An improvement of the discrete method for analyzing the bending problem of plates

Mei HUANG *, Hironobu TAKAHASHI **, Hiroshi MATSUDA ***, Chihiro MORITA ****

An improved method is proposed to analyze the bending problem of plates. The fundamental differential equations are satisfied for the whole plate. By transforming these differential equations into integral equations in a small area, the quantities of an appointed point can be expressed by those of the other three points. By choosing the appointed point according to a regular order, the quantities of these three points can be replaced by the quantities of the boundary points. Finally, the quantities of any point can be expressed by those of the boundary points and the unknown quantities are only on the boundary. That makes the number of the unknown quantities and the computer time of the coefficient reduce greatly. The comparison of the present method with that used early is presented and the advantages of the present method are shown. Some numerical results are given by using uniform or non-uniform divisions. By comparing the numerical results obtained by the present method with those previously published, the efficiency and accuracy of the present method are investigated.

Key Words : bending problem, plate, uniform division, non-uniform division

1. Introduction

Plates are important components in aeronautical, mechanical and ocean structures. The analytical solutions of the plate are limited to only simple plate geometries and boundary conditions. For complex geometries and general boundary conditions, some numerical methods of analysis, such as the finite difference1), finite element2), spline element method3) or boundary element method4), are used.

In this paper, an improved discrete method is proposed for analysing the bending problem of plates based on the Mindlin plate theory. The method is based on the discrete method5),6). Like the method5),6), the present method doesn’t employ the prior assumption of shape of deflection, such as shape function used in the finite element method. So the phenomenon of the shear locking doesn’t happen. Due to the unknown quantities are only on the boundary, the number of these quantities is fewer. That helps to save the computer storage. Compared with the method5), the present method has two advantages. One is that the present method requires less computer time to calculate the coefficients. Another is that non-uniform divisions can be used. Some numerical results are given to show the efficiency and accuracy of the present method for the bending problem of the plates.

2. Discrete method

In this section, the discrete method5),6) for the plate bending problems will be reviewed.

2.1 Fundamental Differential Equations

Consider a rectangular plate of length $a$, width $b$, density $\rho$. An $xyz$ coordinate system is used in the present study with its $x-y$ plane contained in the middle plane of the rectangular plate, the $z$-axis perpendicular to the middle plane of the plate and the origin at one of the corners of the plate.

In this paper, the deflection $w$, the rotations $\theta_x, \theta_y$, the shearing forces $Q_x, Q_y$, the twisting moment $M_{xy}$ and the bending moments $M_x, M_y$ are used as variables.

Considering the equations of equilibrium, the
strain-displacement relations, the stress-strain relations and the load-stress relations, the fundamental differential equations of the plate having uniform load \( \bar{q} \) are as follows:

\[
\begin{align*}
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \bar{q} &= 0, \\
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y &= 0, \\
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= 0, \\
\frac{\partial \theta_x}{\partial x} + \nu \frac{\partial \theta_y}{\partial y} &= \frac{M_x}{D}, \\
\frac{\partial \theta_y}{\partial y} + \nu \frac{\partial \theta_x}{\partial x} &= \frac{M_y}{D}, \\
\frac{\partial \theta_y}{\partial y} + \frac{\partial \theta_x}{\partial x} &= \frac{2 M_{xy}}{(1 - \nu) D}, \\
\frac{\partial w}{\partial x} + \theta_x &= \frac{Q_x}{G t_s}, \\
\frac{\partial w}{\partial y} + \theta_y &= \frac{Q_y}{G t_s}.
\end{align*}
\]

where \( D = E h^3/(12(1 - \nu^2)) \) is the bending rigidity; \( E \) and \( G \) are modulus and shear modulus of elasticity, respectively; \( \nu \) is Poisson’s ratio; \( h \) is the thickness of plate; \( t_s = h/1.2 \) in which 1.2 is the shear correction factor.

By choosing the standard thickness and bending rigidity of the plate as \( h_0 \) and \( D_0 = E h_0^3/(12(1 - \nu^2)) \) and introducing the non-dimensional expressions,

[1, 2, 3, 4, 5, 6],

\[
\begin{align*}
[X_1, X_2] &= \frac{a^2}{D_0(1 - \nu^2)} [Q_y, Q_x], \\
[X_3, X_4, X_5] &= \frac{a}{D_0(1 - \nu^2)} [M_{xy}, M_y, M_x], \\
[X_6, X_7, X_8] &= [\theta_y, \theta_x, w/a, \eta, \zeta] = [x/a, y/b],
\end{align*}
\]

the simple systemized expression of fundamental differential equations of the bending problem of a rectangular plate is as follows

\[
\sum_{s=1}^{8} \{ F_{ts} \frac{\partial X_s}{\partial \eta} + F_{2ts} \frac{\partial X_s}{\partial \eta} + F_{3ts} X_s \} + q \delta_{tt} = 0,
\]

where \( t = 1 \sim 8; q = \bar{q} \mu a^2 / D(1 - \nu^2); \) \( q \) the distributed load; \( \mu = b/a; \) \( \delta_{tt} \) Kronecker’s delta; \( F_{1ts}, F_{2ts}, \) and \( F_{3ts} \) are given in Appendix A.

### 2.2 Fundamental Solutions

As given in Ref. [5], [6], by dividing a rectangular plate vertically into \( m \) equal-length parts and horizontally into \( n \) equal-length parts as shown in Fig. 1, the plate can be considered as a group of discrete points which are the intersections of the \((m+1)\)-vertical and \((n+1)\)-horizontal dividing lines. By integrating Eq. (1) over the area \( 0 \leq \eta \leq \eta_i, 0 \leq \zeta \leq \zeta_j \) and applying the numerical integration method, the simultaneous equation for the unknown quantities \( X_{\text{p}ij} = X_p(\eta_i, \zeta_j) \) at the point \((i, j)\) is obtained as follows:

\[
X_{\text{p}ij} = \sum_{t=1}^{8} \left\{ \beta_{ik} A_{pt} [X_{i0t} - X_{tkj}(1 - \delta_{ik})] \right\} + \sum_{l=0}^{j} \beta_{jl} B_{pt} [X_{t0l} - X_{tlj}(1 - \delta_{jl})] + \sum_{k=0}^{i} \beta_{ikj} C_{ptkl} X_{tkj}(1 - \delta_{ik} \delta_{jl}) \right\} - \sum_{k=0}^{i} \beta_{ikj} B_{ptl} q_{ij},
\]

where \( p = 1 \sim 8, A_{pt}, B_{pt} \) and \( C_{ptkl} \) are given in Appendix A.

By spreading the area according to the order mentioned in Ref. [5], [6], the quantity \( X_{\text{p}ij} \) at the point \((i, j)\) is only related to the quantities \( X_{\text{kn}l} (r=1,3,4,6,7,8) \) and \( X_{\text{sl}l} (s=2,3,5,6,7,8) \) at the boundary dependent points. Eq. (3) is rewritten as follows.

\[
X_{\text{p}ij} = \sum_{d=1}^{6} \left\{ \sum_{f=0}^{i} a_{\text{p}ijdf} X_{r0f} + \sum_{g=0}^{j} b_{\text{p}ijdg} X_{s0g} \right\} + q_{\text{p}ij},
\]

where

\[
\begin{align*}
a_{\text{p}ijdf} &= \sum_{t=1}^{8} \left\{ \beta_{ik} A_{pt} [a_{t0f} - a_{tkj}(1 - \delta_{ki})] \right\} + \sum_{l=0}^{j} \beta_{jl} B_{pt} [a_{t0l} - a_{tlj}(1 - \delta_{jl})] + \sum_{k=0}^{i} \beta_{ikj} C_{ptkl} a_{tkj}(1 - \delta_{ki} \delta_{jl}) \right\} \\
b_{\text{p}ijdg} &= \sum_{t=1}^{8} \left\{ \beta_{ikj} B_{ptl} q_{ij} \right\}.
\end{align*}
\]
Basing the discrete method \(^5\), \(^6\), an improved method for the plate bending problems is proposed.

### 3.1 Fundamental Differential Equations

The simple systemized expression of fundamental differential equations is as the same as Eq. \((2)\).

### 3.2 Fundamental Solutions

By dividing a rectangular plate vertically into \(m\) equal-length parts and horizontally into \(n\) equal-length parts as shown in Fig. (2), the plate can be considered as a group of discrete points which are the intersections of the \((m+1)\)-vertical and \((n+1)\)-horizontal dividing lines. By integrating Eq. \((2)\) over the area \(\eta_{i-1} \leq \eta \leq \eta_i, \zeta_{j-1} \leq \zeta \leq \zeta_j\), the following equation can be obtained.

\[
\int_{\eta_{i-1}}^{\eta_i} \int_{\zeta_{j-1}}^{\zeta_j} \left\{ F_{1\zeta} \frac{\partial X_s}{\partial \zeta} + F_{2\eta} \frac{\partial X_s}{\partial \eta} + F_{3\zeta\eta} X_s \right\} d\eta d\zeta \\
+ \int_{\eta_{i-1}}^{\eta_i} \int_{\zeta_{j-1}}^{\zeta_j} q \delta_{1\zeta} d\eta d\zeta = 0, \quad (8)
\]

By using the Green integration and the trapezoidal integration rule, the simultaneous equation for the unknown quantities \(X_{pij} = X_p(\eta_i, \zeta_j)\) at the point \((i, j)\) is obtained as follows:

\[
X_{pij} = \sum_{t=1}^{8} \left\{ \beta_{ii} A_{pt}[X_{t(i-1)(j-1)} + X_{t(i-1)j} - X_{t(i-1)j}] \\
+ \beta_{jj} B_{pt}[X_{t(i-1)(j-1)} + X_{t(i-1)j} - X_{t(i-1)j}] \\
+ \beta_{ij} \beta_{jj} [C_{pt(i-1)(j-1)} X_{t(i-1)(j-1)}] \\
+ C_{pt(i-1)j} X_{t(i-1)j} + C_{pt(i-1)j} X_{t(i-1)j} \right\} \\
- A_{pt} \left\{ q(i-1)(j-1) + q(i-1)j + q(i-1) + q(i)j \right\}, \quad (9)
\]

where \(\beta_{ii} = \bar{h}_i/2, \bar{h}_i = \eta_i - \eta_{i-1}, \beta_{jj} = \bar{h}_j/2, \bar{h}_j = \zeta_j - \zeta_{j-1}, p = 1 \sim 8, A_{pt}, B_{pt} \) and \(C_{ptkl}\) are given in Appendix A.

In Eq. \((9)\), the quantity \(X_{pij}\) is related to the quantities \(X_{t(i-1)(j-1)}, X_{t(i-1)j}, \) and \(X_{t(i-1)j}\) at the three internal points. By choosing the point \([i, j]\) according to the order as \([1, 1], [1, 2], \ldots, [1, n], [2, 1], [2, 2], \ldots, [2, n], \ldots, [m, 1], [m, 2], \ldots, [m, n]\) and substituting the obtained results into the corresponding terms of the right hand side of Eq. \((9)\), the quantities \(X_{t(i-1)(j-1)}, X_{t(i-1)j}, \) and \(X_{t(i-1)j}\) at the three internal points can be eliminated and the quantity \(X_{pij}\) at the point \((i, j)\) is only related to the quantities \(X_{s0r} (r=1,3,4,6,7,8)\) and \(X_{s0l} (s=2,3,5,6,7,8)\) at the boundary dependent points. Eq. \((9)\) is rewritten as follows.

\[
X_{pij} = \sum_{d=1}^{6} \left\{ \sum_{f=0}^{i} a_{pfjdf} X_{rf0} + \sum_{g=0}^{j} b_{pijgdf} X_{s0g} \right\} + q_{pij}, \quad (10)
\]

where

\[
a_{pfjdf} = \sum_{t=1}^{8} \left\{ \beta_{ii} A_{pt}[a_{t(i-1)(j-1)df} + a_{t(i-1)jfd} - a_{t(i-1)jfd}] \\
+ \beta_{jj} B_{pt}[a_{t(i-1)(j-1)df} + a_{t(i-1)jfd} - a_{t(i-1)jfd}] \right\}
\]
By comparing Eqs.(5) ~ (7) with Eqs.(11) ~ (13), it can be noted that the two summations have been changed to one summation. So by using the Eqs.(11) ~ (13), the computer time of the coefficients can be reduced greatly.

4. Numerical Results

To investigate the validity of the proposed method, numerical results are presented for several specific problems and comparisons are made with previously published results where possible. \( \nu = 0.3 \) is used. All the convergent values are obtained for the plates by using Richardson’s extrapolation formula for two cases of divisional numbers \( m (= n) \).

4.1 Convergence of the present method

In order to examine the convergency, numerical calculation is carried out by varying the number of divisions \( m \) and \( n \) for a square plate with four simply supported edges noted as SSSS. The deflection of the plate for different division number is shown in Fig. 3. It can be found the numerical results converge monotonously from above with increase of the divisional number and the results of the divisional numbers \( m (=n) \) of 12 and 16 are almost same. So it is suitable to obtain the convergent result by using Richardson’s extrapolation formula for two cases of divisional numbers \( m (=n) \) of 12 and 16. By repeating the above procedure, the suitable number of divisions \( m (=n) \) can be determined for the other plates.

4.2 Comparison of the present method with the discrete method

(1) Comparison of the accuracy of the results

Table 1 shows the quantities of the deflection \( w \), the shear force \( Q_y \), the twisting moment \( M_{xy} \) and the moment \( M_z \) at the appointed points shown in Fig 4. The plate with four edges simply supported (SSSS) is considered. In this table, the numerical results obtained by the discrete method \(^5\) and the exact results obtained in Ref. \(^7\) are also shown. It can be seen the present method has the same accuracy as the discrete method. These results obtained by the two methods are in good agreement with exact results.

(2) Comparison of the computer time

Fig. 5 shows the flow chart of computation. The computation is divided into four parts. The ratio of

\[
\begin{align*}
q_{ij} &= \sum_{i=1}^{s} \left\{ \beta_i A_p \left[ q_i(i-1)j - q_{i-1}j \right] \\
&+ \beta_j B_p \left[ q_{i-1}j(i-1) - q_{i-1}j(i-2) \right] \\
&+ \beta_i \beta_j \left[ C_p(i-1)(i-j)q_j\right] \\
&+ C_p(i-1)(i-j)q_j \right\} \\
- A_p \left\{ q_{i-1}(i-1) + q_{i-1}j + q_i(i-1) + q_{ij} \right\}
\end{align*}
\]

\[ (13) \]

By comparing Eqs.(5) ~ (7) with Eqs.(11) ~ (13), it can be noted that the two summations have been changed to one summation. So by using the Eqs.(11) ~ (13), the computer time of the coefficients can be reduced greatly.

Table 1 The quantities of SSSS square plate under uniform load (\( \nu = 0.3, h/a = 0.01 \))

<table>
<thead>
<tr>
<th>Quantities</th>
<th>Pre.</th>
<th>Ref. (^5)</th>
<th>Ref. (^7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( wD_o/q_{a^2} \times 10^5 )</td>
<td>4.07</td>
<td>4.07</td>
<td>4.06</td>
</tr>
<tr>
<td>( Q_y/q_{a} )</td>
<td>0.34</td>
<td>0.34</td>
<td>0.34</td>
</tr>
<tr>
<td>( M_{xy}/q_{a^2} )</td>
<td>-0.033</td>
<td>-0.033</td>
<td>-0.033</td>
</tr>
<tr>
<td>( M_z/q_{a^2} )</td>
<td>0.048</td>
<td>0.048</td>
<td>0.048</td>
</tr>
</tbody>
</table>

Pre.: Present Results
Fig. 4  The deflection $w$, the shear force $Q_y$, the moment $M_x$ and the twisting moment $M_{xy}$ at the appointed points of the simply supported plate.

Fig. 5  Flow chart of computation.

Fig. 6  The ratio of the computer time of each part.

Fig. 7  The computer time ratio of the discrete approximate method to the present method at different parts.

The computer time of these four parts is shown in Fig. 6 by using personal computer (Dell Pentium 4 2.8GHz 2.50GB). For divisional number $m = 12$, the computation time for the four parts is $0.1094s$, $41.1406s$, $0.3438s$, $0.0781s$ by using the discrete method and $0.1094s$, $4.6563s$, $0.0625s$, $0.0781s$ by using the present method, respectively. For each part, the ratio $t_1/t_2$ of computer time $t_1$ of the discrete method to computer time $t_2$ of the present method is shown in Fig. 7. Two divisional numbers $m(= n) = 12$ and $m(= n) = 16$ are considered. From Fig. 7, it can be noted the ratio $t_1/t_2 = 1$ for the first and the forth parts. That means the computer time of the discrete method is as the same as that of the present method. But for the second part, $t_1/t_2 = 8.8$ and $t_1/t_2 = 13.4$ for $m = 12$ and $m = 16$, and for the third part, $t_1/t_2 = 5.5$ and $t_1/t_2 = 11.9$ for $m = 12$ and $m = 16$, respectively. With increase of the divisional number, the ratio $t_1/t_2$ increase. As shown in Fig. 6, the computer time of the second and third parts is about 95 percentage of the whole time. So
the computer time of the present method is much less than that of the discrete method 5), especially for the larger divisional number. It can be understood by comparing Eqs.(11)~(13) with Eqs.(5)~(7). In order to obtain the coefficient at point \((m, n)\), \(((m+1)\times(n+1) - 1)\) points are needed to use by using Eqs.(5)~(7), but only three points are needed by using Eqs.(11)~(13). The numbers of the points needed in the present method and the discrete method 5) are shown in Table 2. It can be noted the number of the points needed in the present method is much less that of the discrete method 5).

### 4.3 Plate with uniform divisions

Table 3 shows the quantities of the deflection \(w\), the shear force \(Q_y\), the moments \(M_x\) and \(M_y\) at the appointed points of CCCC square plate. The square plate with four clamped edges (CCCC) is considered. The results obtained by the present method are compared with the exact results of Ref. 7). It can be seen these results agree well.

Fig 9 shows an isosceles right triangular plate and its equivalent rectangular plate. The thickness of the triangular plate is \(h_0\) and the boundary conditions are the diagonal clamped and the other edges simply supported. The equivalent rectangular is obtained by adding a triangular part to the original plate. The boundary conditions of the added part are clamped and the thickness is \(h_1\). The thickness \(h_1\) is much larger than \(h_0\). The thickness of the diagonal is chosen as \((h_0 + h_1)/2\). The numerical results for the deflection \(w\) and the moment \(M_x\) are presented in Tables 4 and 5. The numerical results obtained by Fletcher 8) and FEM are also shown. From these tables, it can be noted that the present results have enough accuracy.

### Table 2 The number of the points used to obtain the coefficient at point \((m, n)\)

<table>
<thead>
<tr>
<th>Division ((m \times n))</th>
<th>4 × 4</th>
<th>8 × 8</th>
<th>12 × 12</th>
<th>16 × 16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ref. 5)</td>
<td>24</td>
<td>80</td>
<td>168</td>
<td>288</td>
</tr>
<tr>
<td>Present</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

### Table 3 The quantities of CCCC square plate under uniform load \((v = 0.3, h/a = 0.01)\)

<table>
<thead>
<tr>
<th>Quantities</th>
<th>Pre.</th>
<th>Ref. 7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(wD_0/qA^2 \times 10^3)</td>
<td>1.27</td>
<td>1.26</td>
</tr>
<tr>
<td>(Q_y/qA)</td>
<td>0.44</td>
<td>-</td>
</tr>
<tr>
<td>(M_y/qA^2)</td>
<td>-0.051</td>
<td>-0.051</td>
</tr>
<tr>
<td>(M_x/qA^2)</td>
<td>0.023</td>
<td>0.023</td>
</tr>
</tbody>
</table>

### Fig. 8 The deflection \(w\), the shear force \(Q_y\), the moments \(M_x\) and \(M_y\) at the appointed points of CCCC square plate.

### Table 4 The deflection at point \((a/4, b/4)\) of the isosceles right triangular plate with diagonal clamped and the other edges simply supported \((wD_0/qA^2 \times 10^3, v = 0.3, b/a = 1.0, h_0/a = 0.01)\)

<table>
<thead>
<tr>
<th>(h_1/h_0)</th>
<th>Division ((m \times n))</th>
<th>Pre.</th>
<th>Ref. 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>36 × 36</td>
<td>0.369</td>
<td>0.371</td>
</tr>
<tr>
<td>8</td>
<td>40 × 40</td>
<td>0.357</td>
<td>0.360</td>
</tr>
<tr>
<td>10</td>
<td>36 × 36</td>
<td>0.353</td>
<td>0.356</td>
</tr>
</tbody>
</table>

### Table 5 The quantities of the deflection \(w\), the shear force \(Q_y\), the moments \(M_x\) and \(M_y\) at the appointed points of CCCC square plate.

(a)

(b)

Fig. 9 A triangular plate and its equivalent rectangular plate. (a) A triangular plate; (b) An equivalent rectangular plate of a triangular plate.
Table 5 The moment at point \((a/4, b/4)\) of the isosceles right triangular plate with diagonal clamped and the other edges simply supported \((M_z/qa^2, \nu = 0.3, b/a = 1.0, h_0/a = 0.01)\)

<table>
<thead>
<tr>
<th>Division ((m \times n))</th>
<th>(h_1/h_0)</th>
<th>Pre.</th>
<th>FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>(36 \times 36)</td>
<td>6</td>
<td>0.0133</td>
<td>0.014</td>
</tr>
<tr>
<td>(40 \times 40)</td>
<td>8</td>
<td>0.0132</td>
<td>0.013</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.0131</td>
<td>0.013</td>
</tr>
</tbody>
</table>

Table 6 Numerical results at point \((a/40, b/40)\) of SSSS square plate with uniform load

<table>
<thead>
<tr>
<th>Division ((m \times n))</th>
<th>(w_1D_0/qa^4 \times 10^3)</th>
<th>(Q_y/qa)</th>
<th>(M_z/qa^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>uniform</td>
<td>(40 \times 40)</td>
<td>0.288</td>
<td>0.0444</td>
</tr>
<tr>
<td>non-uniform</td>
<td>(10 \times 10)</td>
<td>0.290</td>
<td>0.0444</td>
</tr>
<tr>
<td></td>
<td>(20 \times 20)</td>
<td>0.288</td>
<td>0.0444</td>
</tr>
</tbody>
</table>

4.4 Plate with non-uniform divisions

Comparing with the discrete method \(^5\) which is only suitable for the plate with uniform divisions, one advantage of the present method is that it can be used for the plate with non-uniform divisions. In order to show the efficient of the present method, numerical results are given for the square plates with uniform and non-uniform divisions as shown in Fig. 10. The quantities at point \((a/40, b/40)\) are presented in Table 6. In order to obtain the quantities at point \((a/40, b/40)\), divisional number must be larger than 40 for the plate with uniform division. But for the plate with non-uniform division, the divisional number is not limited. From Table 6, it can be noted the results decrease with increase of the non-uniform divisional number, and the results of the plate with uniform division \(m = 40\) are as the same as those of the plate with non-uniform division \(m = 20\). So the computer time can be saved by using non-uniform division.

In order to show the accuracy of the present method for the plate with non-uniform division, the calculation is carried out for clamped circle plate with uniform and non-uniform divisions as shown in Fig. 11. The numerical results of the deflection \(w\) and the moment \(M_z\) at point \((a/2, a/2)\) are shown in Tables 7 ~ 10. It can be noted for the plate with same
Table 7 The deflection at point \((a/2, a/2)\) of a circle plate with uniform divisions \((wD_0/qa^4 \times 10^3, \nu = 0.3, h_0/a = 0.01)\)

<table>
<thead>
<tr>
<th>(h_1/h_0)</th>
<th>Division ((m \times n))</th>
<th>Pre.</th>
<th>FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.0</td>
<td>28 x 28</td>
<td>0.922</td>
<td>0.920</td>
</tr>
<tr>
<td>8.0</td>
<td>28 x 28</td>
<td>0.920</td>
<td>0.917</td>
</tr>
</tbody>
</table>

Table 8 The moment at point \((a/2, a/2)\) of the circle plate with uniform divisions \((M_x/qa^2, v = 0.3, h_0/a = 0.01)\)

<table>
<thead>
<tr>
<th>(h_1/h_0)</th>
<th>Division ((m \times n))</th>
<th>Pre.</th>
<th>FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.0</td>
<td>28 x 28</td>
<td>0.0196</td>
<td>0.0198</td>
</tr>
<tr>
<td>8.0</td>
<td>28 x 28</td>
<td>0.0196</td>
<td>0.0198</td>
</tr>
</tbody>
</table>

Table 9 The deflection at point \((a/2, a/2)\) of a circle with non-uniform divisions \((wD_0/qa^4 \times 10^3, \nu = 0.3, h_0/a = 0.01)\)

<table>
<thead>
<tr>
<th>(h_1/h_0)</th>
<th>Division ((m \times n))</th>
<th>Pre.</th>
<th>FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.0</td>
<td>28 x 28</td>
<td>0.888</td>
<td>0.903</td>
</tr>
<tr>
<td>8.0</td>
<td>28 x 28</td>
<td>0.883</td>
<td>0.898</td>
</tr>
</tbody>
</table>

Table 10 The moment at point \((a/2, a/2)\) of a circle plate with non-uniform divisions \((M_x/qa^2, v = 0.3, h_0/a = 0.01)\)

<table>
<thead>
<tr>
<th>(h_1/h_0)</th>
<th>Division ((m \times n))</th>
<th>Pre.</th>
<th>FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.0</td>
<td>28 x 28</td>
<td>0.0187</td>
<td>0.0189</td>
</tr>
<tr>
<td>8.0</td>
<td>28 x 28</td>
<td>0.0184</td>
<td>0.0187</td>
</tr>
</tbody>
</table>

divisional number, the deflection at point \((a/2, a/2)\) obtained by using non-uniform division is better than that obtained by using uniform division.

5. Conclusions

An improved method is proposed for analyzing the bending problem of plate. No prior assumption of shape of deflection used in the finite element method are employed in this method. By transforming the differential equations into integral equations in a small area, the quantities of an appointed point can be expressed by those of the other three points. That makes the computer time reduce greatly. The present method is suitable for plate with uniform and non-uniform divisions. Some numerical results are given for the rectangular plate, triangular plate and circle plate. Comparison of the numerical results of the present method with those previously reported is presented. It shows that the present results have a good convergence and satisfactory accuracy.

Appendix A

\[
F_{111} = F_{124} = F_{133} = F_{156} = F_{167} = F_{188} = 1; \quad F_{146} = \nu; \quad F_{212} = F_{223} = F_{235} = F_{247} = F_{266} = \mu; \\
F_{257} = \mu; \quad F_{278} = 1; \quad F_{321} = F_{332} = -\mu; \quad F_{345} = F_{354} = -I; \quad F_{363} = -J; \quad F_{372} = -H; \quad F_{373} = 1; \quad F_{381} = -\mu H; \quad F_{386} = \mu; \quad \text{other } F_{kkk} = 0. \quad A_{p1} = \gamma_{p1}, \quad A_{p2} = 0, A_{p3} = \gamma_{p2}, A_{p4} = \gamma_{p3}, A_{p5} = 0, A_{p6} = \gamma_{p4} + \nu \gamma_{p5}, \quad A_{p7} = \gamma_{p6}, A_{p8} = \gamma_{p7}.
\]

REFERENCES


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