EGALITARIAN SOLUTION FOR GAMES WITH DISCRETE SIDE PAYMENT

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Abstract In this paper, we study the egalitarian solution for games with discrete side payment, where the characteristic function is integer-valued and payoffs of players are integral vectors. The egalitarian solution, introduced by Dutta and Ray in 1989, is a solution concept for transferable utility cooperative games in characteristic form, which combines commitment for egalitarianism and promotion of individual interests in a consistent manner. We first point out that the nice properties of the egalitarian solution (in the continuous case) do not extend to games with discrete side payment. Then we show that the Lorenz stable set, which may be regarded as a variant of the egalitarian solution, has nice properties such as the Davis–Maschler reduced game property and the converse reduced game property. For the proofs we utilize recent results in discrete convex analysis on decreasing minimization on an M-convex set investigated by Frank and Murota.

Keywords: Game theory, discrete optimization, discrete convex analysis, cooperative game, egalitarian solution, discrete side payment

1. Introduction

The egalitarian solution is a solution concept for transferable utility cooperative games in characteristic form, which combines commitment for egalitarianism and promotion of individual interests in a consistent manner. This concept was introduced by Dutta–Ray [16] in 1989.

The egalitarian solution is studied extensively in the literature. For example, Arin et al. [2], Dutta [15], and Klijn et al. [28] axiomatize the egalitarian solution of Dutta–Ray for convex games. Branzei et al. [7], Dietzenbacher et al. [11], Hokari [24, 25], and Llerena–Mauri [33] considered modifications of the egalitarian solution so that the solution exists for a wider class of games. These studies are mostly concerned with the case where the characteristic function is real-valued and, accordingly, the side payment is real-valued.

Substantial connection has been recognized between the egalitarian solution in convex games and the polymatroid theory in optimization. Indeed, the core of a convex game is nothing but the base polyhedron of a polymatroid. In the theory of polymatroids and submodular functions, Fujishige [21] had introduced the concept of *lexicographically optimal base* in 1980, and this concept is essentially equivalent to the egalitarian solution in convex games, as noted by Fujishige [22] and Hokari–Uchida [26]. In particular, the *principal partition* of Fujishige [21] plays the decisive role to clarify the properties of the egalitarian solution in convex games and also to develop algorithms for finding the egalitarian solution.

In this paper we are interested in the egalitarian solution for games with discrete side payment, where the characteristic function is integer-valued and payoffs of players are integral vectors. This study is partly motivated by a recent development in discrete convex analysis, which is a theory of discrete convexity for functions on integer lattice points (see

[38–41]). Frank–Murota [19] recently investigated the discrete decreasing minimization problem, which is concerned with lexicographically minimal integral vectors in an integral base polyhedron. This is a discrete counterpart of the work by Fujishige [21], and, in particular, the discrete counterpart of the principal partition is established as the *canonical partition*. The main objective of this paper is to clarify the properties of the egalitarian solution in games with discrete side payment by making use of these results of lexicographically minimal (decreasingly minimal) integral elements in an integral base polyhedron.

The results of this paper are summarized as follows. First, we show by an example that, unlike the case of $\mathbf{R}$, the egalitarian solution for convex games with discrete side payment is not equivalent to the lexicographically minimal (decreasingly minimal) element. Accordingly, the egalitarian solution in the case of $\mathbf{Z}$ fails to have nice properties of the egalitarian solution in the continuous case. This motivates us to consider the Lorenz stable set (or equivalently Lorenz maximal imputation). The Lorenz stable set is introduced by Arin–Inarra [1] and Hougaard et al. [27] and is defined as the subset of the core consisting of the elements that are not Lorenz-dominated by any other element of the core. We show that the Lorenz stable set has nice properties such as the Davis–Maschler reduced game property [10] and the converse reduced game property [43]. Our analysis of the Lorenz stable set relies heavily on the recent results on the discrete decreasing minimization problem.

This paper is organized as follows. Sections 2 and 3 provide brief reviews on the egalitarian solution and discrete decreasing minimization problem, respectively. In Section 4, we investigate the properties of the egalitarian solution in games with discrete side payment in comparison with the egalitarian solution in continuous variables. In Sections 5 and 6, we clarify the fundamental properties of the Lorenz stable set by utilizing the results on discrete decreasing minimization.

2. Egalitarian Solution in the Continuous Case

We provide a brief summary on the egalitarian solution of Dutta–Ray [16].

2.1. Definition and notation

We consider a transferable utility game in characteristic function form. There are n players and let $N = \{1, 2, \dots, n\}$. A *coalition* is a nonempty subset of N ; in particular, N is called the *grand coalition*. The worth of a coalition S is given by a real number $v(S)$. We assume $v(\emptyset) = 0$ throughout this paper. A pair (N, v) is called a *game*. We denote the set of games by Γ .

Let $\mathbf{R}$ and $\mathbf{Z}$ be the set of reals and integers respectively. Also, we denote by $\mathbf{R}_+$ the set of nonnegative reals and by $\mathbf{Z}_+$ the set of nonnegative integers. For a vector $x \in \mathbf{R}^N$, we sometimes abbreviate $\sum_{i=1}^N x_i$ to $x(N)$. Let (N, v) be a game. The set

$$X(N, v) = \{x \in \mathbf{R}_+^N \mid x(N) \leq v(N)\}$$

is called the set of *feasible payoff vectors* for the game (N, v) . A *solution* on Γ is a function σ that associates with each game $(N, v) \in \Gamma$ a subset $\sigma(N, v)$ of $X(N, v)$.

For a game (N, v) , we call x an *imputation* if $x_i \geq v(\{i\})$ for all $i \in N$ and $x(N) = v(N)$. The restriction of x to $S \subseteq N$ is denoted by x_S . For two vectors $x, y \in \mathbf{R}^N$, we write $x \succeq y$ if $x_i \geq y_i$ for all $i = 1, \dots, n$, with strict inequality for some i . For a vector x , let $x \downarrow$ denote the vector obtained from x by rearranging its components in a decreasing order. The Lorenz-domination is defined as follows:

Definition 2.1. For two vectors x and y in $\mathbf{R}^N$ with $x(N) = y(N)$, we say that x *Lorenz-dominates* y if $\sum_{j=1}^i (x \downarrow)_j \leq \sum_{j=1}^i (y \downarrow)_j$ holds for all $i = 1, \dots, n$, with strict inequality for

some i . □

Also, for $x, y \in \mathbf{R}^N$, we say that x and y are *value-equivalent* if $x \downarrow = y \downarrow$ holds.

Throughout this paper, we define the Lorenz-domination by a decreasing order in accordance with Dutta–Ray [16]. Other papers, however, adopt an increasing order as follows:

Definition 2.2. For a vector x , let $x \uparrow$ denote the vector obtained from x by rearranging its components in an increasing order. For two vectors x and y in $\mathbf{R}^N$ with $x(N) = y(N)$, we say that x *Lorenz-dominates* y if $\sum_{j=1}^i (x \uparrow)_j \geq \sum_{j=1}^i (y \uparrow)_j$ for all $i = 1, \dots, n$, with strict inequality for some i .

The decreasing order is adopted in Dutta [15], Dutta–Ray [16], Llerena [30], and Llerena–Mauri [32], whereas the increasing order is in Arin–Inarra [1], Arin et al. [3], Fei–Fields [18], Hokari [24, 25], Llerena–Mauri [33], and Shaked–Shanthikumar [45]. We note that this difference in the definition of Lorenz-domination does not affect the results of this paper; see Remark 5.2. We also mention that the Lorenz-domination is defined equivalently as follows. For two payoff vectors $x, y \in \mathbf{R}^N$ with $x(N) = y(N)$,

$$x \text{ Lorenz-dominates } y \Leftrightarrow \text{Its Lorenz curve lies nowhere below that of } y.$$

For example, Patrick [42] and Tatiana [47] adopt this definition.

Next, we define the core, which is a central solution concept of cooperative game theory (see [44]).

Definition 2.3. For any coalition S , the *core* of S is defined by

$$C(S, v) = \{x \in \mathbf{R}^S \mid x(S) = v(S), x(T) \geq v(T) \ (\forall T \subsetneq S)\}. \quad \square$$

Definition 2.4. We call a game (N, v) a *convex game* if v is a supermodular function, that is,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

for all $S, T \subseteq N$. We denote the set of convex games by Γ^c . □

The notion of the egalitarian solution will now be described. First, the *Lorenz map* E is defined on the domain

$$\{A \mid A \subseteq \mathbf{R}^k \ (\exists k \in \{1, 2, \dots, n\}), \exists u \in \mathbf{R}, \forall x \in A: \sum_{i=1}^k x_i = u\}.$$

For each such set A , $E(A)$ denotes the set of all elements in A that are not Lorenz-dominated within A .

Next, we define the *Lorenz core* introduced by Dutta–Ray [16]. See [16] for the details about the Lorenz core. The Lorenz core is defined recursively as follows. The Lorenz core of a singleton coalition is $L(\{i\}) = \{v(\{i\})\}$ ($i \in N$). Now suppose that the Lorenz cores for all coalitions of cardinality $k - 1$ or less have been defined, where $2 \leq k < n$. The Lorenz core of coalitions of size k is defined by

$$L(S, v) = \{x \in \mathbf{R}^S \mid x(S) = v(S) \text{ and there is no pair of } T \subsetneq S \text{ and } y \in E(L(T, v)) \text{ such that } x_T \preceq y\}.$$

For a game $(N, v) \in \Gamma$, we call an element of $E(L(N, v))$ an *egalitarian solution*. There is an inclusion between the core and the Lorenz core.

Proposition 2.1. For any $S \subseteq N$,

$$C(S, v) \subseteq L(S, v). \quad (2.1)$$

Proof. Assume that $x \notin L(S, v)$, which implies that $y \succneq x_T$ for some $T \subsetneq S$ ($T \neq \emptyset$) and some $y \in E(L(T, v))$. By this inequality and $y(T) = v(T)$, we obtain $x(T) < y(T) = v(T)$, which implies $x \notin C(S, v)$. Therefore, (2.1) holds. $\square$

Remark 2.1. For the weighted egalitarian solution, see Hokari [25] and Koster [29].

2.2. Properties of egalitarian solution

Here we describe the properties of the egalitarian solution shown by Dutta–Ray [16]. First, the following theorem shows that the egalitarian solution is unique if it exists at all.

Theorem 2.2. (Dutta–Ray [16]) *There is at most one egalitarian solution in any game.* $\square$

Note that Theorem 2.2 does not guarantee the existence of the egalitarian solution. The next theorem reveals that in any convex game, the egalitarian solution always exists and belongs to the core.

Theorem 2.3. (Dutta–Ray [16]) *In convex games, an egalitarian solution exists and it is contained in the core, that is, for any $(N, v) \in \Gamma^c$, we have $\emptyset \neq E(L(N, v)) \subseteq C(N, v)$.* $\square$

Moreover, the egalitarian solution has a nice property as follows.

Theorem 2.4. (Dutta–Ray [16]) *In convex games, the egalitarian solution Lorenz-dominates every other element of the core.* $\square$

Theorem 2.4 raises the question whether the egalitarian solution Lorenz-dominates every other element of the Lorenz core. The following example shows that this is not true even in convex games.

Example 2.1. (Dutta–Ray [16], Example 5) Let $N = \{1, 2, 3\}$, $v(\{1\}) = 4$, $v(\{2\}) = 6$, $v(\{3\}) = 8$, $v(\{1, 2\}) = 11$, $v(\{1, 3\}) = 12$, $v(\{2, 3\}) = 15$ and $v(N) = 21$. This game is convex. The egalitarian solution in this game is $(6, 7, 8)$, which does not Lorenz-dominate $(6.25, 6.5, 8.25) \in L(N, v)$. $\square$

2.3. The algorithm for egalitarian solution in a convex game

We describe an algorithm to locate the egalitarian solution introduced by Dutta–Ray [16]. This algorithm is equivalent to the *decomposition algorithm* of Fujishige [21].

Let $v : 2^N \rightarrow \mathbf{R}$ be a supermodular set function. Define $v_1 = v$.

Step 1: Let S_1 be the coalition that satisfies the following two conditions.

1. $v_1(S_1)/|S_1| \geq v_1(S)/|S|$ for all $S \subseteq N$.
2. $|S_1| > |S|$ for all $S \neq S_1$ such that $v_1(S_1)/|S_1| = v_1(S)/|S|$.

That is, S_1 is the largest coalition having the highest average worth. By using supermodularity of v , we can verify the existence of such an S_1 . Define

$$x_i^* = \frac{v_1(S_1)}{|S_1|} \quad (i \in S_1).$$

Step k ($k \geq 2$): Suppose that $(S_1, v_1), \dots, (S_{k-1}, v_{k-1})$ ($k \geq 2$) have been defined and $S_1 \cup \dots \cup S_{k-1} \neq N$. Define a new game with player set $N \setminus (S_1 \cup \dots \cup S_{k-1})$. For all coalitions S of this new player set, define $v_k(S)$ by

$$v_k(S) = v_{k-1}(S_{k-1} \cup S) - v_{k-1}(S_{k-1}).$$

By the definition of v_k , v_k is a supermodular function. Just as in Step 1, define S_k to be the largest coalition in $N \setminus (S_1 \cup \dots \cup S_{k-1})$ that maximizes $\frac{v_k(S)}{|S|}$, and define

$$x_i^* = \frac{v_k(S_k)}{|S_k|} \quad (i \in S_k).$$

In at most n steps, we can obtain a partition of N into sets $S_1, \dots, S_m$ ($m \leq n$) and the egalitarian solution. By the above construction of x^* , we obtain the following:

$$x_i^* = x_j^* \quad (i, j \in S_l, \quad l = 1, \dots, m), \quad (2.2)$$

$$\sum_{k=1}^l \sum_{j \in S_k} x_j^* = v(S_1 \cup \dots \cup S_l) \quad (l = 1, \dots, m), \quad (2.3)$$

$$x_i^* > x_j^* \quad (i \in S_k, \quad j \in S_l, \quad k < l). \quad (2.4)$$

The equation (2.2) shows that for $l = 1, \dots, m$, each payoff of the players belonging to S_l is the same.

3. Polymatroid Theory and Decreasing Minimization Problem

In this section, we overview the results of the polymatroid theory and decreasing minimization problem from discrete convex analysis.

3.1. Definition and notation

First, we give the basic facts about majorization and decreasing minimization. A vector x is decreasingly smaller than vector y , in notation $x <_{\text{dec}} y$, if $x \downarrow$ is lexicographically smaller than $y \downarrow$ in the sense that they are not value-equivalent and $(x \downarrow)_j < (y \downarrow)_j$ for the smallest subscript j for which $(x \downarrow)_j$ and $(y \downarrow)_j$ differ. We write $x \leq_{\text{dec}} y$ to mean that x is decreasingly smaller or value-equivalent to y . For a set Q of vectors, $x \in Q$ is *decreasingly minimal* (dec-min, for short) in Q if $x \leq_{\text{dec}} y$ holds for every $y \in Q$.

The *decreasing minimization problem* is to find a dec-min element of a given set Q of vectors. Frank–Murota [19] deal with the case where the set Q is an M-convex set, which is to be defined in Section 3.2.

Just as the notion of decreasingly minimality, we can consider a notion of increasingly maximality. A vector x is increasingly larger than vector y , in notation $x >_{\text{inc}} y$, if $x \uparrow$ is lexicographically larger than $y \uparrow$ in the sense that they are not value-equivalent and $(x \uparrow)_j > (y \uparrow)_j$ for the smallest subscript j for which $(x \uparrow)_j$ and $(y \uparrow)_j$ differ. We write $x \geq_{\text{inc}} y$ to mean that x is increasingly larger or value-equivalent to y . For a set Q of vectors, $x \in Q$ is *increasingly maximal* (inc-max, for short) in Q if $x \geq_{\text{inc}} y$ holds for every $y \in Q$.

Let $\bar{x}$ denote the vector whose k -th component $\bar{x}_k$ is equal to the sum of the first k components of $x \downarrow$. A vector x is said to be *majorized* by another vector y , in notation $x \prec y$, if $\bar{x} \leq \bar{y}$ and $\bar{x}_n = \bar{y}_n$ hold. Also, x is said to be *strictly majorized* by y if $\bar{x} \leq \bar{y}$ and $\bar{x}_n = \bar{y}_n$ hold [34]. Let Q be an arbitrary subset of $\mathbf{R}^N$. An element x of Q is said to be *least majorized* in Q if x is majorized by all $y \in Q$.

There exists a relationship between the notion of decreasing minimality and that of being least majorized as follows.

Proposition 3.1. (e.g., Arnold [4], Arnold–Sarabia [5], and Tamir [46]) *Let Q be an arbitrary subset of $\mathbf{R}^N$ and assume that Q admits a least majorized element. Then, an element of Q is least majorized in Q if and only if it is decreasingly minimal in Q . $\square$*

Also, there exists a relationship between being majorized and Lorenz-domination, that is,

$$x \text{ Lorenz-dominates } y \Leftrightarrow x \text{ is strictly majorized by } y.$$

Note that if we replace “strictly majorized” with “majorized” in the above, then $\Rightarrow$ is true but $\Leftarrow$ is not true. Indeed, if $\bar{x}_j = \bar{y}_j$ for all $j = 1, \dots, n$, then x is majorized by y but x does not Lorenz-dominate y . However, if we identify value-equivalent vectors, the notion of Lorenz-domination is equivalent to that of being majorized.

3.2. Polymatroid theory and decreasing minimization on an M-convex set

In polymatroid theory, the concept of base polyhedron plays a central role. A base polyhedron is defined as follows.

Definition 3.1. For a finite-valued supermodular set function g on N with $g(\emptyset) = 0$, the associated *base polyhedron* $B(g)$ is defined by

$$B(g) = \{x \in \mathbf{R}^N \mid x(N) = g(N), x(S) \leq g(S) \ (\forall S \subsetneq N)\}. \quad \square$$

If g is an integer-valued supermodular set function, we call $B(g)$ an integral base polyhedron. Any extreme point of an integral base polyhedron is an integer point and the convex hull of the integer points of $B(g)$ coincides with $B(g)$ itself.

Lexicographically optimal base is defined as follows [21]:

Definition 3.2. For a supermodular function $g: 2^N \rightarrow \mathbf{R}$, $x \in B(g)$ is said to be a *lexicographically optimal base* of $B(g)$ if $x \geq_{\text{inc}} y$ holds for any $y \in B(g)$. $\square$

Remark 3.1. Fujishige [21] deals with the weighted lexicographically optimal base. We mainly treat the unweighted lexicographically optimal base in this paper. $\square$

Next, we define an M-convex set, which plays a central role in discrete convex analysis. Here for two vectors $x, y \in \mathbf{R}^N$, we define

$$\begin{aligned} \text{supp}^+(x - y) &= \{i \in N \mid x_i > y_i\}, \\ \text{supp}^-(x - y) &= \{j \in N \mid x_j < y_j\} \end{aligned}$$

and for each $i \in N$, let characteristic vector $\chi_i \in \{0, 1\}^N$ denote by

$$\chi_i(j) = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j). \end{cases}$$

Then, the concept of M-convex set is defined as follows.

Definition 3.3. (M-convex set, Murota [37–39]) We say that the set $B \subseteq \mathbf{Z}^N$ is an *M-convex set* if for any $x, y \in B$ and for any $i \in \text{supp}^+(x - y)$, there exists some $j \in \text{supp}^-(x - y)$:

$$x - \chi_i + \chi_j \in B, \ y + \chi_i - \chi_j \in B. \quad \square$$

A set $B \subseteq \mathbf{Z}^N$ is an M-convex set if and only if $B = B(g) \cap \mathbf{Z}^N$ holds for some integer-valued supermodular function g . That is, an M-convex set is nothing but the set of integral points of an integral base polyhedron.

Decreasingly minimal elements on an M-convex set can be regarded as a discrete counterpart of the lexicographically optimal base. We note that for any M-convex set B , an element is decreasingly minimal in B if and only if it is increasingly maximal in B (see [19, 46]). Frank–Murota [19] mainly consider the problem of finding a dec-min element of an M-convex set.

M-convex set is characterized by the exchange axiom of Definition 3.3. Similarly, a dec-min element of an M-convex set is characterized by certain exchange operations as follows.

Definition 3.4. (1-tightening step [19]) Let $B \subseteq \mathbf{Z}^N$ be an M-convex set. A *1-tightening step* for $x \in B$ means the operation of replacing x to $x + \chi_i - \chi_j$ for some $i, j \in N$ such that $x_j \geq x_i + 2$ and $x + \chi_i - \chi_j \in B$. $\square$

Theorem 3.2. (Frank–Murota [19], Theorem 3.3) *Let B be an M-convex set. For an element x of B , the following equivalence holds:*

There is no 1-tightening step for $x \Leftrightarrow x$ is decreasingly minimal in B . $\square$

Remark 3.2. A 1-tightening step is called the *Robin Hood transfer* or *Robin Hood operation* in economics and the theory of majorization (see also Arnold [4] and Marshall et al. [34]). Also, a 1-tightening step is called the *progressive transfer* or *rich to poor transfer* in Dutta–Ray [16]. Note that they do not restrict B to an M-convex set in Definition 3.4. $\square$

Finally, we explain the relationship between an M-convex set and a least majorized element. The following theorem shows the existence of a least majorized element in an M-convex set.

Theorem 3.3. (e.g., [46]) *An M-convex set admits a least majorized element.* $\square$

Remark 3.3. According to Frank–Murota [19], the above fact has long been recognized by experts at least since 1995, though it was difficult to identify its origin in the literature. $\square$

3.3. Structure of dec-min elements on an M-convex set

In this section, we introduce a partition and a chain called *canonical partition* and *canonical chain*, respectively, that describe the structure of dec-min elements on an M-convex set. They are introduced by Frank–Murota [19], and the canonical partition is a discrete counterpart of the principal partition considered by Fujishige [21] for the lexicographically optimal base in continuous variables.

The canonical chain and the canonical partition of an M-convex set are constructed as follows [19]. Let $g : 2^N \rightarrow \mathbf{Z}$ be an integer-valued supermodular function with $g(\emptyset) = 0$. Consider the smallest maximizer $L(\beta)$ of $g(X) - \beta|X|$ for all integers β . There are finitely many β for which $L(\beta) \neq L(\beta - 1)$. Denote such integers as $\beta_1 > \beta_2 > \cdots > \beta_q$ and call it the *essential value-sequence*. Furthermore, define $C_k = L(\beta_k - 1)$ for $k = 1, 2, \dots, q$ to obtain a chain: $C_1 \subsetneq C_2 \subsetneq C_3 \subsetneq \cdots \subsetneq C_q$. Call this the *canonical chain*. Finally define a partition $\{S_1, S_2, \dots, S_q\}$ of N by $S_k = C_k \setminus C_{k-1}$ for $k = 1, 2, \dots, q$, where $C_0 = \emptyset$, and call this the *canonical partition*.

Alternatively, the canonical chain and the canonical partition can be defined iteratively as follows. For $k = 1, 2, \dots, q$, define

$$\begin{aligned} \beta_k &= \max \left\{ \left\lceil \frac{g(X \cup C_{k-1}) - g(C_{k-1})}{|X|} \right\rceil \mid \emptyset \neq X \subseteq \overline{C_{k-1}} \right\}, \\ h_k(X) &= g(X \cup C_{k-1}) - (\beta_k - 1)|X| - g(C_{k-1}) \quad (X \subseteq \overline{C_{k-1}}), \\ S_k &= \text{smallest subset of } \overline{C_{k-1}} \text{ maximizing } h_k, \\ C_k &= C_{k-1} \cup S_k, \end{aligned}$$

where $\overline{X} = N \setminus X$ for any $X \subseteq N$.

Then, this chain enables us to construct the set of dec-min elements on an M-convex set as follows. We define the supermodular function $g'_k : 2^{\overline{C_k}} \rightarrow \mathbf{Z}$ by

$$g'_k(X) = g(X \cup C_k) - g(C_k) \quad (X \subseteq \overline{C_k}),$$

which defines the M-convex set $B'_k = B'(g'_k)$ in $\mathbf{R}^{\overline{C_k}}$.

Moreover, we denote the restriction of g'_k to S_k by g_k , which defines the M-convex set $B_k = B'(g_k) \subseteq \mathbf{R}^{S_k}$ for each $k = 1, \dots, q$. Let $B^\oplus$ denote the face of $B(g)$ defined by the canonical chain $\mathcal{C}^* = \{C_1, \dots, C_q\}$, that is, $B^\oplus$ is the direct sum of the M-convex sets B_k ($k = 1, \dots, q$). We note that $B^\oplus$ is an M-convex set because the direct sum of M-convex sets is an M-convex set (see [39]).

We define T_k ($k = 1, \dots, q$) and $T^\oplus$, which is given as the direct sum of T_k ($k = 1, \dots, q$), by using the essential value-sequence as follows:

$$\begin{aligned} T_k &= \{x \in \mathbf{Z}^{S_k} \mid \beta_k - 1 \leq x_i \leq \beta_k \ (i \in S_k)\}, \\ T^\oplus &= \{x \in \mathbf{Z}^N \mid \beta_k - 1 \leq x_i \leq \beta_k \ (i \in S_k), k = 1, \dots, q\}. \end{aligned}$$

The intersection of an M-convex set with an integral box is always an M-convex set, and hence $B^\oplus \cap T^\oplus$ is an M-convex set. The following theorem shows that the set of dec-min elements of an M-convex set forms an M-convex set and is characterized by the canonical partition.

Theorem 3.4. (Frank–Murota [19], Theorem 5.1) *The set of decreasingly minimal elements of $B(g)$ is equal to $B^\oplus \cap T^\oplus$. That is, an element $x \in B(g)$ is decreasingly minimal in $B(g)$ if and only if $x_{S_k} \in B_k \cap T_k$ holds for each $k = 1, \dots, q$. $\square$*

This theorem also implies that for every dec-min element x of $B(g)$, we have

$$x_i \in \{\beta_k - 1, \beta_k\} \quad (i \in S_k). \quad (3.1)$$

4. Egalitarian Solution in the Discrete Case

In this section, we investigate the properties of the egalitarian solution in Case **Z** in comparison with the case **R**. The main properties of the egalitarian solution in Case **R** are as follows:

Property 1: There is at most one egalitarian solution in any game.

Property 2: In convex games, there exists an egalitarian solution and it is in the core.

Property 3: In convex games, the egalitarian solution Lorenz-dominates every other element of the core.

We will investigate whether the egalitarian solution in the discrete cases has these properties. In this section, we show the following (Definition of discrete games will be given in Section 4.1) :

- In discrete games, there may exist multiple egalitarian solutions (Example 4.1).
- In discrete convex games, there exists at least one egalitarian solution (Theorem 4.4).
- In discrete convex games, there may exist an egalitarian solution outside the core (Example 4.2).
- In discrete convex games, every element of egalitarian solutions in the core, if any, Lorenz-dominates every element of the core that is not an egalitarian solution (Theorem 4.3). In addition, the egalitarian solutions outside the core do not necessarily Lorenz-dominate every element of the core that is not an egalitarian solution (Example 4.2).

4.1. Preliminaries on the egalitarian solution in the discrete case

We first define a game with discrete side payment. A *game with discrete side payment* will mean a game (N, v) , where the characteristic function v is integer-valued and payoffs of players are integral vectors. We call a game with discrete side payment a *discrete game* for short and denote the set of discrete games by $\Gamma_{\mathbf{Z}}$. Also, we say that a discrete game is a *discrete convex game* when its characteristic function is supermodular. We denote the set of discrete convex games by $\Gamma_{\mathbf{Z}}^c$. For clarity, we denote the set of games in continuous variables and the set of convex games in continuous variables by $\Gamma_{\mathbf{R}}$ and $\Gamma_{\mathbf{R}}^c$, respectively. See [6, 35, 49] for the details about cooperative games with discrete side payment.

We define the egalitarian solution in discrete games by simply replacing **R** with **Z** in the definitions of Section 2. Specifically, it is defined as follows. For a discrete game $(N, v) \in \Gamma_{\mathbf{Z}}$,

the Lorenz core of a singleton coalition is $L(\{i\}) = \{v(\{i\})\}$ ($i \in N$). We note that since v is integer-valued, $L(\{i\})$ is a set of an integral vector for each $i \in N$. Now suppose that the Lorenz cores for all coalitions of cardinality $k-1$ or less have been defined, where $2 \leq k < n$. The Lorenz core of coalitions S of size k is defined by

$$L(S, v) = \{x \in \mathbf{Z}^S \mid x(S) = v(S) \text{ and there is no } T \subsetneq S \\ \text{and } y \in E(L(T, v)) \text{ such that } x_T \preceq y\}.$$

By the definition, $L(S, v)$ is composed of integral vectors for each $S \subseteq N$.

Then, analogously to the Case **R**, we call an element of $E(L(N, v))$ an *egalitarian solution* for the discrete game $(N, v) \in \Gamma_{\mathbf{Z}}$, where $E(L(N, v))$ is a subset of the Lorenz core $L(N, v)$ that are not Lorenz-dominated by any other element of the Lorenz core.

4.2. Examples of games with discrete side payment

In this subsection, we show three examples of discrete games.

- *Bankruptcy problem*: Side payments are not necessarily divisible goods. For example, if side payments for cooperation are stock, bond, or real assets etc., then the side payments can be integer. Mermelshtejn et al. [35] give the following modified bankruptcy problem as an application of cooperative games with discrete side payment. Assume that a bankrupt firm produced some indivisible goods and creditors prefer to receive finished products because of insufficiency of the firm's monetary compensation. Then the bankruptcy problem can be discrete.
- *Weighted voting game*: The problem of how to assign a given number of cabinet ministries to a committee of political parties is studied in cooperative game theory. Wolff-Karagök [48] study the allocation of cabinet seats in Switzerland's Federal government. They formulate the allocation problem as a coalitional game (N, v) , where N is the finite set of all political parties and the characteristic function $v(S)$ for a coalition $S \subseteq N$ is defined to be 0 if the coalition loses, and otherwise (i.e., the coalition wins) $v(S)$ is defined to be the number of all government seats. Wolff-Karagök [48] analyze a nucleolus of their voting game over real-valued and integer-valued imputation sets, where an element of the imputation sets is interpreted as the number of government seats assigned to each party.
- *Communication for omniscience (CO) problem*: In this problem, we consider a set of users who exchange their observations of a multiple source and the goal is to attain omniscience, i.e., the state that all users recover the entire source [8, 12–14, 36]. An optimization problem that naturally arises is how to minimize the overall source coding rate to attain omniscience. In particular, some papers [12–14, 36] discuss the problem of finding an optimal coding rate vector that also attains fairness, and formulate the problem as a cooperative game. Ding et al. [13] study CO problems formulated as a coalitional game in continuous variables, where the broadcast rates are real-valued, and analyze the egalitarian solution and shapley value in this game. In contrast, the papers [12, 14, 36] consider the same coalitional game in the setting with discrete variables, where the communication rates are restricted to be integral.

4.3. Relationship between the egalitarian solution and polymatroid theory

In this subsection, we describe the connection between the egalitarian solution in convex games and the polymatroid theory.

First, it is obvious from the definitions that if (N, v) is a convex game, then the core $C(N, v)$ coincides with the base polyhedron. Then, the following relationship holds between the lexicographically optimal base and the egalitarian solution.

Theorem 4.1. ([22, 26]) *In convex games $\Gamma_{\mathbf{R}}^c$, the lexicographically optimal base is equivalent to the egalitarian solution.* $\square$

On the other hand, as is shown in Section 4.4, the lexicographically optimal base (decreasingly minimal element) is not equivalent to the egalitarian solution in discrete convex games.

Next, we investigate the relationship between the core and an M-convex set. For a game (N, v) , if v is an integer-valued supermodular function, then the core of the game is an integral base polyhedron. Since an M-convex set is the set of integral members of an integral base polyhedron, the core of a discrete convex game is an M-convex set. Therefore, the following property holds in discrete convex games by Theorem 3.3.

Proposition 4.2. *In discrete convex games $\Gamma_{\mathbf{Z}}^c$, the core admits a least majorized element.* $\square$

4.4. Properties of the egalitarian solution in Case $\mathbf{Z}$

We first consider Property 1. Example 4.1 below shows that there can exist multiple egalitarian solutions in Case $\mathbf{Z}$. That is, Property 1 does not hold in Case $\mathbf{Z}$.

Example 4.1. Let $N = \{1, 2\}$, $v(\{i\}) = 0$ ($i \in N$) and $v(N) = 1$. The egalitarian solutions are $E(L(N, v)) = \{(1, 0), (0, 1)\}$, which implies the non-uniqueness of the egalitarian solution in Case $\mathbf{Z}$. $\square$

Next, we consider Property 2. The non-uniqueness of the egalitarian solution in discrete games suggests two separate problems in Case $\mathbf{Z}$. The first question is whether there exists at least one egalitarian solution for any discrete convex game. The second is what is the relationship between the core and the egalitarian solution.

We first consider the existence of the egalitarian solution in discrete convex games. The following fundamental result is a key property in this paper.

Theorem 4.3. *For any discrete convex game $(N, v) \in \Gamma_{\mathbf{Z}}^c$, there exists some $x \in C(N, v)$ that Lorenz-dominates any $y \in C(N, v)$ with $y \downarrow \neq x \downarrow$.* $\square$

Theorem 4.3 is derived from Proposition 4.2 as follows. Proposition 4.2 and the fact that the core in discrete convex games is an M-convex set imply that there exists a least majorized element in the core. By these facts and the relationship between being majorized and Lorenz-domination as seen in Section 3.1, we obtain that for any discrete convex game, there exists an element of the core that Lorenz-dominates every element of the core not value-equivalent to the element.

Next, we show the existence of the egalitarian solution in any discrete convex game. Note that the following theorem does not state that all egalitarian solutions are contained in the core.

Theorem 4.4. *In discrete convex games, there exists an egalitarian solution, that is, for any $(N, v) \in \Gamma_{\mathbf{Z}}^c$, $E(L(N, v)) \neq \emptyset$ holds.* $\square$

Proof. Assume, to the contrary, that $E(L(N, v)) = \emptyset$. Take any $x \in C(N, v)$ in Theorem 4.3. If there is no element of the Lorenz core that Lorenz-dominates x , then $x \in E(L(N, v))$ holds, which contradicts the assumption that $E(L(N, v)) = \emptyset$. Therefore there exists some $y \in L(N, v)$ that Lorenz-dominates x . Here we obtain $y \notin C(N, v)$, since otherwise x is Lorenz-dominated by the core element y , which contradicts the fact that x is not Lorenz-dominated by any element of the core by Theorem 4.3. Thus we have $y \in L(N, v) \setminus C(N, v)$. Note that $y \neq x$.

Then, $E(L(N, v)) = \emptyset$ shows that $y \notin E(L(N, v))$. Hence there exists some $y_1 \in L(N, v)$ that Lorenz-dominates y . By the above argument, we have $y_1 \notin C(N, v)$ and $y_1 \neq y$. Moreover, $y_1 \neq x$ holds since if y_1 Lorenz-dominates y , then y_1 Lorenz-dominates x . By

repeating the above arguments, we arrive at the Lorenz core element y_k that is not Lorenz-dominated by any element of $L(N, v)$ because $L(N, v)$ is bounded. Note that $y, y_1, \dots, y_k, x$ are all distinct. However, this contradicts the assumption that $E(L(N, v)) = \emptyset$. $\square$

Remark 4.1. From the proof of Theorem 4.4, we obtain that for any discrete convex game $(N, v) \in \Gamma_{\mathbf{Z}}^c$ and for each $y \in L(N, v) \setminus E(L(N, v))$, there exists an $x \in E(L(N, v))$ that Lorenz-dominates y . Llerena–Mauri [33] calls this property the *external Lorenz stability*. $\square$

Next, we consider the relationship between the core and the egalitarian solution in Case **Z**. In Case **R**, the egalitarian solution always belongs to the core for any convex game. The following example reveals that, in Case **Z**, there can exist an egalitarian solution outside the core even in convex games.

Example 4.2. Let $N = \{1, 2, 3\}$ and define v as in the following table. This example is based on Example 5 in Dutta–Ray [16].

Table 1: Values of v and egalitarian solutions in Example 4.2

S	$v(S)$	$EL(S, v)$
$\{1\}$	40	$\{40\}$
$\{2\}$	60	$\{60\}$
$\{3\}$	80	$\{80\}$
$\{1, 2\}$	110	$\{(50, 60)\}$
$\{1, 3\}$	120	$\{(40, 80)\}$
$\{2, 3\}$	150	$\{(70, 80)\}$
$\{1, 2, 3\}$	210	$\{(60, 70, 80), (64, 65, 81), (65, 64, 81)\}$

This game is convex and $(60, 70, 80) \in E(L(N, v))$ is in the core. However, $(64, 65, 81)$ and $(65, 64, 81) \in E(L(N, v))$ are not contained in the core because for each vector, the sum of its second and third components is as follows respectively.

$$65 + 81 = 146 < 150 = v(\{2, 3\}), \quad 64 + 81 = 145 < 150 = v(\{2, 3\}). \quad \square$$

Example 4.2 poses the question whether there always exists an egalitarian solution in the core. However, this question remains unsolved in this paper.

As mentioned in Theorem 4.1, in convex games, lexicographically optimal base and the egalitarian solution are equivalent in Case **R**. However, the fact that not all egalitarian solutions belong to the core even in discrete convex games shows that, the set of egalitarian solutions do not necessarily coincide with the set of dec-min elements in an M-convex set in Case **Z**.

Remark 4.2. We have not yet obtained an algorithm that can find an egalitarian solution for any discrete convex game. $\square$

Finally, we consider Property 3. In discrete convex games, we have to consider two problems:

- (a) Whether every egalitarian solution Lorenz-dominates every element of the core that is not an egalitarian solution.
- (b) Whether all egalitarian solutions satisfying (a) Lorenz-dominate every other element of the Lorenz core.

We first demonstrate that the egalitarian solutions outside the core do not satisfy Property 3, and then we reveal that egalitarian solutions of the core have this property.

We consider Example 4.2 again. The egalitarian solutions outside the core of the game of Example 4.2 are $(64, 65, 81)$ and $(65, 64, 81)$ (Table 1) and we take $(64, 65, 81) \in E(L(N, v))$. For example, vector $(59, 71, 80)$ is in the core. Since its largest component 80 is smaller than that of $(64, 65, 81)$, $(64, 65, 81)$ does not Lorenz-dominate $(59, 71, 80)$. Note that $(59, 71, 80) \notin E(L(N, v))$ because $(60, 70, 80)$ Lorenz-dominates $(59, 71, 80)$. Thus, in discrete convex games, egalitarian solutions outside the core do not necessarily Lorenz-dominate every element of the core except for the egalitarian solution. This is the distinction between the case **R** and **Z**.

We next consider the egalitarian solutions in the core. Since all least majorized elements in the core Lorenz-dominate every element of the core that is not a least majorized element, the set of the egalitarian solutions in the core coincides with the set of the least majorized elements in the core. Therefore, in discrete convex games, Property 3 holds for the egalitarian solutions in the core.

In contrast, Property 3 does not hold for the elements of the Lorenz core as follows. For the game of Example 4.2, the egalitarian solution $(60, 70, 80) \in E(L(N, v))$ does not Lorenz-dominate $(64, 64, 82) \in L(N, v) \setminus E(L(N, v))$. Recall that $(60, 70, 80) \in C(N, v)$. This fact shows that the egalitarian solutions of the core do not necessarily Lorenz-dominate every element of the Lorenz core that are not contained in $E(L(N, v))$.

5. Reduced Game Property

By Example 4.2, we see that there can exist an egalitarian solution outside the core even in discrete convex games. Therefore, in Case **Z**, the egalitarian solution and the dec-min element of the core are not equivalent. Also, we do not know about the existence of the egalitarian solution of the core in discrete convex games.

Thus, we are motivated to consider the Lorenz stable set introduced by Arin–Inarra [1] and Houggaard et al. [27], a subset of the core consisting of the elements that are not Lorenz-dominated by any other element of the core. Their approach is based on the fact that the core is considered to be the set of natural stable allocations. For example, in a class of cost and surplus sharing games, the core plays a crucial role (see e.g., [44]). In particular, this class is contained in a class of games arising from combinatorial optimization problems including the polymatroid theory, where the core also plays a central role (see [9, 27, 44]). We follow their approach and show that the Lorenz stable set in discrete convex games has nice properties such as the Davis–Maschler reduced game property and the converse reduced game property in Sections 5 and 6.

Dutta [15] has already shown that the egalitarian solution in convex games in continuous variables has these nice properties. He derives these results by making use of the properties of the principal partition explained in Section 2.3. The point in our study is that we can give the proofs of these properties by utilizing the canonical partition and the canonical chain due to Frank–Murota [19].

The results of Sections 5 and 6 are summarized as follows.

1. In discrete convex games, the Lorenz stable set is nonempty, and every element of it Lorenz-dominates every element of the core not contained in the Lorenz stable set (Theorems 5.1 and 5.2).
2. In discrete convex games, the Lorenz stable set has the Davis–Maschler reduced game property and the converse reduced game property (Theorems 5.6 and 6.2).

5.1. Lorenz stable set

First, we give the definition of the Lorenz stable set.

Definition 5.1. (Lorenz stable set [1, 27]) For a game $(N, v) \in \Gamma_{\mathbf{R}}$, the *Lorenz stable set* $\text{LSS}(N, v)$ is defined as follows:

$$\text{LSS}(N, v) = \{x \in C(N, v) \mid \nexists y \in C(N, v) : y \text{ Lorenz-dominates } x\}. \quad (5.1)$$

□

We define the Lorenz stable set in discrete games by replacing $\Gamma_{\mathbf{R}}$ with $\Gamma_{\mathbf{Z}}$ in the above definition.

Remark 5.1. Hougaard et al. (2001) [27] introduced the notion of the *Lorenz maximal imputation*, whose definition is exactly the same as the Lorenz stable set. In this paper, we use the term “Lorenz stable set” following Arin–Inarra (2001) [1]. □

The Lorenz stable set is contained in the core by its definition. That is,

$$\text{LSS}(N, v) \subseteq C(N, v) \quad (5.2)$$

holds for each game $(N, v) \in \Gamma$ regardless of the case $\mathbf{R}$ and $\mathbf{Z}$.

The following properties hold for the Lorenz stable set.

Theorem 5.1. For any discrete convex game, the Lorenz stable set is nonempty. □

Theorem 5.2. For any discrete convex game $(N, v) \in \Gamma_{\mathbf{Z}}^c$, if $x \in \text{LSS}(N, v)$, then x Lorenz-dominates every element of the core except for the elements value-equivalent to x . □

Theorem 5.2 shows that the Lorenz stable set has Property 3. This is one of the reasons that we consider the Lorenz stable set instead of the egalitarian solution in Case $\mathbf{Z}$.

5.2. Davis–Maschler reduced game property

In this subsection, we consider the Davis–Maschler reduced game property of the Lorenz stable set by using the properties of the canonical chain and the canonical partition describing the structures of dec-min elements of an M-convex set.

We first show that for every discrete convex game, the Lorenz stable set coincides with the set of dec-min elements of the core. This enables us to apply the results of Frank–Murota [19] to the study of the Lorenz stable set.

Proposition 5.3. In discrete convex games, the Lorenz stable set coincides with the set of dec-min elements in the core. □

Proof. Note first that since every least majorized element of the core Lorenz-dominates every element of the core that is not a least majorized element, the Lorenz stable set coincides with the set of the least majorized elements of the core (see also Definition 5.1). The existence of a least majorized element of the core is guaranteed by Proposition 4.2. Since if the core admits a least majorized element, then an element is the least majorized in the core if and only if it is a dec-min element in the core by Proposition 3.1, the Lorenz stable set coincides with the set of dec-min elements of the core. □

Remark 5.2. As noted in Section 2.1, even when we define the notion of Lorenz-domination in an increasing order, its change does not affect the results of Sections 5 and 6. This is justified by the following property (e.g., Frank–Murota [19] and Tamir [46]). Let Q be an arbitrary subset of $\mathbf{R}^N$ and assume that Q admits a least majorized element. For any $x \in Q$ the following three conditions are equivalent.

- (A) x is least majorized in Q .
- (B) x is decreasingly minimal in Q .
- (C) x is increasingly maximal in Q .

Since, in discrete convex games, the Lorenz stable set coincides with the set of dec-min elements of the core by Proposition 5.3, the equivalence between (B) and (C) implies that the Lorenz stable set coincides with the set of inc-max elements of the core. $\square$

Remark 5.3. It follows from Proposition 5.3 that we can obtain the Lorenz stable set in the core for any discrete convex game by using the algorithm of Frank–Murota [20] that can find a decreasingly minimal element in an M-convex set. $\square$

Next we define the reduced game and the Davis–Maschler reduced game property.

Definition 5.2. (Reduced game (Davis–Maschler [10])) Let $(N, v) \in \Gamma_{\mathbf{R}}$ be a game, $S \subsetneq N$, and $x \in \mathbf{R}^N$ be a payoff vector. The *reduced game* with respect to S and x is the game (S, v_S^x) where

$$v_S^x(T) = \begin{cases} 0 & (T = \emptyset), \\ v(N) - x(N \setminus S) & (T = S), \\ \max_{Q \subseteq N \setminus S} \{v(T \cup Q) - x(Q)\} & (T \subsetneq S). \end{cases} \quad (5.3)$$

$\square$

Definition 5.3. (Davis–Maschler reduced game property [10]) Let σ be a solution over a class $\Gamma_{\mathbf{R}}$ of games. Then σ is said to have the *Davis–Maschler reduced game property* over $\Gamma_{\mathbf{R}}$, when for all $(N, v) \in \Gamma_{\mathbf{R}}$, for all $x \in \sigma(N, v)$, and for all $S \subsetneq N$, $(S, v_S^x) \in \Gamma_{\mathbf{R}}$ and $x_S \in \sigma(S, v_S^x)$ hold. $\square$

We define the reduced game and the Davis–Maschler reduced game property in Case **Z** by replacing **R** with **Z** in the above definitions.

Dutta [15] shows the following fact in Case **R**. This also holds for any discrete convex game. For completeness, we give the proof.

Lemma 5.4. *For any discrete convex game $(N, v) \in \Gamma_{\mathbf{Z}}^c$, for all $S \subseteq N$, and for all $y \in \text{LSS}(N, v)$, (S, v_S^y) is a discrete convex game.* $\square$

Proof. For any $T_i \subseteq S$ ($i = 1, 2$), there exists some $R_i \subseteq N \setminus S$ such that

$$v_S^y(T_i) = \max\{v(T_i \cup R) - y(R) \mid R \subseteq N \setminus S\} = v(T_i \cup R_i) - y(R_i).$$

By using the supermodularity of v , we have

$$\begin{aligned} & v_S^y(T_1) + v_S^y(T_2) \\ &= v(T_1 \cup R_1) - y(R_1) + v(T_2 \cup R_2) - y(R_2) \\ &= v(T_1 \cup R_1) + v(T_2 \cup R_2) - y(R_1 \cup R_2) - y(R_1 \cap R_2) \\ &\leq v((T_1 \cup R_1) \cup (T_2 \cup R_2)) + v((T_1 \cup R_1) \cap (T_2 \cup R_2)) - y(R_1 \cup R_2) - y(R_1 \cap R_2) \\ &= v((T_1 \cup T_2) \cup (R_1 \cup R_2)) - y(R_1 \cup R_2) + v((T_1 \cap T_2) \cup (R_1 \cap R_2)) - y(R_1 \cap R_2) \\ &\leq \max\{v((T_1 \cup T_2) \cup Q) - y(Q) \mid Q \subseteq N \setminus S\} \\ &\quad + \max\{v((T_1 \cap T_2) \cup Q) - y(Q) \mid Q \subseteq N \setminus S\} \\ &= v_S^y(T_1 \cup T_2) + v_S^y(T_1 \cap T_2), \end{aligned}$$

which shows the supermodularity of v_S^y . $\square$

Peleg [43] has already shown the Davis–Maschler reduced game property of the core in Case **R**. This also holds for any discrete game. Its proof is exactly the same as that of Peleg, but, we give the proof for the sake of completeness. Note that we do not assume the convexity of games in the following theorem.

Theorem 5.5. *The core has the Davis–Maschler reduced game property for any discrete game. That is, for all $(N, v) \in \Gamma_{\mathbf{Z}}$, for all $x \in C(N, v)$, and for all $S \subsetneq N$, $(S, v_S^x) \in \Gamma_{\mathbf{Z}}$ and $x_S \in C(S, v_S^x)$ hold. $\square$*

Proof. Take any $x \in C(N, v)$ and $S \subseteq N$ ($S \neq \emptyset$). We want to show that $x_S \in C(S, v_S^x)$. First we note that $x(S) = v_S^x(S)$. Indeed, if $T = S$, then using $x(N) = v(N)$, we have $v_S^x(T) - x(T) = v(N) - x(N \setminus S) - x(S) = v(N) - x(N) = 0$, which shows that $x(S) = v_S^x(S)$. If $T \subsetneq S$, then the following inequality holds:

$$\begin{aligned} v_S^x(T) - x(T) &= \max\{v(T \cup Q) - x(Q) \mid Q \subseteq N \setminus S\} - x(T) \\ &= \max\{v(T \cup Q) - x(T \cup Q) \mid Q \subseteq N \setminus S\} \\ &\leq 0. \end{aligned}$$

This inequality implies that $x(T) \geq v_S^x(T)$ for any $T \subsetneq S$. Therefore we obtain $x_S \in C(S, v_S^x)$. $\square$

Here we show the Davis–Maschler reduced game property of the Lorenz stable set in discrete convex games. We emphasize that the proof of the following theorem relies heavily on the properties of the canonical chain and the canonical partition.

Theorem 5.6. *For any discrete convex game $(N, v) \in \Gamma_{\mathbf{Z}}^c$, the Lorenz stable set has the Davis–Maschler reduced game property. That is, if $x \in \text{LSS}(N, v)$, then $x_S \in \text{LSS}(S, v_S^x)$ holds for all $S \subseteq N$ ($S \neq \emptyset$). $\square$*

Proof. Assume, to the contrary, that for some $y \in \text{LSS}(N, v)$ and for some $T \subsetneq N$ ($T \neq \emptyset$), y_T is Lorenz-dominated by some $x \in \text{LSS}(T, v_T^y)$. Note that $x \in C(T, v_T^y)$ by (5.2). Then, we can prove the following claim, which is proved later.

Claim 1.

$$\exists k \in \{1, \dots, q\} : \sum_{i \in T \cap C_k} y_i > \sum_{i \in T \cap C_k} x_i, \quad (5.4)$$

where $\{C_1, \dots, C_q\}$ is the canonical chain for N constructed by the iterative procedure in Section 3.2.

For k in Claim 1, let $R = T \cap C_k$ and $R_k = C_k \setminus T$. Then, we have

$$v(R_k \cup R) = v(C_k) = \sum_{i \in C_k} y_i. \quad (5.5)$$

The second equality follows from $y(C_k) = v(C_k)$ for each $k = 1, \dots, q$ (see Theorem 3.4). By the definition of v_T^y , (5.5) and the inequality of Claim 1, we obtain

$$\begin{aligned} v_T^y(R) &= \max\{v(Q \cup R) - y(Q) \mid Q \subseteq N \setminus T\} \geq v(R \cup R_k) - \sum_{i \in R_k} y_i \\ &= \sum_{i \in R} y_i > \sum_{i \in R} x_i, \end{aligned}$$

which contradicts $x \in \text{LSS}(T, v_T^y) \subseteq C(T, v_T^y)$ (see also (5.2)).

We now prove Claim 1. Assume, to the contrary, that

$$\forall k \in \{1, \dots, q\} : \sum_{i \in T \cap C_k} y_i \leq \sum_{i \in T \cap C_k} x_i. \quad (5.6)$$

$$\begin{array}{l}
x : \overbrace{\beta_1 = \cdots = \beta_1}^r = \beta_1 > \overbrace{\beta_1 - 1 = \cdots = \beta_1 - 1}^{|T \cap C_1| - r} \\
y : \overbrace{\beta_1 = \cdots = \beta_1}^{\text{at most } r-1} > \beta_1 - 1 \geq \dots
\end{array}$$

Figure 1: Values of x and y on $T \cap C_1$

Under this assumption, we will show the value-equivalence of x and y_T on T , which contradicts the assumption that x Lorenz-dominates y_T . Then we are done.

First, we show that x and y_T are value-equivalent on $T \cap C_1$. We may assume that $T \cap C_1 \neq \emptyset$. By Theorem 3.4 and (3.1), $y_j = \beta_1$ or $y_j = \beta_1 - 1$ holds for all $j \in T \cap C_1$. Two cases are to be distinguished.

- (1) The case where $y_j = \beta_1$ for all $j \in T \cap C_1$. Since x Lorenz-dominates y_T , we have $x_j \leq \beta_1$ for all $j \in T \cap C_1$. Then, this fact and (5.6) show that $x_j = \beta_1$ holds for all $j \in T \cap C_1$. Therefore, x and y_T are value-equivalent on $T \cap C_1$.
- (2) The case where $y_j = \beta_1 - 1$ for some $j \in T \cap C_1$. We show that

$$\sum_{i \in T \cap C_1} y_i = \sum_{i \in T \cap C_1} x_i. \quad (5.7)$$

Assume that $\sum_{i \in T \cap C_1} y_i < \sum_{i \in T \cap C_1} x_i$ holds. Then, since x Lorenz-dominates y_T , $x_j \leq \beta_1$ holds for all $j \in T \cap C_1$. Hence, this inequality implies that the number of β_1 -valued components of y_T is strictly smaller than that of x (see Figure 1), which contradicts the assumption that x Lorenz-dominates y_T . Therefore, we have (5.7). This equation, together with the facts that $x_i \leq \beta_1$ holds for all $i \in T \cap C_1$ and either $y_i = \beta_1$ or $y_i = \beta_1 - 1$ holds, shows that x and y_T are value-equivalent on $T \cap C_1$.

Since x and y are value-equivalent on $T \cap C_1$ as above, (5.6) implies the following:

$$\sum_{i \in T \cap (C_2 \setminus C_1)} y_i \leq \sum_{i \in T \cap (C_2 \setminus C_1)} x_i. \quad (5.8)$$

Next, we show the value-equivalence between x and y_T on $T \cap C_2$, that is, x and y_T are value-equivalent on $T \cap (C_2 \setminus C_1)$. Note first that either $y_j = \beta_2$ or $y_j = \beta_2 - 1$ holds for all $j \in T \cap (C_2 \setminus C_1)$ by Theorem 3.4 and (3.1).

- (1) The case where $y_j = \beta_2$ for all $j \in T \cap (C_2 \setminus C_1)$. Then, since x and y_T are value-equivalent on $T \cap C_1$ and x Lorenz-dominates y_T , we obtain $x_j \leq \beta_2$ ($\forall j \in T \cap (C_2 \setminus C_1)$). This statement and (5.8) show that $x_j = \beta_2$ holds for all $j \in T \cap (C_2 \setminus C_1)$, which implies that x and y_T are value-equivalent on $T \cap (C_2 \setminus C_1)$.
- (2) The case where $y_k = \beta_2 - 1$ for some $k \in T \cap (C_2 \setminus C_1)$. We will first show that $\sum_{i \in T \cap (C_2 \setminus C_1)} y_i = \sum_{i \in T \cap (C_2 \setminus C_1)} x_i$ holds. Assume that

$$\sum_{i \in T \cap (C_2 \setminus C_1)} y_i < \sum_{i \in T \cap (C_2 \setminus C_1)} x_i. \quad (5.9)$$

By the facts that y_j is either β_2 or $\beta_2 - 1$ for any $j \in T \cap (C_2 \setminus C_1)$, x Lorenz-dominates y_T , and x and y_T are value-equivalent on $T \cap C_1$, we have $x_j \leq \beta_2$ for all $j \in T \cap (C_2 \setminus C_1)$.

$$\begin{array}{c}
\begin{array}{c}
\overbrace{\beta_1 = \dots = \beta_1}^{r_1} > \overbrace{\beta_1 - 1 = \dots = \beta_1 - 1}^{|T \cap C_1| - r_1} \geq \overbrace{\beta_2 = \dots = \beta_2}^{r_2} > \overbrace{\beta_2 - 1 = \dots = \beta_2 - 1}^{T \cap (C_2 \setminus C_1)} \\
\overbrace{\beta_1 = \dots = \beta_1}^{r_1} > \overbrace{\beta_1 - 1 = \dots = \beta_1 - 1}^{|T \cap C_1| - r_1} \geq \overbrace{\beta_2 = \dots = \beta_2}^{\text{at most } r_2 - 1} > \beta_2 - 1 \geq \dots
\end{array}
\end{array}$$

Figure 2: Values of x and y on $T \cap C_2$

Therefore, if (5.9) is true, then the number of β_2 -valued components of y_T is strictly smaller than that of x on $T \cap (C_2 \setminus C_1)$ (see Figure 2), which contradicts the assumption that x Lorenz-dominates y_T together, since x and y_T are value-equivalent on $T \cap C_1$.

From the above arguments, we obtain that $\sum_{i \in T \cap C_2} y_i = \sum_{i \in T \cap C_2} x_i$ and x and y_T are value-equivalent on $T \cap C_2$. By repeating this argument until $k = q$, we have that x and y_T are value-equivalent on T . Thus the proof of Claim 1 is completed. $\square$

Remark 5.4. Here, we demonstrate that the egalitarian solution in discrete variables fails to have the Davis–Maschler reduced game property even in discrete convex games. We reconsider the game of Example 4.2. The value of v_S^x and the set of egalitarian solutions of the reduced game with respect to $S = \{2, 3\}$ and $x = (64, 65, 81) \in E(L(N, v))$ are given as in Table 2.

Table 2: Values of v_S^x and the egalitarian solution of (S, v_S^x)

T	$v_S^x(T)$	$E(L(T, v_S^x))$
$\{2\}$	60	$\{60\}$
$\{3\}$	80	$\{80\}$
$\{2, 3\}$	146	$\{(66, 80)\}$

The values of v_S^x are calculated as follows:

$$\begin{aligned}
v_S^x(\{2\}) &= \max\{v(\{2\}), v(\{1, 2\}) - x_1\} = \max\{60, 110 - 64\} = 60, \\
v_S^x(\{3\}) &= \max\{v(\{3\}), v(\{1, 3\}) - x_1\} = \max\{80, 120 - 64\} = 80, \\
v_S^x(\{2, 3\}) &= v(\{1, 2, 3\}) - x_1 = 210 - 64 = 146.
\end{aligned}$$

Then, we obtain $x_S = (65, 81) \notin E(L(S, v_S^x))$, which shows that the egalitarian solutions outside the core fail to have the Davis–Maschler reduced game property even in discrete convex games. $\square$

Remark 5.5. For convex games in continuous variables, the Lorenz stable set is characterized by the Davis–Maschler reduced game property and the constrained egalitarianism (e.g., Arin–Inarra [1]) and, on the domain of essential games, the only single-valued solution satisfying the constrained egalitarianism and the projection consistency* is the Lorenz stable set [31]. Axiomatic characterizations of the Lorenz stable set in discrete convex games remains unsolved in our paper. $\square$

*The concept of projection consistency is introduced by Funaki [23].

6. Converse Reduced Game Property

In this section, we consider the converse reduced game property of the Lorenz stable set in discrete convex games. Peleg [43] defines the converse reduced game property as follows.

Definition 6.1. (Converse reduced game property (Peleg [43])) Let σ be a solution on $\Gamma_{\mathbf{R}}$. A solution σ is said to have the *converse reduced game property* if the following condition is satisfied: For $x \in \mathbf{R}^N$ with $x(N) = v(N)$, if $(N, v) \in \Gamma_{\mathbf{R}}$ and $x_S \in \sigma(S, v_S^x)$ for every $S \subseteq N$ with $|S| = 2$, then $x \in \sigma(N, v)$ holds. $\square$

We define the converse reduced game property in Case **Z** by replacing **R** with **Z** in the above definition.

Peleg [43] has already shown the converse reduced game property of the core. This is also true for any discrete game. Its proof is exactly the same as that of Peleg, but, we give the proof for the sake of completeness. Note that we do not assume the convexity of games in the following lemma.

Lemma 6.1. For any discrete game $(N, v) \in \Gamma_{\mathbf{Z}}$, the core satisfies the converse reduced game property. $\square$

Proof. For a discrete game $(N, v) \in \Gamma_{\mathbf{Z}}$, let $x \in \mathbf{Z}^N$ be a vector satisfying $x(N) = v(N)$ and $x_S \in C(S, v_S^x)$ for every S with $|S| = 2$. Take any $T \subsetneq N$ ($T \neq \emptyset$), $i \in T$, and $j \in N \setminus T$. Let $Q = \{i, j\}$. Then, by using $x_Q \in C(Q, v_Q^x)$, we obtain the following inequalities:

$$\begin{aligned} 0 \geq v_Q^x(\{i\}) - x_i &\geq v((T \setminus \{i, j\}) \cup \{i\}) - x(T \setminus \{i, j\}) - x_i \\ &= v(T) - x(T), \end{aligned}$$

where the equality is due to $j \notin T$. Therefore, $x(T) \geq v(T)$ holds for every $T \subsetneq N$. Also, we have $x(N) = v(N)$ by the hypothesis of x . Thus, we obtain $x \in C(N, v)$. $\square$

6.1. Converse reduced game property in Case **Z**

In this subsection, we prove the converse reduced game property of the Lorenz stable set in Case **Z**.

Theorem 6.2. For any discrete convex game $(N, v) \in \Gamma_{\mathbf{Z}}^c$, the Lorenz stable set has the converse reduced game property, that is, for $x \in \mathbf{Z}^N$ with $x(N) = v(N)$, if $(N, v) \in \Gamma_{\mathbf{Z}}^c$ and $x_S \in \text{LSS}(S, v_S^x)$ for every $S \subseteq N$ with $|S| = 2$, then $x \in \text{LSS}(N, v)$ holds. $\square$

Proof. Note first that $x \in C(N, v)$ by Lemma 6.1. Suppose that $x \notin \text{LSS}(N, v)$. Since the Lorenz stable set coincides with the set of dec-min elements of $C(N, v)$ by Proposition 5.3, x is not a dec-min element of $C(N, v)$. Therefore, by Theorem 3.2, there is a 1-tightening step for x , that is, there exist some $i, j \in N$ such that $x_j \geq x_i + 2$ and $x' = x + \chi_i - \chi_j \in C(N, v)$. Since the core satisfies the Davis–Maschler reduced game property by Theorem 5.5, we have $x'_{\{i, j\}} \in C(\{i, j\}, v_{\{i, j\}}^{x'})$. Note that $x_{\{i, j\}} \in \text{LSS}(\{i, j\}, v_{\{i, j\}}^x) \subseteq C(\{i, j\}, v_{\{i, j\}}^x)$ by the hypothesis.

Let $R = \{i, j\}$. Then, using the equation $x_{N \setminus R} = x'_{N \setminus R}$, we can show that

$$v_R^{x'} = v_R^x. \quad (6.1)$$

Indeed, for any $T \subsetneq R$ ($T \neq \emptyset$), we obtain

$$\begin{aligned} v_R^{x'}(T) &= \max\{v(T \cup Q) - x'(Q) \mid Q \subseteq N \setminus R\} \\ &= \max\{v(T \cup Q) - x(Q) \mid Q \subseteq N \setminus R\} \\ &= v_R^x(T), \end{aligned}$$

where the first equality is due to (5.3) and the second equality follows from $x_{N \setminus R} = x'_{N \setminus R}$. Also, if $T = \emptyset$, then we have $v_R^{x'}(\emptyset) = v_R^x(\emptyset) = 0$ by the definition of $v_R^{x'}$ (see Definition 5.2). Similarly, for $T = R$, we can show $v_R^{x'}(R) = v_R^x(R)$ as follows:

$$\begin{aligned} v_R^{x'}(R) &= v(N) - x'(N \setminus R) \\ &= v(N) - x(N \setminus R) \\ &= v_R^x(R). \end{aligned}$$

From the above arguments we obtain (6.1). Hence we have

$$x'_R \in C(R, v_R^{x'}) = C(R, v_R^x). \quad (6.2)$$

It follows from $x_j \geq x_i + 2$ that $x'_R = (x_i + 1, x_j - 1)$ Lorenz-dominates $x_R = (x_i, x_j)$, which contradicts $x_R \in \text{LSS}(R, v_R^x)$ (see also Definition 5.1). $\square$

7. Conclusion

In this paper, we have pointed out that the egalitarian solution does not have nice properties in games with discrete side payment. Then, we have focused on the Lorenz stable set and shown that it has nice properties such as the Davis–Maschler reduced game property and the converse reduced game property. The existence of the egalitarian solution of the core and axiomatizations of the Lorenz stable set in discrete convex games are left for the future.

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