Non-convex Mechanical Response of Pressure Dependent Yielding Materials within Associated and Non-associated Plasticity

Seiichiro TSUTSUMI*, Masahiro TOYOSADA** and Koichi HASHIGUCHI***

* Dr. of Agr., Research Associate, Dept. of Marine System Eng., Kyushu University
(Hakozaki 6-10-1, Higashi-ku, Fukuoka 812-8581)
** Dr. of Eng., Professor, Dept. of Marine System Eng., Kyushu University
*** Dr. of Eng. and Dr. of Agr., Professor, Dept. of Agr. Eng., Kyushu University

The occurrence of a nose, i.e. non-convexity, in the stress rate response envelope of the elastoplastic constitutive model to certain stretching directions can be explained as the result of the natural property of the model adopting the pressure dependent yield/plastic potential function, whilst the elastoplastic constitutive model assuming a non-associated plastic flow violates a condition of the plastic relaxation of the second work rate or stiffness moduli, representing the so-called Drucker's postulate on stability. In this article, the characters of the noses formed by the model involving associated and non-associated flow rule with pressure dependent yield/plastic potential function are examined focusing of the Drucker's postulate on stability. The existence of the stable softening mechanism is also pointed out for the non-associative elastoplastic materials.

Key Words: elastoplastic constitutive equation, plastic flow rule, non-associate, nose

1. Introduction

The phenomenological elastoplastic constitutive equation based on a plastic flow rule requires the plastic potential function and the yield function. If the plastic potential function different from the yield function is adopted, the model falls within the framework of the non-associated plasticity. In particular for materials with frictional characteristics like sand, rock, and concrete, the yield surface and the plastic potential are often chosen to be different to be able to simulate well the deformation behavior, while the elastoplastic stiffness matrix of the model becomes non-symmetric\(^{(1\text{-}3)}\).

To archives elaborated constitutive model under the non-proportional loading behavior the response envelopes of stress rate/stretching in the stress rate/stretching space has been considered clarifying the mechanical properties of the models\(^{(4\text{-}8)}\). It has been pointed out that the response envelopes of the models adopting a non-associate flow rule exhibit a peculiar nose, i.e. non-convexity of the response envelope, to some particular directions of stretching\(^{(5, 9, 10)}\), while some noses are observed for the models adopting an associated flow rule\(^{(9)}\). The existence of a nose in the stress rate space means that the material in an elastoplastic state behaves stiffer than in an elastic state to certain stretching directions.

Although its occurrence in the stress rate response envelope has been miss-understandably explained as the result of the adoption of the non-associated flow rule\(^{(9)}\), the authors\(^{(5, 11)}\) explained its occurrence as the natural property of the model adopting the pressure dependent yield function.

On the other hand, it has been clarified that the elastoplastic constitutive model adopting the non-associated plastic flow does not fulfill the condition for the plastic relaxation of the second work rate or stiffness moduli, and looses positive definiteness of elastoplastic matrix in the hardening regime, i.e. so-called Drucker's postulate on stability\(^{(1\text{-}3), 13, 13)}\).

In this article, focusing on not only the nose observed in the stress rate response envelope but also the stability condition in a sense of Drucker's postulate\(^{(13)}\), the
fundamental features on the mechanical properties of the models adopting the associated and the non-associated flow rule are examined for the pressure dependent yielding and plastic flow materials, in both hardening and softening states under triaxial condition.

The tensile stress (rate) and stretching (a symmetric component of velocity gradient) are taken to be positive throughout this article.

2. Elastoplastic Constitutive Equation

Let us summarize the constitutive equation for isotropic elastoplastic materials.

Denoting the current configuration of material particle as \( x \) and the current velocity as \( v \), the velocity gradient is described as \( L = \nabla v / \nabla x \), by which the stretching is defined as \( D = (L + L^T)/2 \), \( L^T \) standing for the transpose.

It is assumed that the stretching \( D \) (the symmetric part of the velocity gradient) is additively decomposed into the elastic stretching \( D_e \) and the plastic stretching \( D_p \), i.e.

\[
D = D_e + D_p \tag{1}
\]

where the elastic stretching \( D_e \) is given by

\[
D_e = \mathbf{E} : \dot{\sigma} \tag{2}
\]

\( \sigma \) is the Cauchy stress tensor and (\( \cdot \)) indicates the proper co-rotational rate with the objectivity. The fourth-order tensor \( \mathbf{E} \) is the elastic modulus, and given as usual in Hooke’s type as

\[
E_{ijkl} = \frac{1}{2} (K \delta_{ij} \delta_{kl} + G \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \tag{3}
\]

where \( K \) and \( G \) are the elastic bulk and shear moduli, respectively, which are functions of the stress and internal state variables in general. \( \delta_{ij} \) is the Kronecker’s delta, i.e. \( \delta_{ij} = 1 \) for \( i=j \) and \( \delta_{i\neq j} = 0 \) for \( i \neq j \). The functions for the yield surface \( f \) is assumed by the isotropic-hardening/softening yield condition:

\[
f(\sigma) = F(H) \tag{4}
\]

where the scalars \( F \) and \( H \) are isotropic hardening/softening function and variable, respectively, which control the size of the current yield surface. Let it be assumed that the function \( f \) is homogeneous of degree-one in the stress tensor \( \sigma \).

The time-differentiation of equation (4) leads to the consistency condition given by

\[
\frac{\partial f}{\partial \sigma} = \dot{F} \frac{\partial f}{\partial H} \tag{5}
\]

where

\[
\dot{F} = \frac{dF}{dH} \tag{6}
\]

and \( (\cdot) \) stands for the material-time derivative. Here, note that \( \dot{F} \) includes \( D_p \) in homogenous degree one since it has a dimension of time in degree minus one and fulfills \( \dot{H} = 0 \) for \( D_p = 0 \). For deriving equation (5) the following relation is used.

\[
\text{tr} \left( \frac{\partial f}{\partial \sigma} \sigma \right) = \text{tr} \left( \frac{\partial f}{\partial \sigma} \sigma \right) \tag{7}
\]

since it holds for the function \( f \), which is homogeneous of degree-one in the stress tensor \( \sigma \), that

\[
\text{tr} \left( \frac{\partial f}{\partial \sigma} (\sigma W - W) \right) = 0 \tag{8}
\]

where \( W \) is an arbitrary skew-symmetric second-order tensor.

Here, adopt the plastic flow rule assuming a non-associated flow rule.

\[
\mathbf{D}_p = \lambda \mathbf{M} \tag{9}
\]

where \( \lambda \) is the positive proportionality factor. The second-order tensor \( \mathbf{M} \) is the normalized outward-normal to the plastic potential surface \( g(\sigma) = g(\sigma_c) \) (\( \sigma_c : \) current stress):

\[
\mathbf{M} = \frac{\nabla g(\sigma)}{\nabla \sigma} \tag{10}
\]

\( \| \| \) standing for the magnitude.

The substitution of equation (9) into the consistency condition (5) leads to

\[
\lambda = \frac{\text{tr}(\mathbf{N} \dot{\sigma})}{M_p} \tag{11}
\]

where the second order tensor \( \mathbf{N} \) is the normalized outward-normal of the yield surface, i.e.

\[
\mathbf{N} = \frac{\nabla f(\sigma)}{\nabla \sigma} \| \frac{\nabla f(\sigma)}{\nabla \sigma} \| \tag{12}
\]

and the plastic modulus \( M_p \) is given as follows:

\[
M_p = \text{tr}(\mathbf{N} \dot{\sigma}) \frac{\dot{F}}{F} \tag{13}
\]

using the Euler’s theorem for a homogeneous function. \( h \) is the function of the stress, plastic internal state variables and \( \mathbf{M} \) in degree one, which is related to \( \dot{H} \) as

\[
h = \frac{\dot{H}}{\lambda} \tag{14}
\]

since \( \dot{H} \) is homogeneous degree-one in \( \mathbf{D}_p (=\lambda \mathbf{M}) \) and thus we can write \( \dot{H}(\sigma, H, \mathbf{D}_p) = \lambda h(\sigma, H, \mathbf{M}) \).

The plastic stretching is obtained from Eqs. (9) and (11) as

\[
\mathbf{D}_p = \frac{\text{tr}(\mathbf{N} \dot{\sigma})}{M_p} \mathbf{M} \tag{15}
\]

The substitution of Eqs. (2) and (15) into equation (1) gives rise to the stretching:
The positive proportionality factor $\lambda$ in the flow rule of Eq. (9) is expressed in terms of stretching, rewriting $\lambda$ by $A$, as follows:

$$A = \frac{\text{tr}(\text{NED})}{M^p + \text{tr}(\text{NEM})}$$

(17)

The inverse expression of equation (16) is given as follows

$$\hat{\sigma} = ED - \frac{\text{tr}(\text{NED})}{M^p + \text{tr}(\text{NEM})} EM$$

(18)

The elastoplastic stiffness tensor and the tangent compliance tensor obtained from Eq. (18) are not symmetric for a non-associated flow rule ($M \neq N$).

Now the stress rate can be additively decomposed into the elastic stress rate and the plastic-relaxation stress rate, i.e.

$$\dot{\sigma} = \dot{\sigma}^e + \dot{\sigma}^p$$

(19)

where

$$\dot{\sigma}^e = ED$$

(20)

$$\dot{\sigma}^p = - \frac{\text{tr}(\text{NED})}{M^p + \text{tr}(\text{NEM})} EM$$

(21)

The loading criterion is given as follows:

$$\mathbf{D}^p = 0: f(\sigma) - F(H) = 0 \text{ and } \text{tr}(\text{NED}) > 0,$$

$$\mathbf{D}^p = 0: \text{ otherwise.}$$

(22)

3. Material Functions for Soils

The particular forms of the material functions for soils are given for the calculation of the response envelopes.

Let Eq. (4) of the yield surface for soils be given by the same form as the modified Cam-clay model $^{14, 15}$ (see Fig. 1):

$$f = p(1 + \chi^2) = F$$

(23)

where

$$\chi = \frac{\|\eta\|}{m}$$

(24)

$$\eta = \frac{\sigma^*}{p}$$

(25)

$$\sigma^* = \sigma + pI$$

(26)

$$p = -\frac{\text{tr}(\sigma)}{3}$$

(27)

$m$ is the material parameter which exhibits the stress ratio at the critical state. $(\cdot)^*$ stands for the deviator component. The concrete form of the normalized outward-normal $\mathbf{N}$ of the yield surface of Eq. (23) is given as

$$\mathbf{N} = \frac{1}{3} \left( 1 - \chi^2 \right) \mathbf{I} - \frac{2 \chi \sigma^*}{m \|\sigma^*\|}$$

(28)

Let the plastic potential function $g(\sigma) = g(\sigma_p)$ for soils be simply given by the same form as the yield surface adopting the modified Cam-clay model. Then, the plastic potential surface is given by

$$g = p(1 + \chi^2) = F'$$

(29)

where

$$\chi' = \frac{\|\mathbf{n}\|}{m'}$$

(30)

$m'$ is the material parameter which controls the direction of the normalized outward-normal $\mathbf{M}$ to the plastic potential surface $g(\sigma)$. The normalized outward-normal of the plastic potential surface (29) is given as

$$\mathbf{M} = \frac{1}{3} \left( 1 - \chi'^2 \right) \mathbf{I} - \frac{2 \chi' \sigma^*}{m' \|\sigma^*\|}$$

(31)

If the material parameter $m'$ in Eq. (30) is replaced by $m$, the associativity of the plastic flow is induced.

The elastic bulk modulus $K$ is given as follows:

$$K = \frac{E}{\gamma}$$

(32)

The elastic shear modulus $G$ is assumed to be given by the elastic bulk modulus $K$ and the Poisson’s ratio $\nu$ as

$$G = \frac{3(1 - 2\nu)}{2(1 + \nu)} K$$

(33)
4. Material Constants and Conditions

The mechanical responses of the models proposed in section 3 with an associated and a non-associated flow rule, i.e. \( m = m' \) and \( m \neq m' \), respectively, are examined not only in the hardening (\( M^P > 0 \)) but also in the softening (\( M^P < 0 \)) state. Hereafter the stretching-controlled loading regimes are only considered, since load control precludes softening response.

The stress rate responses under triaxial condition (\( D_1, D_2 = D_3 \)) are calculated to the imposed stretching \( D \) with the identical magnitude, i.e. \( \|D\| = D = \text{const.} \), and various directions \( \alpha \) in the stretching space (see Figs. 2 and 3). Connection of the end points of stress rate vectors for the imposed stretching \( D \) forms the response envelope in the corresponding stress rate space \( (\beta, [\sigma^*]') \).

Stretchings with the various directions and the identical magnitude \( D \) are imposed at the stress state \( \chi = 2.0 \) and \( \chi' = (m/m')\chi = 1.2 \) (see Fig. 1). The stretchings imposed under triaxial condition (\( D_1, D_2 = D_3 \)) are expressed by isotropic \( p_D \) and deviatoric \( D^* \) components, respectively, where

\[
\begin{align*}
D^* &= \sqrt{\frac{2}{3}} (D_2 - D_1) \quad (|D^*| = \|D^*\|) \quad (35)
\end{align*}
\]

In Fig. 2 the imposed stretch are depicted in the stretching space \( (\sqrt{3}p_D, D^*) \) with the imposed stretching direction \( \alpha \) and the magnitude \( D \).

Now the material constants are assumed for the following calculations as

\[
\begin{align*}
&\nu = 0.2, \quad m = 1.2, \quad \chi = 2.0 \quad (36) \\
&a) \ m' = m \quad \text{for associated flow rule,} \\
&b) \ m' = 2.0 \quad \text{for non-associated flow rule} \quad (37)
\end{align*}
\]

Archiving hardening and softening responses of the model, the plastic modulus are assumed for \( H) \) hardening and \( S) \) softening, respectively, as follows:

\[
\begin{align*}
&H) \ M^P = 4G \quad \text{for hardening responses,} \\
&S) \ M^P = -G/2 \quad \text{for softening responses} \quad (38)
\end{align*}
\]

The relationships between the direction \( \alpha \) and the stretching components \( (D_1, D_2) \) together with the loading condition \( \text{tr}(\text{NED})/2G \) in Eq. (22) are shown in Fig. 3.

5. Response Envelopes in Stress Rate Space

The response envelopes formed by connecting the end points of stress rate vectors at the stress state \( \chi = 2.0 \) (\( \chi' = (m/m')\chi = 1.2 \)) are depicted. Here, note from Eq. (21) that the plastic-relaxation stress rate \( \bar{\sigma}^P \) of the model adopting the non-associated flow rule is not directed toward the inward-normal direction \(-M\) of the plastic potential surface but is directed toward the direction of \( -M-(\frac{K/2G-1/3}{\text{tr}(M)})\text{tr}(M)I \) (\( = -EM/2G \)) except for the cases of \( \text{tr}(M) = 0 \) and/or \( K/2G = 3/2 \), i.e. \( \nu = 0 \) from Eqs. (32) and (33). The same is true for the case of the associated flow rule, where the plastic-relaxation stress rate \( \bar{\sigma}^P \) is not directed toward the inward-normal direction \(-N\) of the yield surface except for the cases of \( \text{tr}(N) = 0 \) and/or \( \nu = 0 \). In other words, the direction of the plastic-relaxation stress rate \( \bar{\sigma}^P \) generally deviates from the
inward-normal direction of the yield surface $-N$ or the plastic potential surface $-M$.

In Fig. 4 the response envelopes in the $(\dot{\sigma}, |\sigma^*|)$ plane for the models adopting (a) associated and (b) non-associated flow rule in both hardening $(\dot{\sigma}^H)$ and softening $(\dot{\sigma}^S)$ state are depicted, while the elastic stress rate envelopes cut the vertical and horizontal axis at $2GD$ and $\sqrt{3}KD$, respectively.

As can be seen from Fig. 4 (a) the plastic-relaxation stress rate $\dot{\sigma}^P$ is not directed towards the inward-normal direction $-N$ of the yield surface. Therefore, as explained by the authors $^{8,11}$, the response envelope of stress rates by the associated plasticity model exhibits a nose, while the appearance of a nose in the response envelope of the stress rates only by the non-associated models has been miss-understandably pointed out as the peculiar response $^{4,5,9,10}$. Further, the response envelope in the softening state exhibits the crescent form, since a large stress relaxation into the inward direction of the yield surface is induced by the stress rate component normal to the yield surface.

Therefore, they generally exhibit a nose in each mechanical response, while the larger nose is predicted in Fig. 4 (b) for the model adopting the non-associated plastic flow. The nose and the crescent form in the stress rate response envelopes in the stress rate space seem peculiar, however. The experimental evidence and physical considerations are required in order to clarify.

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Fig. 4 Mechanical response $(\dot{\sigma}, |\sigma^*|)$ to the imposed stretching calculated by the model adopting (a) associated and (b) non-associated plastic flow for $\chi = 2.0$

Fig. 5 Mechanical response $(\dot{\sigma}, |\sigma^*|)$ to the imposed stretching calculated by the model adopting (a) associated and (b) non-associated plastic flow for $\chi = 1.0$
For stress state $\chi = 1.0$ (tr $N = 0.0, |\eta| = m$), as shown in Fig. 5(a), the stress rate envelops for the associated flow rule exhibit no nose, since the plastic-relaxation stress rate is directed towards the inward-normal direction $-N(-=EN/2G)$. On the other hand, they exhibit a nose for non-associated flow rule as depicted in Fig. 5(b).

In Figs. 6 and 7 the mechanical responses given in Figs. 4 and 5 together with the characteristic regions are depicted, respectively, for (a) associated and (b) non-associated plasticity models. For the associated plasticity model the hardening and the softening responses directly corresponds to the stable $\text{tr}(\mathbf{D}\mathbf{\hat{\sigma}}) > 0$ and the unstable conditions $\text{tr}(\mathbf{D}\mathbf{\hat{\sigma}}) < 0$, respectively, in a sense of Drucker's postulate, which corresponds to the condition for the plastic relaxation of the second work rate $\omega^p = \text{tr}(\mathbf{D}\mathbf{\hat{\sigma}}) = \lambda \text{tr}(\mathbf{M}\mathbf{\hat{\sigma}})$.

Fig. 8 shows the relationships between, the imposed stretching angle $\alpha$ and normalized second-order work for hardening and softening responses. For the non-associated model, as shown Figs. 6(b) and 7(b) and Fig. 8, there exist a region, which does not fulfill the condition on the stability even for the hardening stress rate. On the other hand, it should be noted that the model adopting the non-associated flow has the domain, which fulfills the condition of the stability even for the softening stress rate response. In other words, it shows the existence of the stable softening (see. Fig. 8).
6. Results and Discussion

The fundamental features of the associated and non-associated plasticity are examined for the prediction of plastically compressible materials from the viewpoints of the theoretical pertinence. The mechanical responses of the model for the stretch probe test exhibit the following characteristics. The response envelope of the stress rates to the imposed stretching exhibits a small nose in the stress rate plane. The direction of the plastic relaxation stress rate is affected by the elastic property of materials. It is not inward-normal of the yield surface and/or plastic potential surface, not only for the non-associativity but also the associativity of the flow rule. The appearance of the nose is caused substantially by the pressure-dependency of the yield surface. Nose can be appeared even for the model adopting associated flow rule as far as the yield surface depends on the pressure. Further, it is pointed out that the nose of response envelopes would be predicted much larger for softening states than for hardening states. Also, it should be noted that there exists the stable softening behavior for the non-associated material.

References

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