Reflection and Refraction of Plane SH Waves
at Irregular Interfaces. II.

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Summary

The effect of an irregular interface with an isolated irregularity like a trough or a depression on the reflection and refraction of plane harmonic SH waves is investigated on the assumption that the maximum depth of the depression is small compared with the wavelength of the incident wave and that the slope of the surface is everywhere small. It is found that besides the specularly reflected and refracted SH waves there exists various scattered SH waves. In particular the scattered field contains the following secondary wave types whose amplitudes are proportional to the depth of the depression: (i) Direct reflected and refracted SH waves which appear to come from the point of intersection of the axis of symmetry of the depression and the horizontal plane asymptotic to the interface. They have cylindrical divergence. (ii) A diffracted wave which travelled along the interface with the higher of the two shear wave velocities for the media and is finally refracted into the medium with the slower velocity at the critical angle. (iii) An interface wave travelling with the slower of the two velocities in the media. Its energy is confined to the neighbourhood of the interface in the higher velocity halfspace.

§ 1. Introduction

In Part I of this paper it was found that when a plane harmonic SH wave is incident on a periodic interface separating two homogeneous elastic halfspaces its energy is scattered in various directions and in general the phase angles of the scattered waves differ from that of the incident wave. In this paper we extend the investigation to the study of the effect of an isolated irregularity in the interface on the reflection and refraction of plane harmonic SH waves. The notation of paper I and the results derived in it will be used throughout.

§ 2. Symmetrical Curvilinear Interface

Let the equation of the interface surface be

\[ z = \varepsilon f(x) = \varepsilon \exp(-\alpha x^2) \]  

(1.1)

where \( \alpha \varepsilon \) is assumed small so that the slope of the surface is everywhere small. (1.1) represents the cross-section in the \( x-z \) plane of a cylindrical surface whose generators are normal to that plane. It is further assumed that the maximum depth of the interface is small compared with the incident wavelength.

§ 3. First Order Approximation

In paper I we obtained the expressions for \( \widetilde{A}_0, \widetilde{B}_0 \) (I, 4.12). These are the amplitudes of the reflected and refracted waves whose amplitudes are independent of the irregularity of the interface. We now proceed to evaluate the first order amplitudes \( A_1, B_1 \). Thus inserting the value of \( f \) from (1.1) into the equations (I, 4.15) and (I, 4.16) which determine \( A_1, B_1 \) we obtain

\[ A_1 - B_1 = \frac{ie\widetilde{B}_0}{2\pi} (1 - \rho) (k_z^2 - k_i^2 \sin^2 \theta)^{1/2} \int_{-\infty}^{\infty} \exp(-a^2 x^2 + iax \sin \theta - i\zeta x) dx, \]  

(3.1)

\[ \nu_1 A_1 + \sigma_1 B_1 = \frac{ie\widetilde{B}_0}{2\pi} \int_{-\infty}^{\infty} [k_i^2 \cos^2 \theta - \rho (k_z^2 - k_i^2 \sin^2 \theta)] + 2a^2 ik_x (\sigma - 1) \sin \theta \exp(-a^2 x^2 + iax \sin \theta - i\zeta x) dx. \]  

(3.2)
Integrating these in closed form yields

\[ A_1 - B_1 = \frac{\lambda}{4a} \cdot B_0 \cdot [1 - \sigma(k_1^2 - k_1^2 \sin^2 \theta)^{1/2}] \cdot \exp \left[ -\frac{(\zeta - k_1 \sin \theta)^2}{2a^2} \right], \]  

(3.3)

\[ \nu_+ A_1 + \nu_+ B_1 = \frac{\lambda}{4a} \cdot B_0 \cdot [k_1^2 \cos^2 \theta - \sigma(k_1^2 - k_1^2 \sin^2 \theta) \cdot k_1 (\sigma - 1) \sin \theta] \cdot \exp \left[ -\frac{(\zeta - k_1 \sin \theta)^2}{2a^2} \right]. \]  

(3.4)

Solving the two simultaneous equations we obtain

\[ A_1(\zeta) = \frac{\lambda}{4ad(\zeta)} \cdot \nu_+ [\sigma \cdot (1 - \sigma)(k_1^2 - k_1^2 \sin^2 \theta)^{1/2} + k_1 (\sigma - 1) \sin \theta] \cdot \exp \left[ -\frac{(\zeta - k_1 \sin \theta)^2}{2a^2} \right], \]

(3.5)

\[ B_1(\zeta) = \frac{\lambda}{4ad(\zeta)} \cdot \nu_+ [\sigma \cdot (1 - \sigma)(k_1^2 - k_1^2 \sin^2 \theta)^{1/2} + k_1 (\sigma - 1) \sin \theta] \cdot \exp \left[ -\frac{(\zeta - k_1 \sin \theta)^2}{2a^2} \right]. \]

(3.6)

where

\[ d(\zeta) = \nu_+ - \nu_-, \]  

(3.7)

and we recall from (I, 4.13) that

\[ B_0 = 2k_1 \cos \theta |[k_1 \cos \theta + \sigma(k_1^2 - k_1^2 \sin^2 \theta)^{1/2}]|. \]  

(3.8)

Inserting the values of \( A_1(\zeta), B_1(\zeta) \) given by (3.5) and (3.6) into the integrals (I, 2.4) and (I, 2.5) respectively we obtain the displacements of the reflected and refracted waves whose amplitudes are proportional to \( \varepsilon \) as

\[
\begin{align*}
V_1 &= \varepsilon \int_{-\infty}^{\infty} A_1(\zeta) \exp (i \zeta x - i \nu_1 z) d\zeta, \\
V_2 &= \varepsilon \int_{-\infty}^{\infty} B_1(\zeta) \exp (i \zeta x + i \nu_2 z) d\zeta.
\end{align*}
\]  

(3.9)

This is the formal solution to this order of approximation. Exact evaluation of the integrals (3.9) along the \( \zeta \)-axis is impossible. We, therefore, regard \( \zeta \) as complex, and distort the path of integration so as to concentrate the important contributions in certain sections.

§ 4. Singularities and Distortion of Path of Integration

The only singularities of the integrands of (3.9) are branch points at \( \zeta = \pm k_1, \zeta = \pm k_2 \) at which \( \nu_1 = 0, \nu_2 = 0 \) respectively. We regard \( \omega \) as complex with its real and imaginary parts both positive. To make the integrands uniform-valued over the path of integration we make cuts along \( \text{Im} \nu_1 = 0, \text{Im} \nu_2 = 0 \) as shown in Fig. 1.

Henceforth \( \varepsilon \) will be regarded as positive.

\[ \zeta \text{-plane} \]

![Fig. 1. Positions of singularities and branch cuts in the \( \zeta \)-plane.](image)

\[ \zeta \text{-plane} \]

![Fig. 2. The loops around the cuts.](image)
We deform the path of integration from the real \( \zeta \)-axis into the infinite semicircle in the upper half \( \zeta \)-plane and going round the cuts along the contours \( \Gamma_1 \) and \( \Gamma_2 \) (Fig. 2.). The integrals vanish on the infinite semicircles and we are left with the branch line integrals

\[
V_{ij} = \epsilon \int_{r_j} A_i(\zeta) \exp(i\zeta x - i\nu_j z) d\zeta, \quad (4.1)
\]

\[
V_{kj} = \epsilon \int_{r_j} B_k(\zeta) \exp(i\zeta x + i\nu_k z) d\zeta, \quad (4.2)
\]

where \( j = 1 \) or \( 2 \).

§ 5. Interpretation of the Integrals

We assume that \( |\omega| \) is large enough for the exponential factor to vary faster than its multiplier in each integrand and that \( |x/z| \) is large enough for the variation of \( \exp(i\zeta x) \) to dominate that of other exponential factors in the integrands in the neighbourhood of the branch points.

As the point \( \zeta \) describes the contours \( \Gamma_1 \), \( \Gamma_2 \) around the cuts, the modulus of \( \exp(i\zeta x) \) will assume its largest value at the branch points, and will decrease rapidly as \( \zeta \) recedes from the branch points. So the major contribution to the integrals will arise from the neighbourhood of the branch points. Thus to a first approximation of the phases we have

(i) \( V_{11} \sim \exp(-i\omega t + ik_1 x) \). This represents secondary reflected SH wave.

(ii) \( V_{12} \sim \exp[-i\omega t + ik_1 x + iz(k_2^2 - k_1^2)^{1/2}] \). This is a secondary SH wave refracted into the upper medium. If \( k_2 > k_1 \), that is \( \beta_1 > \beta_2 \), it represents a wave refracted at an angle \( \sin^{-1}(\beta_2/\beta_1) \). If \( k_2 < k_1 \) it represents a wave whose energy is confined to the neighbourhood of the interface and may be regarded as an interface wave. This interface wave travels with the slower of the velocities of shear waves characteristic of the two media.

(iii) \( V_{12} \sim \exp[-i\omega t + ik_2 x - i\pi(k_1^2 - k_2^2)^{1/2}] \). This represents a secondary reflected plane wave or an interface wave according as \( k_1 \geq k_2 \).

(iv) \( V_{22} \sim \exp(-i\omega t + ik_2 x) \). This represents a secondary refracted wave in the upper halfspace.

§ 6. Evaluation of the Branch Integrals

In the preceding section by considering the contribution to the integrals from the neighbourhood of the branch points we have seen the type of waves which the integrals (4.1) and (4.2) represent. But we have been unable to make any statement concerning the amplitudes of the various waves. To this end we now proceed to approximate these integrals along the contours \( \Gamma_1 \) and \( \Gamma_2 \) in more detail after the method of Lapwood (1949).

Since \( \Gamma_1 \) lies near the cut \( Im\omega = 0 \), we may write \( \nu_1 = \pm u \) on \( \Gamma_1 \), where \( u \) is real and positive. It is readily determined that the positive sign refers to the right bank of the cut and the negative sign to its left bank as shown in Fig. 2.

Since, as discussed above, the major contributions to the integrals come from the neighbourhood of the branch points \( u = 0 \), we will regard \( u \) as small in making approximations to the integrands. Thus we have

\( \zeta = (k_2^2 - u^2)^{1/2} \) and therefore \( \zeta d\zeta = -udu \),

\( \nu_2 = (k_2^2 - \zeta^2)^{1/2} = (k_2^2 - k_1^2)^{1/2} = \gamma \), say.

In the exponential phase retardation of the integrands we put \( \zeta = k_1 u^2/2k_1 \), and in its multiplier we put \( \zeta \). The denominator can be approximated to by \( \gamma \). Hence the first integral taken along \( \Gamma_1 \) yields
Reinserting the hitherto omitted time factor \( \exp(-i\omega t) \) into (6.1) we have from (6.1)

\[
V_{11} \sim \exp \left[ -i\omega t + ik_1(x + z^2/2a) \right] \\
\equiv \exp \left[ -i\omega(t - r/\beta_1) \right],
\]

where

\[
r = (x^2 + z^2)^{1/2} \equiv x + z^2/2a,
\]

since it is assumed that \(|x/z| \ll 1\).

Thus (6.1) is the displacement due to a body wave apparently diverging from the origin of coordinates with velocity \( \beta_1 \). Its amplitude is inversely proportional to the square root of the distance traversed. The amplitude is a function of \( k_1/a \) which is proportional to the ratio of the halfwidth of the depression to the incident wavelength. For non-grazing angles of incidence the larger the ratio the smaller the amplitude. This, as will be seen subsequently, applies to all the secondary reflected waves at least to the first order approximation. This is in agreement with their absence in the limiting case of a trough with an infinite halfwidth which is equivalent to a plane boundary.

\[
V_{21} = \int_{\Gamma_1} B_1(\zeta) \exp(i\zeta x + i\zeta z) d\zeta
\]

\[
= \frac{ik_{1/2}B_2}{-2\pi k_1\sigma_1} \left( 1 - \sigma (k_2^2 - k_1^2 \sin^2 \theta)^{1/2} \cos \theta \right)
\]

\[
\times \exp \left[ -\frac{k_1(1 - \sin \theta)^2}{4a^2} + ik_1x + i\zeta z \right] \int_0^{2k_1/iz} u^2 \exp \left( -\frac{iuz^2}{2k_1} \right) du
\]

\[
= \frac{B_1}{4\pi a\sigma_1} \left( 1 - \sigma (k_2^2 - k_1^2 \sin^2 \theta)^{1/2} \cos \theta \right)
\]

\[
\times \exp \left[ -\frac{k_1(1 - \sin \theta)^2}{4a^2} + ik_1x + iz(k_2^2 - k_1^2)^{1/2} \right].
\]

(6.2)

If \( k_2 < k_1 \) that is \( \beta_1 < \beta_2 \), (6.2) represents an interface wave travelling with velocity \( \beta_1 \). Its amplitude is proportional to \((\text{distance})^{-1/2}\).

If \( k_2 > k_1 \) it is a real or plane wave. It is the diffracted wave which travelled along the interface with speed \( \beta \) (the velocity in the lower medium) and finally emerged into the upper medium at the critical angle \( \sin^{-1}(\beta_2/\beta_1) \).

To see its physical picture from Fig. 3 the phase retardation of the wave which travelled from \( O \) to \( A \) with speed \( \beta_1 \) and from \( A \) to \( B \) with speed \( \beta_2 \) is

\[
k_1(x - z \tan \sin^{-1}(\beta_2/\beta_1)) + k_2z \sec \sin^{-1}(\beta_2/\beta_1)
\]

\[
= k_1x + z(k_2^2 - k_1^2)^{1/2},
\]

Fig. 3. Suggested path of \( V_{21} \).
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which is the same as the phase retardation of the wave (6.2).

On \( I_2 \) we write \( \nu_1 = \pm u \) where \( u \) is real and positive. Then

\[
\zeta = (k_2^2 - u^2)^{1/2}
\]

and therefore \( \zeta d\zeta = -u du \),

\[
\nu_1 \equiv i\gamma .
\]

In the exponential phase retardation of the integrands we put \( \zeta = k_2 - u^2/2k_2 \) and in the multipliers we put \( \zeta = k_2 \). The denominator is approximately equal to \( i\gamma \). Hence we have

\[
V_{1\pm} = \frac{1}{\gamma k_2^{1/2}} \int_0^\infty u^2 \exp (-izu^2/2k_2) du
\]

If \( k_1 < k_2 \), that is \( \beta_1 > \beta_2 \), (6.3) represents an interface wave travelling with velocity \( \beta_2 \) and its amplitude is proportional to \( (\text{distance})^{-3/2} \). If \( k_1 > k_2 \) (6.3) represents a diffracted wave which travelled along the interface with velocity \( \beta_2 \) and was finally refracted at the critical angle \( \sin^{-1}(\beta_1/\beta_2) \) into the medium with shear velocity \( \beta_1 \).

Thus from (6.2) and (6.3) we conclude that the interface wave and the diffracted wave always exist. The interface wave travels with the smaller of the two velocities \( \beta_1, \beta_2 \); the diffracted wave travels along the interface with the higher of the two velocities and is refracted at the critical angle into the lower velocity medium.

\[
V_{1\pm} = \frac{1}{\gamma k_2^{1/2}} \int_0^\infty u^2 \exp (-izu^2/2k_2) du
\]

Hence reinserting the hitherto omitted time factor \( \exp (-i\omega t) \) and using

\[
r = (x^2 + z^2)^{1/2} = x + z^2/2x ,
\]

since \( |z/x| \) is assumed small

\[
V_{1\pm} \sim \exp [-i\omega (t - r/\beta_2)] .
\]

This represents a body wave which appears to radiate from the origin of coordinates. It travels with velocity \( \beta_2 \). Its amplitude depends on the angle of its refraction \( \cot^{-1}(z/x) \). The wave has a cylindrical divergence.

\[7. \text{Conclusion}\]

Thus when a plane harmonic SH wave is incident on the curvilinear interface

\[
z = \epsilon \exp (-a^2 x^2) \]

separating two homogeneous
isotropic elastic halfspaces besides the specul-"arily reflected and refracted waves which occur in the case of plane interface there appear the following secondary \( SH \) waves whose amplitudes are proportional to the maximum depth of the irregularity:

(i) A reflected \( SH \) wave which appears to come from the point of intersection of the axis of symmetry of the surface and its asymptote at infinity. It has a cylindrical divergence. The amplitude of the wave diminishes as \( (\text{distance})^{-1/2} \) and it is also proportional to the cotangent of the angle of its reflection.

(ii) A refracted \( SH \) wave with similar remarks as above.

(iii) An interface wave travelling with the smaller of the two velocities of \( SH \) waves in the two media. Its energy is confined to the neighbourhood of the interface in the higher velocity halfspace. Its amplitude is proportional to \( (\text{distance})^{-3/2} \) and so it diminishes rapidly with increasing distance.

(iv) A diffracted wave which travelled along the interface with the higher velocity and is finally refracted into the medium with the lower velocity at the critical angle for the medium. Its amplitude is proportional to \( (\text{distance})^{-3/2} \).

Reference