The First and Second Vertical Derivatives of Gravity.

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Abstract

A method has been shown to calculate the distribution of $\partial g/\partial z$ from that of $g$. The second derivative method has been critically reviewed.

§1. A paper by H.M. Evjen (1936) was likely to be the first to direct our interest anew to the problems of the vertical derivatives of gravity. More recently, the second derivatives have become the subject of many studies. Particularly Thomas A. Elkins (1951) has given a detailed description of the second derivative method of gravity interpretation and by several examples, has shown how this method is effective for detecting small irregularities in gravity anomalies and is therefore useful for figuring out minute underground mass distribution which are liable to be overlooked by the ordinary method. There can be no objection to Elkins' statements expressed in his paper. It is interesting to note that he wrote "its (the method) use is justified only on data of high accuracy".

§2. The second derivative method consists essentially of calculating $\partial^2 g/\partial x^2$ and $\partial^2 g/\partial y^2$ from a given distribution of $g$ and of summing up the two. It is evident from the Laplace's equation, that this sum will give $\partial^2 g/\partial z^2$ with the algebraic sign reversed. In the Laplace's equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0,$$

where $U$ is the gravity potential, the last term on the left side is nothing but $\partial g/\partial z$, while the first two have no direct connection with $g$ and can therefore not be calculated from $g$ at once. Here lies the reason why Elkins differentiated each term of the Laplace's equation with respect to $z$ and obtained the relation

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = 0.$$

This is a very important relation in that a possibility is thus opened to calculate a vertical derivative (second) of $g$ from its horizontal derivative (second). But at the same time, this was made possible at the cost of accuracy, because the second derivatives are too much sensitive to small errors in gravity measurements as Elkins admitted. In our opinion, what was aimed at by the second derivative method has been not so much to obtain the very quantity $\partial^2 g/\partial z^2$ as to get one which has a definite physical meaning and is reasonably sensitive to small irregularities in gravity distributions and which can be calculated directly from them.

§3. Let us consider a thin sheet of mass at the depth $d$ from the earth's surface and the mass along the sheet be expressed by

$$M(xy) = \sum \sum C_{mn} \alpha m \alpha n x^m y^n.$$

The gravitational potential due to this sheet is given by

$$U(xyz) = 2\pi k^2 \sum \sum C_{mn} \alpha m \alpha n x^m y^n e^{-\sqrt{m^2 + n^2 + (d+z)^2}},$$
where \( z \) is taken vertically upward with the origin at the earth's surface. \( k^2 \) is the universal constant of gravitation. The potential is so taken that its derivative in a certain direction will at once give the force in that direction. From the expression of the potential, it is evident that the horizontal distributions of gravity and its vertical derivatives are given by

\[
g(x,y) = 2\pi k^2 \sum C_m n \cos mx \sin ny e^{-\sqrt{m^2+n^2} d},
\]

\[
\frac{\partial g}{\partial z}(x,y) = -2\pi k^2 \sum \sqrt{m^2+n^2} C_m n \cos mx \sin ny e^{-\sqrt{m^2+n^2} d},
\]

\[
\frac{\partial^2 g}{\partial z^2}(x,y) = 2\pi k^2 \sum (m^2+n^2) C_m n \cos mx \sin ny e^{-\sqrt{m^2+n^2} d}.
\]

If the observed horizontal distribution of gravity is expressed by

\[
g(x,y) = \sum \sum B_m n \cos mx \sin ny,
\]

then

\[
\frac{\partial g}{\partial z} = -\sum \sqrt{m^2+n^2} B_m n \cos mx \sin ny,
\]

and

\[
\frac{\partial^2 g}{\partial z^2} = \sum (m^2+n^2) B_m n \cos mx \sin ny.
\]

Since these expressions do not involve \( d \), they can be extended to the case of any three dimensional underground mass distribution. There are two points to be noticed. The one is that the first vertical derivative of gravity can also be calculated from the horizontal distribution of gravity, if arithmetical labours in doing so are not spared. If the horizontal distribution of gravity is expressed by a double Fourier series and if each coefficient is multiplied by the corresponding \( \sqrt{m^2+n^2} \) and again synthesized, the results will at once give the horizontal distribution of the first vertical derivative of gravity. The computational labour is not so great as it at first appears to be. With well designed stencils for harmonic analyses and a calculating machine, the computations can be carried out in a relatively short time. In a forthcoming paper, one of the authors is going to describe his results of calculation of \( \frac{\partial g}{\partial z} \) from an actual distribution of gravity.

The second point to be noticed is that by the expressions given above, we can see to what degree \( \frac{\partial g}{\partial z} \) and \( \frac{\partial^2 g}{\partial z^2} \) are sensitive against local horizontal gravity variations. If these variations are regarded to consist of various components with different wave-lengths, the former is sensitive in the first power of the reciprocal of the wave-length, while the latter is so in its second power. This is also the very reason that \( \frac{\partial^2 g}{\partial z^2} \) is sensitive to observation errors.

After all, if the object of the second derivative method is really to see the distribution of \( \frac{\partial g}{\partial z^2} \), the method of Elkins is to be adopted. But it is to be remembered that \( \frac{\partial g}{\partial z} \) can also be calculated from gravity distribution. If the object of the second derivative method is merely to get a quantity which is reasonably sensitive to small irregularities in gravity distribution, the first power quantities seem to be preferable. An example of these is \( \sqrt{(\frac{\partial g}{\partial x})^2 + (\frac{\partial g}{\partial y})^2} \) and this quantity can be calculated easily from a given gravity distribution.

References
