POLYHEDRA AND OPTIMIZATION RELATED TO A WEAK ABSOLUTE MAJORIZATION ORDERING

Ping Zhan*
Edogawa University

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Abstract A vector $x \in \mathbf{R}^n$ is weakly k -majorized by a vector $q \in \mathbf{R}^k$ if the sum of r largest components of x is less than or equal to the sum of r largest components of q for $r = 1, 2, \dots, k$ and $k \leq n$. In this paper we extend the components of x to their absolute values in the above description and generalize some results in [2] and [3] by G. Dahl and F. Margot.

Keywords: Majorization, bisubmodular functions, polyhedra

1. Introduction

For $p, q \in \mathbf{R}^n$ we say that p is *weakly sub-majorized* by q if $\sum_{j=1}^r p_{[j]} \leq \sum_{j=1}^r q_{[j]}$ for $r = 1, 2, \dots, n$. Here $p_{[j]}$ denotes the j th largest component of p . If $\sum_{j=1}^n p_j = \sum_{j=1}^n q_j$ also holds, p is *majorized* by q and we write $p \prec q$. Furthermore, for integers k, n with $k \leq n$ and $x \in \mathbf{R}^n, q \in \mathbf{R}^k$, we say that x is *weakly k -majorized* by q and write $x \prec_k q$ if

$$\sum_{j=1}^r x_{[j]} \leq \sum_{j=1}^r q_{[j]} \text{ for } r = 1, 2, \dots, k. \quad (1.1)$$

Majorization is a concept appearing in several branches of mathematics and applied mathematics as indicated in [2]. Here, we extend the components of x in (1.1) to their absolute values as follows:

$$\sum_{j=1}^r |x_{[j]}| \leq \sum_{j=1}^r q_{[j]} \text{ for } r = 1, 2, \dots, k. \quad (1.2)$$

We say that x is *weakly absolutely k -majorized* by q and write $x_{\text{abs}} \prec_k q$. In the following we investigate properties induced by (1.2) and generalize some results in [2] and [3].

Hereafter, we assume that *majorant* $q \in \mathbf{R}^k$ satisfies

$$q_1 \geq q_2 \geq \dots \geq q_k \geq 0. \quad (1.3)$$

2. Polyhedra Induced by Absolute Majorization

Let

$$P(q; k) := \{x \in \mathbf{R}^n | x \prec_k q\}, \quad (2.1)$$

and denote $\mathbf{N}_t := \{1, 2, \dots, t\}$. Define $x(X) := \sum_{j \in X} x_j$. Then we have [2]

$$P(q; k) = \{x \in \mathbf{R}^n | x(X) \leq q(\mathbf{N}_r) \text{ for all } X \subseteq \mathbf{N}_n \text{ with } r = |X| \leq k\}, \quad (2.2)$$

so $P(q; k)$ is a polyhedron.

In the case of absolute majorization, we first denote by $3^{\mathbf{N}_n}$ the set of all ordered pairs of disjoint subsets of $\mathbf{N}_n$, i.e., $3^{\mathbf{N}_n} := \{(X, Y) \mid X, Y \subseteq \mathbf{N}_n, X \cap Y = \emptyset\}$. Similarly, let

$$P_S(q; k) := \{x \in \mathbf{R}^n \mid x_{\text{abs}} \prec_k q\}. \quad (2.3)$$

Then we have

$$P_S(q; k) = \{x \in \mathbf{R}^n \mid x(X) - x(Y) \leq q(\mathbf{N}_r) \text{ for all } (X, Y) \in 3^{\mathbf{N}_n} \text{ with } r = |X \cup Y| \leq k\}. \quad (2.4)$$

Hence $P_S(q; k)$ is also a polyhedron, or more precisely, a polytope. Note here that the absolute-value notation disappears, and instead we have a signed linear form description.

Define the set function $f : 3^{\mathbf{N}_n} \rightarrow \mathbf{R}$ by

$$f(X, Y) = \sum_{i=1}^r q_i \quad \text{for all } (X, Y) \in 3^{\mathbf{N}_n} \text{ where } r = |X \cup Y|. \quad (2.5)$$

We call a function $f : 3^{\mathbf{N}_n} \rightarrow \mathbf{R}$ *bisubmodular* if f satisfies

$$f(X_1, Y_1) + f(X_2, Y_2) \geq f(X_1 \cap X_2, Y_1 \cap Y_2) + f((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)). \quad (2.6)$$

for any $(X_1, Y_1), (X_2, Y_2) \in 3^{\mathbf{N}_n}$.

Proposition 2.1 *The function f defined by (2.5) is a bisubmodular function.*

(Proof) For any $(X_1, Y_1), (X_2, Y_2) \in 3^{\mathbf{N}_n}$, we show that the function satisfies bisubmodular inequality (2.6).

Rewrite the inequality (2.6) as

$$f(X_1, Y_1) - f(X_1 \cap X_2, Y_1 \cap Y_2) \geq f((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)) - f(X_2, Y_2). \quad (2.7)$$

Let $s = |X_1 \cap X_2| + |Y_1 \cap Y_2|$, $t = |X_2| + |Y_2|$, and $l_1 = |X_1 \cup Y_1 - X_2 \cup Y_2|$, $l_2 = |X_1 \cap Y_2| + |X_2 \cap Y_1|$. By the definition of function $f(X, Y)$ in (2.5), the left-hand side of (2.7) is

$$\sum_{i=s+1}^{s+l_1+l_2} q_i, \quad (2.8)$$

and the right-hand side of (2.7) is

$$\begin{cases} \sum_{i=t+1}^{t+l_1-l_2} q_i & (\text{if } l_1 \geq l_2) \\ \sum_{i=t+l_2-l_1+1}^t -q_i & (\text{if } l_1 < l_2) \end{cases} \quad (2.9)$$

Since $t \geq s$, it follows from the assumption (1.3) that the left-hand side of (2.7) is, indeed, greater than or equal to the right-hand side of (2.7) (see Figure 1). Here note that $|L'_1 \cup L''_1| = l_1$ and $|L'_2 \cup L''_2| = l_2$. $\square$

Figure 1: Lengths of sets and Venn diagram of Proposition 2.1

From the above proposition, we have a diameter description of $P_S(q; n)$.

Let P be a convex polytope. The *diameter* of P , denoted by $\delta(P)$, is the smallest number $\delta(P)$ such that any two vertices in P can be connected by a path with at most $\delta(P)$ edges. Let ϕ be a linear form defined on P . Suppose that any v can be connected to v^* by a nonincreasing (nondecreasing) path induced by ϕ with at most $\delta(P, \phi)$ edges, where v^* minimizes (maximizes) ϕv . The *monotonic diameter* of P is defined as

$$\delta^*(P) = \max\{\delta(P, \phi) \mid \phi \text{ is a linear form on } P\}. \quad (2.10)$$

Now we have the following result ([5], [1]).

Proposition 2.2 *If all the components of q take different values, the diameter and the monotonic diameter of $P_S(q, n)$ are n^2 .*

(Outline of the proof): From the proof of 2.1, we can easily see that different components of q guarantee strict inequalities of bisubmodular function. Therefore, each vertex of $P_S(q; n)$ can be characterized by a signed *chain*, a list of signed elements, or a special signed Hasse diagram [1]. Then we traverse to adjacent vertices along the edges of $P_S(q; n)$ by two types of operations.

- (1) Exchange the pair of two adjacent elements of a chain.
- (2) Change the sign of an element at one fixed terminal of a chain.

Hence, what we have to do is counting the number of operations of types (1) and (2). Without loss of generality, we suppose that a maximally distant pair of vertices are characterized by following two chains.

$$(-x_1, -x_2, \dots, -x_n), \quad (2.11)$$

$$(x_1, x_2, \dots, x_n). \quad (2.12)$$

Hence, the total number of operations to obtain from one chain to another is

$$1 + (2 \times 2 - 1) + \dots + (2i - 1) + \dots + (2n - 1) = n^2. \quad (2.13)$$

The monotonic property can be obtained from the definition of the bisubmodularity inequality. $\square$

3. Optimization Problems and Integrality

Let $c \in \mathbf{R}^n$ and consider the following optimization problem,

$$\max\{c^T x \mid x_{\text{abs}} \prec_k q\}. \quad (3.1)$$

We know from (2.4) that the above problem is equivalent to

$$\max\{c^T x \mid x \in P_S(q; k)\}, \quad (3.2)$$

a linear programming (LP). And it is clear that when $k = n$, i.e., in the bisubmodular case, the above LP can be solved by greedy algorithm [4].

In the following, we generalize some results in [3] and [2] for $k < n$.

First, for $g \in \mathbf{R}^k$, we define *tail average* of g by $\bar{g}_{s:k} := 1/(k-s+1) \sum_{i=s}^k |g_i|$.

Suppose that $c \in \mathbf{R}^k$ satisfies $|c_1| \geq |c_2| \geq \cdots \geq |c_{k-1}|$ (note that c_k may be arbitrary). Then there is an $m \in \{1, 2, \dots, k\}$ such that [2]

$$\bar{c}_{1:k} \geq \cdots \geq \bar{c}_{m:k} \leq \bar{c}_{m+1:k} \leq \cdots \leq \bar{c}_{k:k} = |c_k|. \quad (3.3)$$

For $0 \leq s \leq k-1$ and $c \in \mathbf{R}^n$ with $|c_1| \geq |c_2| \geq \cdots \geq |c_n|$, we define *signed sth q -average* $w^s \in \mathbf{R}^n$ by

$$w^s := ((-1)^{i_1} q_1, \dots, (-1)^{i_s} q_s, (-1)^{i_{s+1}} \bar{q}_{s+1:k}, \dots, (-1)^{i_n} \bar{q}_{s+1,k}), \quad (3.4)$$

where $i_l = 0$ or 1 for $l = 1, 2, \dots, n$, and define $w_c^s \in \mathbf{R}^n$, *signed sth q -average related to c* by

$$w_c^s := (\text{sign}(c_1) q_1, \dots, \text{sign}(c_s) q_s, \text{sign}(c_{s+1}) \bar{q}_{s+1:k}, \dots, \text{sign}(c_n) \bar{q}_{s+1,k}). \quad (3.5)$$

Theorem 3.1 *The optimal solution of Problem (3.2) can be obtained as a permutation of w_c^s , where $s = m-1$.*

(Proof) To prove the optimality, we only consider the case when for $i = 1, 2, \dots, n$, (1) $x_i \geq 0$ if $c_i \geq 0$ and (2) $x_i < 0$ if $c_i < 0$. Otherwise, reversing the signs of components of x would increase the value of $c^T x$ without violating the constraints $x_{\text{abs}} \prec_k q$. Our proof is only based on the form of $c^T x$, we may suppose $c_i \geq 0$ and $x_i \geq 0$ for $i = 1, 2, \dots, n$. Otherwise, let $c_i x_i = (-c_i)(-x_i)$ and omit the signs before them for convenience.

For the second part of the proof, see appendix. $\square$

Conversely, by the same arguments as the proof of Theorem 3.1 and by an appropriate choice of c (see Theorem 5 of [3]), we can prove that each w^s ($s = 0, 1, \dots, k-1$) is a unique optimal solution of (3.2) on $P_S(q; k)$, and therefore a vertex of $P_S(q; k)$. Combining it with Theorem 3.1, we have the following result.

Theorem 3.2 *The set of vertices of $P_S(q; k)$ is precisely the set of vectors that can be obtained by permutations of w^s , $s = 0, 1, \dots, k-1$.*

By Theorem 3.1, the simple (greedy) algorithm for solving problem (3.1) is similar to that described in [2] and its time complexity is $O(n^2)$. The difference is that we compute s by taking absolute values of components of c and we take minus components of w^s when the corresponding components of c are less than zero.

Now we consider the following integer programming problem:

$$\max\{c^T x \mid x_{\text{abs}} \prec_k q, x \text{ is integral}\}. \quad (3.6)$$

We represent the integer hull of $P_S(q; k)$ by

$$Q_S(q; k) := \text{conv}\{x \in \mathbf{R}^n \mid x_{\text{abs}} \prec_k q, x \text{ is integral}\}. \quad (3.7)$$

We assume that q is an integer vector, otherwise round down each component of q without changing $Q_S(q; k)$. Note here that $Q_S(q; k)$ is full-dimensional if $q \neq 0$ (together with the assumption that q is integral and non-negative).

In [3] and [2], a complete description of vertices and facets of polyhedra $Q(q; k) = \text{conv}\{x \in \mathbb{R}^n | x \prec_k q, x \text{ is integral}\}$ are provided. By the symmetry, these results can be generalized to polytope $Q_S(q; k)$ without much modifications. We summarize two main results here.

Let $\alpha \in [q_k, \bar{q}_{1,k}]$ and define $s(\alpha) = \max\{0 \leq s < k | \bar{q}_{s+1,k} \geq \alpha\}$ and $\Delta(\alpha) = \sum_{i=s+1}^k q_i - (k - s(\alpha) - 1)\alpha$. We define the vector $x(\alpha) \in \mathbb{R}^n$ by

$$x(\alpha) := ((-1)^{i_1} q_1, \dots, (-1)^{i_s} q_s, (-1)^{i_{s+1}} \Delta(\alpha), (-1)^{i_{s+2}} \alpha, \dots, (-1)^{i_n} \alpha), \quad (3.8)$$

for $i_l = 0$ or 1 and $l = 1, 2, \dots, n$, where $\alpha \leq \bar{q}_{1,k}$ is a round down or round up of $\bar{q}_{s+1,k}$ and $s = 0, 1, \dots, k-1$. And we say $x(\alpha)$ a *signed rounded q -average*.

Theorem 3.3 *The set of vertices of $Q_S(q; k)$ is precisely the set of vectors that can be obtained by permutations of signed rounded q -averages.*

Let s and t be integers with $0 \leq s < k < t \leq n$, and put $\delta^s := \sum_{i=s+1}^k q_i - (k-s)\lfloor \bar{q}_{s+1:k} \rfloor = \sum_{i=s+1}^k q_i \bmod (k-s)$. Then, define $\alpha_0^{s,t} := (t-k)/(k-s-\delta^s) \sum_{i=1}^s q_i + \sum_{i=1}^k q_i + (t-k)\lfloor \bar{q}_{s+1:k} \rfloor$. We call

$$(t-s-\delta^s)/(k-s-\delta^s)(x(X_1) - x(Y_1)) + x(X_2) - x(Y_2) \leq \alpha_0^{s,t} \quad (3.9)$$

a *signed q -average inequality* if $\bar{q}_{s+1:k}$ is fractional, where X_1, Y_1, X_2, Y_2 are pairwise disjoint, $|X_1 \cup Y_1| = s$ and $|X_2 \cup Y_2| = t-s$.

We call

$$x(X) - x(Y) \leq \sum_{i=1}^r q_i \quad \text{for } (X, Y) \in 3^{\mathbb{N}^n}, |X \cup Y| = r \quad (3.10)$$

a *signed set size inequality* if $r=1$ or $q_1 > q_r$.

Theorem 3.4 *A complete and non-redundant facet description of $Q_S(q; k)$ is given by signed set size inequalities and signed q -average inequalities.*

4. Appendix

For the second part of the proof of Theorem 3.1 we first assume $c_1 \geq c_2 \geq \dots \geq c_n$. Let $x \geq 0$ be a feasible solution. Without loss of generality, we may assume that $x_1 \geq \dots \geq x_k = \dots = x_n$. Let $d_i = c_i$ for $i = 1, 2, \dots, k-1$, and $d_k = \sum_{j=k}^n c_j$. Let $x^* = w_c^s$, as indicated in the theorem.

$$\begin{aligned} \sum_{i=1}^n c_i(x_i^* - x_i) &= \sum_{i=1}^k d_i(x_i^* - x_i) = \sum_{i=1}^s d_i(q_i - x_i) + \bar{q}_{s+1:k} \sum_{i=s+1}^k d_i - \sum_{i=s+1}^k d_i x_i \\ &= \sum_{i=1}^s d_i(q_i - x_i) + (k-s)\bar{d}_{s+1:k}\bar{q}_{s+1:k} - \sum_{i=s+1}^k d_i x_i, \end{aligned} \quad (4.1)$$

where,

$$\begin{aligned}
 - \sum_{i=s+1}^k d_i x_i &= - \sum_{i=s+1}^{k-1} \left(\sum_{j=i}^k d_j - \sum_{j=i+1}^k d_j \right) x_i - d_k x_k \\
 &= - \sum_{i=s+1}^{k-1} ((k-i+1)\bar{d}_{i:k} - (k-i)\bar{d}_{i+1:k}) x_i - d_k x_k \\
 &= - \left(\sum_{i=s}^{k-2} (k-i)\bar{d}_{i+1:k} x_{i+1} - \sum_{i=s+1}^{k-1} (k-i)\bar{d}_{i+1:k} x_i \right) - d_k x_k \\
 &= \sum_{i=s+1}^{k-2} (k-i)\bar{d}_{i+1:k} (x_i - x_{i+1}) - (k-s)\bar{d}_{s+1:k} x_{s+1} + \bar{d}_{k:k} x_{k-1} - d_k x_k. \quad (4.2)
 \end{aligned}$$

Note that $x_i - x_{i+1} \geq 0$. From the definition of s we obtain

$$\begin{aligned}
 &\sum_{i=s+1}^{k-2} (k-i)\bar{d}_{i+1:k} (x_i - x_{i+1}) \\
 &\geq \bar{d}_{s+1:k} \sum_{i=s+1}^{k-2} (k-i)(x_i - x_{i+1}) \\
 &= \bar{d}_{s+1:k} \sum_{i=s+1}^{k-2} (k-i)x_i - \bar{d}_{s+1:k} \sum_{i=s+2}^{k-1} (k-i+1)x_i \\
 &= -\bar{d}_{s+1:k} \sum_{i=s+2}^{k-2} x_i + (k-s-1)\bar{d}_{s+1:k} x_{s+1} - 2\bar{d}_{s+1:k} x_{k-1} \\
 &= -\bar{d}_{s+1:k} \sum_{i=s+1}^{k-1} x_i + (k-s)\bar{d}_{s+1:k} x_{s+1} - \bar{d}_{s+1:k} x_{k-1}. \quad (4.3)
 \end{aligned}$$

Now, the last whole row of equation (4.2) (after deleting $(k-s)\bar{d}_{s+1:k} x_{s+1} - (k-s)\bar{d}_{s+1:k} x_{s+1}$) is greater than

$$\begin{aligned}
 &-\bar{d}_{s+1:k} \sum_{i=s+1}^{k-1} x_i - \bar{d}_{s+1:k} x_{k-1} + \bar{d}_{k:k} x_{k-1} - d_k x_k \\
 &= -\bar{d}_{s+1:k} \sum_{i=s+1}^k x_i + \bar{d}_{s+1:k} x_k - \bar{d}_{s+1:k} x_{k-1} + \bar{d}_{k:k} x_{k-1} - d_k x_k \\
 &= -\bar{d}_{s+1:k} \sum_{i=s+1}^k x_i + (d_k - \bar{d}_{s+1:k})(x_{k-1} - x_k) \\
 &\geq -\bar{d}_{s+1:k} \sum_{i=s+1}^k x_i. \quad (4.4)
 \end{aligned}$$

Finally, since $q_i - x_i \geq 0$ for all $i = 1, 2, \dots, s$, we have

$$\sum_{i=1}^s d_i (q_i - x_i) \geq \bar{d}_{s+1:k} \sum_{i=1}^s (q_i - x_i). \quad (4.5)$$

Summarizing the above equations and inequalities, we have

$$\sum_{i=1}^k d_i(x_i^* - x_i) \geq \bar{d}_{s+1:k} \left(\sum_{i=1}^k q_i - \sum_{i=1}^k x_i \right) \geq 0. \quad (4.6)$$

□

Note that this is a direct proof of Theorem 2 in [2].

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Ping Zhan
 Edogawa University
 474 Komaki, Nagareyama-shi, Chiba,
 270-0198, Japan
 E-mail: zhan@edogawa-u.ac.jp