the next section.

4. The Class $\mathcal{F}(\mu, \sigma^2, M)$

In this section, we shall derive the maximal bound ϕ^u and the saddle value ϕ^s in the more restrictive and interesting class $\mathcal{F} = \mathcal{F}(0, \sigma^2, M)$ (see (1.10)).

$$\mathcal{F}(0, \sigma^2, M) = \{F \mid \int_A dF(t) = 1, \int_A t dF(t) = 0, \int_A t^2 dF(t) = \sigma^2, A = [-M, M], \sigma \leq M\}.$$

If $\sigma = M > 0$, the class $\mathcal{F}$ contains only one cdf $G_2(t; 1) = (< -M, M > < 1/2, 1/2 >)$. So, it is assumed that $\sigma < M$.

We calculate ϕ^u and the maximizing point (x^u, F^u) in the same fashion as (3.6) with the additional constraint $\sigma/M \leq q \leq M/\sigma$,

$$(4.1) \quad \phi^u = \max_{\sigma/M \leq q \leq M/\sigma} [\sigma q - c(1 + q^2)] .$$

Theorem 4. For a class $\mathcal{F}(0, \sigma^2, M)$ of cdf's, $\sigma < M$, the maximal bound ϕ^u and all the maximizing points $(x^u, F^u) \in (\mathbf{x}^u, \mathcal{F}^u)$ are as follows:

(i) When $0 \leq c \leq \sigma^2/2M$,

$$\begin{aligned} \phi^u &= M - c(1 + \frac{M^2}{\sigma^2}) , \quad \mathbf{x}^u = [-\frac{\sigma^2}{M}, M) , \\ F^u(t) &= G_2(t; \frac{M}{\sigma}) = (< -\frac{\sigma^2}{M}, M > < \frac{M^2}{\sigma^2 + M^2}, \frac{\sigma^2}{\sigma^2 + M^2} >) . \end{aligned}$$

(ii) When $\sigma^2/2M \leq c \leq M/2$, the same result as Theorem 1 holds, i.e.,

$$\begin{aligned} \phi^u &= \frac{\sigma^2}{4c} - c , \quad \mathbf{x}^u = [-2c, \frac{\sigma^2}{2c}) , \\ F^u(t) &= G_2(t; \frac{\sigma}{2c}) = (< -2c, \frac{\sigma^2}{2c} > < \frac{\sigma^2}{\sigma^2 + 4c^2}, \frac{4c^2}{\sigma^2 + 4c^2} >) . \end{aligned}$$

(iii) When $M/2 \leq c \leq M$,

$$\begin{aligned} \phi^u &= \frac{\sigma^2}{M} - c(1 + \frac{\sigma^2}{M^2}) , \quad \mathbf{x}^u = [-M, \frac{\sigma^2}{M}) , \\ F^u(t) &= G_2(t; \frac{\sigma}{M}) = (< -M, \frac{\sigma^2}{M} > < \frac{\sigma^2}{\sigma^2 + M^2}, \frac{M^2}{\sigma^2 + M^2} >) . \end{aligned}$$

Remark on Theorem 4. When $c > M$, i.e. the observation cost is larger than the maximal value of the observation, it is natural for the player 1 to stop immediately, i.e., $\phi^u = -c$, $\mathcal{F}^u = \mathcal{F}$ and $\mathbf{x}^u = (-\infty, -M)$. The shape of ϕ^u as the function of c is illustrated in Figure 1 at the end of this paper. Furthermore, if the results for the class $\mathcal{F}(\mu, \sigma^2, M)$ are required, Lemma 2 is useful for the theorems in this section.

Now, we shall derive the saddle value ϕ^s . By calculating $T_F(x)$ of (1.2) for the cdf's $G_2(\cdot; M/\sigma)$ and $G_2(\cdot; \sigma/M)$, we have, from (2.8) and Lemma 6,

$$(4.2) \quad T_{\mathcal{F}}^{\ell}(x) \geq (-x)^+ = -x = T_{G_2(\cdot; M/\sigma)}(x) \text{ for } x \in [-M, -\sigma^2/M] ,$$

$$(4.3) \quad T_{\mathcal{F}}^{\ell}(x) \geq (-x)^+ = 0 = T_{G_2(\cdot; \sigma/M)}(x) \text{ for } x \in [\sigma^2/M, M] .$$

If the observation cost c is a constant such that $M \geq c \geq \sigma^2/M$, the equality $\phi^* = (T_{\mathcal{F}}^{\ell})^{-1}(c) = -c$ is satisfied from (4.2). In this case, the saddle value ϕ^s and the saddle point (x^s, F^s) are the same ones given in Theorem 3, because the player 2 can choose a cdf $G_2(\cdot, M/\sigma)$ in this restrictive class $\mathcal{F} = \mathcal{F}(0, \sigma^2, M)$, i.e., $G_2(\cdot; M/\sigma) \in \mathcal{F}(0, \sigma^2, M) \subset \mathcal{F}(0, \sigma^2)$. So that we confine our consideration to the case:

$$(4.4) \quad 0 < c < \sigma^2/M .$$

On the other hand, from (4.2) and (4.3), it holds that

$$(4.2') \quad \inf_{F \in \mathcal{F}} \phi(x, F) \leq \phi(x, G_2(\cdot; M/\sigma)) = -c \text{ for } x \in [-M, -\sigma^2/M],$$

$$(4.3') \quad \inf_{F \in \mathcal{F}} \phi(x, F) \leq \phi(x, G_2(\cdot; \sigma/M)) = -\infty \text{ for } x \in [\sigma^2/M, M],$$

because the player 1 stops immediately in the case of (4.2') or he cannot stop in the case of (4.3'). Then, the player 1 must decide his stopping level x in the interval

$$(4.5) \quad \mathbf{x}^M \stackrel{\text{def}}{=} \left[-\frac{\sigma^2}{M}, \frac{\sigma^2}{M}\right),$$

in order not to make his reward $\inf_{F \in \mathcal{F}} \phi(x, F) \leq -c$, where $-c$ is the reward of immediately stopping or the saddle value $\phi^s = -c$ in Section 3.

Hereafter, we consider the stopping problem $\phi(x, F)$ under the restriction of (4.4) and (4.5). To begin with, we prepare the following three lemmas.

Lemma 7. If F is an arbitrary cdf defined on a finite interval $[a, b]$ with the mean μ ($a \leq \mu \leq b$, $a < b$), then

$$\int_{[a,b]} (t - \mu)^2 dF(t) \leq (\mu - a)(b - a).$$

Furthermore, the equality holds if and only if

$$(4.6) \quad F = \left(\left\langle a, b \right\rangle \left\langle \frac{b - \mu}{b - a}, \frac{\mu - a}{b - a} \right\rangle\right).$$

Proof. Since $f(t) = (t - \mu)^2$ is strictly convex, we have

$$\int_{[a,b]} (t - \mu)^2 dF(t) \leq (a - \mu)^2 \int_{[a,\mu]} dF(t) + (b - \mu)^2 \int_{(\mu,b]} dF(t).$$

The equality holds iff F satisfies $\int_{[a,\mu]} dF(t) = \int_{\{a\}} dF(t)$ and $\int_{(\mu,b]} dF(t) = \int_{\{b\}} dF(t)$. Such a cdf is uniquely determined by (4.6). This completes the proof. $\square$

Lemma 8. For any strategy $(x, F) \in (\mathbf{x}^M, \mathcal{F})$, if F has a positive probability on the interval (x, M) and satisfies $\phi(x, F) \geq -c$, then there exists a cdf $F'' \in \mathcal{F}$ such that F'' has no probability on the interval $(x + \varepsilon, M)$ for an arbitrary small $\varepsilon > 0$, and it satisfies $\phi(x, F'') \leq \phi(x, F)$.

Proof. Let $x' = x + \varepsilon$, $p = \int_{(x',M)} dF(t) > 0$ and $y = p^{-1} \int_{(x',M)} t dF(t)$, $x' < y < M$. From Lemma 7, the probability p of F is allotted to the points x' and M without changing its mean 0. Let a cdf F' be

$$(4.7) \quad F'(t) = \begin{cases} F(t) & , \quad t < x' \\ F(x') + \left(\left\langle x', M \right\rangle \left\langle p \frac{M-y}{M-x'}, p \frac{y-x'}{M-x'} \right\rangle\right) & , \quad x' \leq t < M \\ 1 & , \quad M \leq t \end{cases}$$

which has the variance greater than σ^2 . The difference γ^2 of the variances is given by

$$\begin{aligned} \gamma^2 &= \int_A t^2 dF'(t) - \int_A t^2 dF(t) \\ &= p \left[(x')^2 \frac{M-y}{M-x'} + M^2 \frac{y-x'}{M-x'} \right] - \int_{(x',M)} t^2 dF(t) > 0. \end{aligned}$$

Putting $\lambda = \sigma^2/(\sigma^2 + \gamma^2)$, let us define

$$(4.8) \quad F''(t) = \lambda F'(t) + (1 - \lambda)G_1(t), \quad 0 < \lambda < 1,$$

where G_1 is the cdf degenerated at the origin 0. Then we find F'' is in $\mathcal{F}$, i.e., F'' has the mean 0 and the variance σ^2 . From Lemma 4 and (4.8) it holds that

$$(4.9) \quad \phi(x, F'') = \frac{\lambda \bar{F}'(x)}{\bar{F}''(x)} \phi(x, F') + \frac{(1 - \lambda) \bar{G}_1(x)}{\bar{F}''(x)} \phi(x, G_1),$$

where $\bar{G}_1(x) = 1$ and $\phi(x, G_1) = -c$ for $x < 0$, $\bar{G}_1(x) = 0$ for $x \geq 0$. Since $\bar{F}'(x) = \bar{F}(x)$ and $\phi(x, F') = \phi(x, F)$ from (4.7) and (1.1), we have $\phi(x, F'') \leq \phi(x, F)$ because $\phi(x, F'')$ in (4.9) is a convex combination of $\phi(x, F)(\geq -c)$ and $-c$.

Now, we shall redefine the cdf F'' which has no probability on the interval (x', M) .

When $x \geq 0$, F'' in (4.8) has no probability on (x', M) and the result holds. When $x < 0$, let $x' = x + \varepsilon < 0$ and $F''_{(1)} \stackrel{\text{def}}{=} F''$ in (4.8) has a probability mass $p_{(1)} = F''_{(1)}(0) - F''_{(1)}(0 - 0)$ at the origin 0 on (x', M) . If the mass $p_{(1)}$ of $F''_{(1)}$ is allotted to the points x' and M , then we can define $F'_{(2)}$ and $F''_{(2)}$ similarly as (4.7) and (4.8):

$$(4.7') \quad F'_{(2)}(t) = F'_{(1)}(t) + (< x', 0, M > < p_{(1)} \frac{M}{M + |x'|}, -p_{(1)}, p_{(1)} \frac{|x'|}{M + |x'|} >),$$

$$(4.8') \quad F''_{(2)}(t) = \lambda_{(2)} F'_{(2)}(t) + (1 - \lambda_{(2)}) G_1(t), \quad \text{where } \lambda_{(2)} = \frac{\sigma^2}{\sigma^2 + p_{(1)} M |x'|}.$$

Let us define $F'_{(n+1)} \in \mathcal{F}$ and $F''_{(n+1)} \in \mathcal{F}$ recursively by (4.7') and (4.8') with the subindexes (2) and (1) replaced by $(n + 1)$ and (n) respectively ($n \geq 1$). The mass $p_{(n+1)}$ of F''_{n+1} is given successively by

$$(4.10) \quad p_{(n+1)} = \frac{p_{(n)}}{k + p_{(n)}}, \quad k = \frac{\sigma^2}{M |x'|} > 0, \quad n \geq 1,$$

where $\lim_{n \rightarrow \infty} p_{(n)} = 0$ for any $p_{(1)}$. By the similar argument about $F'' = F''_{(1)}$ in (4.9), $F''_{(n)} \in \mathcal{F}$, $n \geq 1$, $F'' \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} F''_{(n)} \in \mathcal{F}$ and $\phi(x, F'') \leq \phi(x, F''_{(n)}) \leq \phi(x, F)$, $n \geq 1$. This completes the proof. $\square$

Lemma 9. For any strategy $(x, F) \in (\mathbf{x}^M, \mathcal{F})$, if F has a positive probability on the interval $(-M, x)$ and it satisfies $\phi(x, F) \geq -c$, then there exists a cdf $F'' \in \mathcal{F}$ such that F'' has no probability on the interval $(-M, x)$, and it satisfies $\phi(x, F'') \leq \phi(x, F)$.

Proof. Let $p = \int_{(-M, x)} dF(t) > 0$ and $y = p^{-1} \int_{(-M, x)} t dF(t)$, $-M < y < x$. From Lemma 7, the probability p of F is allotted to the point $-M$ and x without changing its mean 0. Let a cdf F' be

$$F'(t) = \begin{cases} F(-M) + (< -M, x > < p \frac{x-y}{M+x}, p \frac{M+y}{M+x} >) & , \quad -M \leq t \leq x \\ F(t) & , \quad x < t \end{cases}$$

which has the variance greater than σ^2 . The difference γ^2 of the variances is given by

$$\gamma^2 = p \left[x^2 \frac{M+y}{M+x} + M^2 \frac{x-y}{M+x} \right] - \int_{(-M, x)} t^2 dF(t) > 0.$$

Putting $\lambda = \sigma^2/(\sigma^2 + \gamma^2)$, let a cdf F'' be defined similarly by (4.8). Then, in the same fashion as in Lemma 8, the result holds. $\square$

Let us define for any $x \in [-\sigma^2/M, \sigma^2/M]$, a three-point cdf $G_3^M(\cdot; x) \in \mathcal{F}$ which has all the mass at three points $-M, x, M$ with the mean 0 and the variance σ^2 . This cdf is uniquely determined by

$$(4.11) \quad G_3^M(t; x) = (< -M, x, M > < \frac{Mx + \sigma^2}{2M(M+x)}, \frac{M^2 - \sigma^2}{M^2 - x^2}, \frac{\sigma^2 - Mx}{2M(M-x)} >),$$

and let $\mathcal{G}_3^M = \{G_3^M(t; x) \mid -\sigma^2/M \leq x \leq \sigma^2/M\}$. Note that if $x = \sigma^2/M$ or $-\sigma^2/M$, $G_3^M(t; x)$ becomes the two-point cdf $G_2(t; \sigma/M)$ or $G_2(t; M/\sigma)$ respectively.

Now, we shall derive a saddle point. It holds, from Theorem 3, that

$$(4.12) \quad -c = \sup_{x \in R} \inf_{F \in \mathcal{F}(0, \sigma^2)} \phi(x, F) \leq \sup_{x \in R} \inf_{F \in \mathcal{F}} \phi(x, F).$$

Then the player 1 would decide a stopping level x in the following set

$$(4.13) \quad \{x \mid \phi(x, F) \geq -c \text{ for all } F \in \mathcal{F}\} \cap \mathbf{x}^M \stackrel{\text{def}}{=} \mathbf{x}_c^M.$$

This set is not empty because $x = -c$ is contained in it.

Remark on Lemmas 8 and 9. Lemmas mean the following : for any $x \in \mathbf{x}_c^M$, if the player 1(he) decides his stopping level x , the player 2(she) may choose her cdf which has no probability on the interval $(-M, x) \cup (x + \varepsilon, M)$ for an arbitrary small $\varepsilon > 0$, where her cdf converges to $G_3^M(\cdot; x)$ of (4.11) as ε goes to 0. In this situation, his stopping set becomes $\{X \in (x, x + \varepsilon] \cup \{M\}\}$ for an arbitrary small $\varepsilon > 0$, i.e., $\{X \in \{M\}\}$ or $\{X \in \{x\} \cup \{M\}\}$.

From the above remark, we have for any $x \in \mathbf{x}_c^M$,

$$(4.14) \quad \inf_{F \in \mathcal{F}} \phi(x, F) = \min[\phi(x, G_3^M(\cdot; x)), \phi(x - 0, G_3^M(\cdot; x))].$$

If there exists a point $x^s \in \mathbf{x}_c^M$ such that

$$(4.15) \quad \phi(x^s, G_3^M(\cdot; x^s)) = (T_{\mathcal{G}_3^M}^\ell)^{-1}(c) (= \phi(x^s - 0, G_3^M(\cdot; x^s))) \geq -c,$$

the strategy $(x^s, G_3^M(\cdot; x^s))$, $x \in \mathbf{x}_c^M$, $G_3^M(\cdot; x^s) \in \mathcal{G}_3^M \subset \mathcal{F}$, is the saddle point and $\phi^s = (T_{\mathcal{G}_3^M}^\ell)^{-1}(c)$ is the saddle value. Because, from (4.14) and (2.10), the following relation is satisfied.

$$\begin{aligned} \min[\phi(x^s, G_3^M(\cdot; x^s)), \phi(x^s - 0, G_3^M(\cdot; x^s))] &\leq \sup_{x \in \mathbf{x}_c^M} \inf_{F \in \mathcal{F}} \phi(x, F) \\ &\leq \inf_{F \in \mathcal{F}} \sup_{x \in \mathbf{x}_c^M} \phi(x, F) \leq \inf_{F \in \mathcal{G}_3^M} \sup_{x \in \mathbf{x}_c^M} \phi(x, F) = (T_{\mathcal{G}_3^M}^\ell)^{-1}(c) = \phi(x^s, G_3^M(\cdot; x^s)). \end{aligned}$$

For any $x, y \in [-\sigma^2/M, \sigma^2/M]$, it is shown, by the graph of $T_{\mathcal{G}_3^M(\cdot; y)}(x)$, that

$$(4.16) \quad T_{\mathcal{G}_3^M(\cdot; y)}(x) = \int_x^M [1 - G_3^M(t; y)] dt \geq T_{\mathcal{G}_3^M(\cdot; x)}(x) = \int_x^M [1 - G_3^M(t; x)] dt = \frac{\sigma^2 - Mx}{2M}.$$

This line segment $T = (\sigma^2 - Mx)/2M$ connects the points $(-\sigma^2/M, \sigma^2/M)$ and $(\sigma^2/M, 0)$ in (x, T) -plane, and the equality in (4.16) holds only when $y = x$.

Then, we have proved (4.15).

$$(4.17) \quad (T_{G_3^M}^\ell)^{-1}(c) = \frac{\sigma^2}{M} - 2c = \left(\frac{\sigma^2}{M} - c\right)^+ - c \geq -c,$$

$$(4.18) \quad x^s = \left(\frac{\sigma^2}{M} - c\right)^+ - c, \quad F^s = G_3^M(\cdot; x^s).$$

Hence, we have the main theorem of our problem.

Theorem 5. For a class $\mathcal{F}(0, \sigma^2, M)$ of cdf's, $\sigma \leq M$, we have the following solution:

(i) When $\sigma^2/M \leq c \leq M$, the same result as Theorem 3 holds, that is, the saddle value ϕ^s and all the saddle points $(x^s, F^s) \in (\mathbf{x}^s, \mathcal{F}^s)$ are given by

$$\phi^s = -c, \quad \mathbf{x}^s = [-M, -c] \quad \text{and} \quad \mathcal{F}^s = \left\{ F \mid \int_{[-c, M]} dF(t) = 1, F \in \mathcal{F}(0, \sigma^2, M) \right\}.$$

(ii) When $0 < c < \sigma^2/M$, the saddle value ϕ^s and the saddle point (x^s, F^s) are given by

$$\phi^s = (\sigma^2/M - c)^+ - c, \quad x^s = (\sigma^2/M - c)^+ - c \quad \text{and} \quad F^s(t) = G_3^M(t; x^s) \text{ defined by (4.11).}$$

Unfortunately, we could not show the uniqueness of the saddle point (x^s, F^s) in Case (ii). Finally, we conclude this paper by illustrating ϕ^u of Theorem 4 and ϕ^s of Theorem 5 in Figure 1.

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Figure 1. The maximal bound ϕ^u and the saddle value ϕ^s in the class $\mathcal{F}(0, \sigma^2, M)$ when $M = 12$, $\sigma = 8$.